2024 Raffles Institution H3 Mathematics Preliminary Examinations

1	The Chebyshev polynomial of the first kind, $T_n(x)$, is defined by $$T_n(x)=\cos\left(n\cos^{-1}x\right)$$ so that $T_n(\cos\theta)=\cos\left(n\theta\right)$.	
	(a)	Write down $T_0(x)$ and $T_1(x)$, and show that $T_2(x)=2x^2-1$. [2]
	(b)	Show that $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$ for $n\ge 1$. [3]
	(c)	Hence find $T_n(0)$ in terms of *n*, describing carefully all the possible cases that arise. [3]
	(d)	Write down the roots of the equation $T_n(x)=0$. [2]
	(e)	Hence for positive integers *n*, evaluate $\prod_{k=0}^{n-1}\cos\left(\frac{2k+1}{2n}\pi\right)$, describing carefully all the possible cases that arise. [4]

2	A set of positive integers is called *well-spaced* if it contains at most one out of any three consecutive integers. For example, the set $\{2,5,8\}$ is *well-spaced* while $\{2,5,7,10\}$ is not.	
	Define S_n to be the number of *well-spaced* subsets, including the empty set, of the set $\{1,2,3,...,n\}$.	
	(a)	Find S_1, S_2 and S_3. [2]
	(b)	Explain why, for any positive integer *n*, $$S_{n+3}=S_{n+2}+S_n.$$ Hence obtain S_8. [4]
	Define $T_{n,k}$ to be the number of *well-spaced* subsets of the set $\{1,2,3,...,n\}$ that has *k* elements.	
	(c)	Use the bijection principle to show that $T_{n,k}=\binom{n-2k+2}{k}$. [4]
	(d)	Hence express S_n in terms of binomial coefficients. [2]

3	**(a)**	Let g and h be continuous functions defined on the real numbers such that $\mathrm{g}(x)+\mathrm{g}(-x)=1$ and $\mathrm{h}(x)=\mathrm{h}(-x)$ for all real numbers x.
		(i) Show that $\int_{-a}^{a} \mathrm{g}(x)\mathrm{h}(x)\, \mathrm{d}x = \int_{0}^{a} \mathrm{h}(x)\, \mathrm{d}x$ for any real number a. [3]
		(ii) Determine the exact value of $\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+2^x}\, \mathrm{d}x$. [4]
	(b)	Let f be defined on $\mathbb{R}$ such that $\mathrm{f}(x)=\lvert x\rvert$ for $\lvert x\rvert \le 1$ and $\mathrm{f}(x)=\mathrm{f}(x+2)$ for all real numbers x. Determine $\int_{0}^{\infty} \mathrm{f}(x)\mathrm{e}^{-x}\, \mathrm{d}x$. [8]

4	The sequence of real numbers $u_1, u_2, u_3, \dots,$ is defined by $$u_{n+2} = \frac{u_{n+1}}{u_n}\left(2u_n - u_{n+1}\right)$$ where k is a constant. It is given that $u_1 = a$ and $u_2 = b$, where a and b are non-zero real numbers.	
	(a)	Prove that $$u_{2n} = \frac{b}{a} u_{2n-1}$$ $$u_{2n+1} = c u_{2n}$$ for positive integers n, where c is a constant to be found in terms of a and b. [6]
	(b)	Determine the range of $\frac{b}{a}$ such that the series $\sum_{n=1}^{N} u_n$ converges. [5]
	(c)	Find the value(s) of $\frac{b}{a}$ such that the sequence $u_1, u_2, u_3, \dots$ is periodic with period 4. [3]

5 **(a)** Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a convex function. Using a sketch and by considering the vector equation of the line segment l joining the two points $(x,\varphi(x))$ and $(y,\varphi(y))$, explain why if x and y are any real numbers and $\lambda\in[0,1]$,

$$\varphi\big((1-\lambda)x+\lambda y\big)\le(1-\lambda)\varphi(x)+\lambda\varphi(y).$$

[3]

(b) Let $n\ge 1$ be a real number, and a and b be nonnegative real numbers. Show that

$$\left(\frac{a+b}{2}\right)^n\le\frac{a^n+b^n}{2}$$

and determine when equality holds. [4]

(c) Show that if a and b are nonnegative real numbers, and p and q are real numbers greater than 1 such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\le\frac{a^p}{p}+\frac{b^q}{q}.$$

[4]

6 A *Latin square of order n* is an $n\times n$ array, each cell containing one entry from the set $\{1,2,...,n\}$, with the property that each element of $\{1,2,...,n\}$ occurs once in each row and once in each column.

The following is an example of a Latin square of order 4.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

A *Latin rectangle* is a $k\times n$ array, where $k\le n$, with entries from $\{1,2,...,n\}$, with the property that each element of $\{1,2,...,n\}$ occurs once in each row and at most once in each column.

The following is an example of a 2×4 Latin rectangle.

1	2	3	4
3	4	2	1

	(a)	Explain why a Latin square of order *n* exists for each positive integer *n*. [1]
	(b)	Determine the number of Latin squares of order 2 and 3. [3]
	(c)	Given an $(n-1)\times n$ Latin rectangle, show that it is possible to add one more row to obtain a Latin square of order *n*. [3]
	(d)	Use the principle of inclusion and exclusion to show that the number of $2\times n$ Latin rectangles is $(n!)^2\sum_{i=0}^{n}\frac{(-1)^i}{i!}$. [5]

7	A postman needs to distribute letters to houses situated along a street. The houses along this street are regularly spaced apart and numbered 1, 2, …, *n*, where $n\geq 2$ is an integer. The postman has exactly one letter to distribute to each house, and he does so by first delivering the letter to house 1, before distributing the subsequent letters randomly and then returning to house 1. The postman thus follows a route, represented by the successive numbers of the houses where he drops off the letters. For example, if *n* = 5, a possible route is 1, 5, 2, 4, 3, 1. The total distanced travelled, which we call the length of the route, is 12, since $\lvert 1-5\rvert+\lvert 5-2\rvert+\lvert 2-4\rvert+\lvert 4-3\rvert+\lvert 3-1\rvert=12$. Another possible route is 1, 3, 5, 4, 2, 1, whose length is 8.	
	(a)	State the total number of possible routes. [1]
	(b)	Show that the minimum length of a route is $2(n-1)$. [3]
	(c)	Determine the number of routes of minimum length, explaining your reasoning clearly. [2]
	(d)	Determine the maximum length of a route. [5]

8	Let n be a positive integer and denote by $\sigma(n)$ the sum of all the positive divisors of n. We say that a number n is almost perfect if $\sigma(n) = 2n - 1$.	
	(a)	Show that if n is a power of 2, it is almost perfect. [1]
	(b)	Show that if a and b are coprime, then $\sigma(ab) = \sigma(a)\sigma(b)$. [3]
	(c)	Deduce that if n is odd and is almost perfect, then n must be a perfect square. [3]
	(d)	Show that for all prime numbers p and positive integers a, $$1 + \frac{1}{p} + \ldots + \frac{1}{p^a} < \frac{p}{p-1}.$$ [1]
	(e)	Let n be an odd integer greater than 1. Show that if n is almost perfect, then n must contain at least 3 distinct prime factors. [3]