Mark Scheme

Logarithm and Exponential Equations

1 (a)

$5(5^x)^2 - 18(5^x) + 16 = 0$

Let $y = 5^x$,

$5y^2 - 18y + 16 = 0$ M1

$(5y - 8)(y - 2) = 0$

$y = \frac{8}{5}$ or $y = 2$ M1

$5^x = \frac{8}{5}$ $\quad$ $5^x = 2$

$x = \frac{\ln\frac{8}{5}}{\ln 5}$ $\quad$ $x = \frac{\ln 2}{\ln 5}$ M1

$x = 0.292$ $\quad$ $x = 0.431$ A1

(b)

$\log_x 9 - \frac{1}{3}\log_x 3 + \log_x 27 = \frac{7}{6}$ M1 (change to base x)

$\log_x \frac{9 \times 27}{3^{\frac{1}{3}}} = \frac{7}{6}$ M1 (combine single log)

$\log_x 3^{\frac{14}{3}} = \frac{7}{6}$

$\log_x 3 = \frac{1}{4}$ M1

$x^{\frac{1}{4}} = 3$

$x = 81$ A1

OR

$\frac{1}{\frac{\log_3 x}{\log_3 9}} - \frac{1}{3\log_3 x} + \frac{\log_3 27}{\log_3 x} = \frac{7}{6}$ M1

$\frac{2}{\log_3 x} - \frac{1}{3\log_3 x} + \frac{3}{\log_3 x} = \frac{7}{6}$

$\frac{14}{3}\left(\frac{1}{\log_3 x}\right) = \frac{7}{6}$ M1

OR $2\log_x 3 - \frac{1}{3}\log_x 3 + 3\log_x 3 = \frac{7}{6}$

$\frac{14}{3}\log_x 3 = \frac{7}{6}$ M1

2

Solve $3^{2x-1} = 4^{2-x}$ and show that $x = \frac{\lg 48}{2\lg 6}$.

$3^{2x}(3^{-1}) = 4^2(4^{-x})$ — (M1)

$\frac{3^{2x}}{3} = \frac{16}{4^x}$

$9^x(4^x) = 16(3)$

$36^x = 48$ — (M1)

$\lg 36^x = \lg 48$

$x \lg 36 = \lg 48$

$x = \frac{\lg 48}{\lg 36}$

$= \frac{\lg 48}{2\lg 6}$ — (A1)

3

$\lg(x+3) + \lg(x-2) = 1$

$\lg[(x+3)(x-2)] = 1$ — (M1)

$(x+3)(x-2) = 10$ — (M1)

$x^2 + x - 6 - 10 = 0$

$x^2 + x - 16 = 0$

$x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-16)}}{2}$

$= \dfrac{-1 \pm \sqrt{65}}{2}$

$= [3.53 \text{ or } -4.53 \text{ (rej)}]$ — (A1)

Alternative

$\lg(x+3) = \lg\left(\dfrac{10}{x-2}\right)$ — (M1)

$x+3 = \dfrac{10}{x-2}$

$(x+3)(x-2) = 10$ — (M1)

4

Given that $\log_2 p = a$, $\log_q 8 = b$ and $\dfrac{p}{q} = 2^c$, express c in terms of a and b. [3]

$\log_q 8 = b$

$\dfrac{\log_2 8}{\log_2 q} = b$

$\dfrac{3}{\log_2 q} = b$

$\log_2 q = \dfrac{3}{b}$

$q = 2^{\frac{3}{b}}$ — (M1)

$\dfrac{p}{q} = \dfrac{2^a}{2^{\frac{3}{b}}}$ — (M1)

$= 2^{a - \frac{3}{b}}$

$\therefore c = a - \dfrac{3}{b}$. — (A1)

accept also: $c = \dfrac{ab-3}{b}$.

