RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 6 (Summary and Tutorial)

Topic: Vectors 1 (Vector Algebra, Ratio Theorem, Scalar and Vector Product)

Summary for Vectors 1

Vector Algebra

With reference to an origin $O(0, 0, 0)$, given points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$, we have the corresponding (position vectors) expressed in column form $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

$$\mathbf{a} \pm \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix} \text{ and}$$

$$k\mathbf{a} = k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \text{ with } k \text{ a real number.}$$

The **magnitude** (or modulus) of a vector, $|\mathbf{a}|$, is the non-negative number

$|\mathbf{a}| = \left| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. This value is equal to the distance from O to A.

We say that $\mathbf{a}$ is parallel to $\mathbf{b}$, denoted by $\mathbf{a} \mathbin{//} \mathbf{b}$, if and only if $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, that is, $\mathbf{b}$ is a (non-zero) scalar multiple