Alternative #1

$\log_q 8 = b$

$q^b = 8$

$q = 8^{\frac{1}{b}}$

$= 2^{\frac{3}{b}}$ — (M1)

$\dfrac{p}{q} = \dfrac{2^a}{2^{\frac{3}{b}}}$ — (M1)

$= 2^{a - \frac{3}{b}}$

$c = a - \dfrac{3}{b}$ — (A1)

Alternative #2

$\dfrac{p}{q} = 2^c$

$\log_2 \dfrac{p}{q} = c$

$\log_2 p - \log_2 q = c$

$a - \log_2 q = c$ — (M1)

$\log_q 8 = b$

$\dfrac{\log_2 8}{\log_2 q} = b$ — (M1)

$\dfrac{3}{\log_2 q} = b$

$\log_2 q = \dfrac{3}{b}$

$\therefore c = a - \dfrac{3}{b}$ — (A1)

Alternative #3

$\dfrac{\log_2 8}{\log_2 q} = b$.

$\dfrac{3}{b} = \log_2 q$.

$\log_2 \dfrac{p}{q} = \log_2 2^c$

$\log_2 p - \log_2 q = c$.

$a - \dfrac{3}{b} = c$.

5	**(a)**	$e\sqrt{e^x} = e^{2y}$ $e^{1+\frac{x}{2}} = e^{2y}$ $1+\frac{x}{2} = 2y$ $2+x = 4y$ $x = 4y-2$ ---(1) $\log_4(x+2) = 1+\log_2 y$ $\frac{\log_2(x+2)}{\log_2 4} = \log_2 2+\log_2 y$ $\frac{\log_2(x+2)}{2} = \log_2(2y)$ $\log_2(x+2) = 2\log_2(2y)$ $\log_2(x+2) = \log_2 4y^2$ $x+2 = 4y^2$ $x = 4y^2-2$ ---------------------------------------(2) $4y^2-2 = 4y-2$ $y^2-y=0$ $y(y-1)=0$ $y=0$ (reject) or $y=1$ $x = 4(1)-2 = 2$
	(b)	$2(100^y)-10^y = 6$ Let $10^y = x$ $2x^2-x-6=0$ -- $(2x+3)(x-2)=0$ $x = -\frac{3}{2}$ (reject) $x=2$ $10^y = 2$ $\lg 10^y = \lg 2$ $y = \lg 2$ or 0.301 (3 s.f.)
6	**(a)**	

$3^{x+1} \times 2^{2x+1} = 2^{x+2}$

$3^{x+1} \times \frac{2^{2x+1}}{2^{x+2}} = 1$

$3^{x+1} \times 2^{2x+1-(x+2)} = 1$

$3^{x+1} \times 2^{x-1} = 1$

$(3^x)3 \times \frac{2^x}{2} = 1$

$3^x \times 2^x = \frac{2}{3}$

$6^x = \frac{2}{3}$

(b)

$$\begin{aligned}\log_2 y &= \log_8 x - \log_2 4 \\ &= \frac{\log_2 x}{\log_2 8} - \log_2 4 \\ &= \frac{\log_2 x}{\log_2 2^3} - \log_2 4 \\ &= \frac{\log_2 x}{3} - \log_2 4\end{aligned}$$

$3\log_2 y = \log_2 x - 3\log_2 4$

$\log_2 y^3 = \log_2 x - \log_2 4^3$

$\log_2 y^3 = \log_2 \frac{x}{64}$

$y^3 = \frac{x}{64}$

$y = \frac{1}{4}x^{\frac{1}{3}}$

7 It is given that $\log_2\left(4-x^2\right)-\log_{\sqrt{2}}(x-1)=1$.

(a) Explain clearly why $1<x<2$. [4]

$4-x^2>0$

$(x-2)(x+2)<0$ [M1] and $x-1>0$ [M1]

$-2<x<2$ [A1] $x>1$

Reasonable conclusion using words or number line. [A1]

$\therefore\ 1<x<2$

(b) Hence, solve the equation and show that it has only one solution. [5]

$\log_2\left(4-x^2\right)-\dfrac{\log_2(x-1)}{\log_2\sqrt{2}}=1$ [M1]: Change base

$\log_2\left(4-x^2\right)-\log_2(x-1)^2=1$ [M1]: Power Law

$\log_2\dfrac{4-x^2}{(x-1)^2}=1$ [M1]: Quotient Law

$\dfrac{4-x^2}{(x-1)^2}=2$ [M1]: Convert form/Equivalence

$3x^2-4x-2=0$

$x=\dfrac{4\pm\sqrt{16-4(-6)}}{6}$ [A1]

$=1.72$ or -0.387 (rejected)

Logarithm and Exponential Word Problem

1 **(i)** Find the initial population of the insect. [1]

$$p = \frac{3500}{1 + 4e^{-k(0)}}$$

$$= 700 \; \# \; \text{— (B1)}$$

(ii) A group of scientists concluded that the population of the insect increased by 50% after 30 months. Find the value of *k*. [4]

$$p = \frac{150}{100} \times 700$$

$$= 1050. \; \text{— (M1)}$$

$$1050 = \frac{3500}{1 + 4e^{-k(2.5)}} \; \text{— (M1) : for } n = 2.5$$

$$= \frac{3500}{1 + 4e^{-2.5k}}$$

$$1 + 4e^{-2.5k} = \frac{3500}{1050}$$

$$= \frac{10}{3}$$

$$4e^{-2.5k} = \frac{7}{3}$$

$$e^{-2.5k} = \frac{7}{12}$$

$$\ln e^{-2.5k} = \ln \frac{7}{12} \; \text{— (M1) (ecf)}$$

$$-2.5k = -0.53899$$

$$k = 0.21559$$

$$= 0.216 \; \# \; \text{— (A1)}$$

	(iii) Another group of scientists argued that the population of the insects will be more than 4000 in the future. Do you agree with them? Explain your answer. [3]

when $n \to \infty$, $4e^{-0.216n} \to 0$. — (M1)

$\therefore P = \frac{3500}{1+4e^{-0.216n}} \to \frac{3500}{1} = 3500$. — (M1)

Hence, since the maximum number of the insects is 3500, it cannot be more than 4000. — (A1)

[Alt.]:

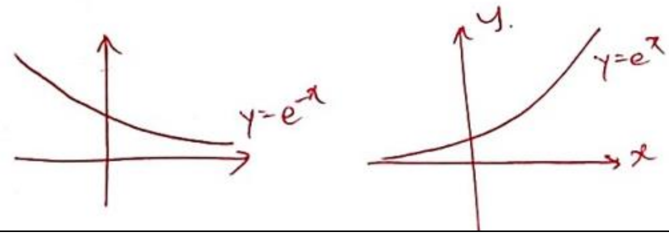
$\frac{3500}{1+4e^{-0.216n}} > 4000$ — [M1]

$1+4e^{-0.216n} < \frac{3500}{4000}$

$4e^{-0.216n} < -\frac{1}{8}$

$e^{-0.216n} < -\frac{1}{32}$. — [M1]

Since $e^x > 0$, it is not possible to be more than 4000. — [A1]



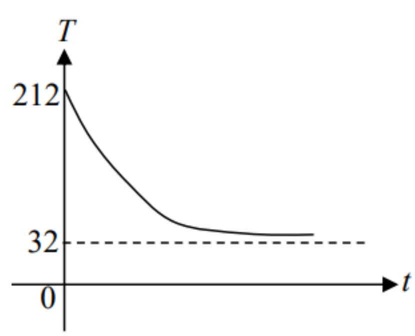
2	$P = 2^{-kn}$ Let $P = 0.5$, $n = 7200$ $0.5 = 2^{-7200k}$ $\lg 0.5 = \lg 2^{-7200k}$ $-7200k = \dfrac{\lg 0.5}{\lg 2}$ $k = 0.000138888$ / $\dfrac{1}{7200}$ When $P = 0.34$, $0.34 = 2^{-0.000138888n}$ $\lg 0.34 = \lg 2^{-0.000138888n}$ $-0.000138888n = \dfrac{\lg 0.34}{\lg 2}$ $n = 11206.03$ $n = 11206$ years	

3 **(a)**

Solution
$P = 212 - 180 = 32$ $32 + 180e^{-5k} = 185$ $e^{-5k} = 0.85$ $-5k = \ln 0.85$ $k = 0.0325$ (3 s.f.)

(b)

Solution



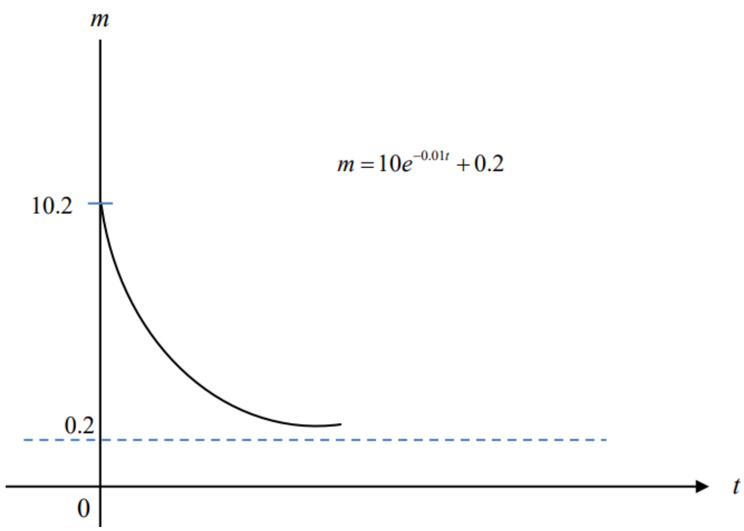
From the graph, the graph is completely above $T = 32$, hence, T cannot be 30.

4

(a) Find the initial mass.

$$\text{Initial mass} = 10e^{-0.01(0)} + 0.2$$
$$= 10.2 \text{ g}$$

(b) Sketch the graph $m = 10e^{-0.01t} + 0.2$ for $t \geq 0$.



(c) Find the least number of days it takes before the amount of substance is reduced to 5% of its initial mass.

$10e^{-0.01t} + 0.2 < 0.05(10.2)$ M1

$10e^{-0.01t} + 0.2 < 0.51$

$10e^{-0.01t} < 0.31$

$e^{-0.01t} < 0.031$

$-0.01t < \ln(0.031)$ M1

$t > 347.37$

Least number of days = 348 A1

5

(a) When $t = 0$,

$$B(0) = \frac{300}{1 + e^{5-0}}$$
$$= 2.01 \text{ (3 s.f.)}$$

Initial number of infected ducks is 2.

	(b)	$\frac{300}{1+e^{5-t}} = 100$ $300 = 100 + 100e^{5-t}$ $100e^{5-t} = 200$ $e^{5-t} = 2$ $5 - t = \ln 2$ $t = 4.31$ (to 3 s.f.) The number of infected ducks first reached 100 on Day 4.
	(c)	For all values of t, $e^{5-t} > 0$. $1 + e^{5-t} > 1$ $1 > \frac{1}{1+e^{5-t}}$ $\frac{1}{1+e^{5-t}} < 1$ $B(t) = \frac{300}{1+e^{5-t}} < 300$ Therefore the number of infected ducks will never exceed 300.

Logarithm and Exponential Graphs

1

$x = 2\ln\left(\frac{7-x}{5}\right)$

$\frac{x}{2} = \ln\left(\frac{7-x}{5}\right)$

$e^{\frac{x}{2}} = \frac{7-x}{5}$ —(M1)

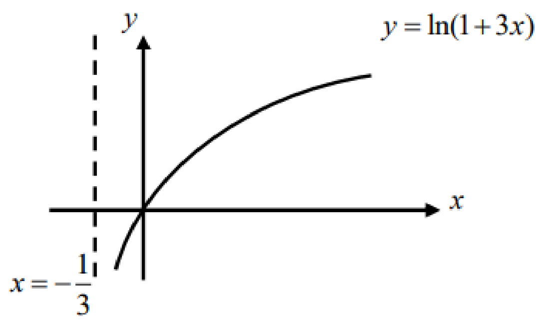
$5e^{\frac{x}{2}} = 7-x$

$5e^{\frac{x}{2}} - 4 = 7 - x - 4$

$5e^{\frac{x}{2}} - 4 = 3 - x$

$\therefore$ Equation: $y = 3 - x$. —(A1)

2 **(i)**



(ii)

$e^{\frac{6-3x}{4}} = 1+3x$

$\ln e^{\frac{6-3x}{4}} = \ln(1+3x)$

$\frac{6-3x}{4} = \ln(1+3x)$

Draw the line $y = \frac{6-3x}{4}$ or $4y = 6-3x$ or $y = -\frac{3}{4}x + \frac{3}{2}$

The **number of intersections** of the line and the curve is the number of solutions to the equation $e^{\frac{6-3x}{4}} - 1 - 3x = 0$. Hence, the student is correct.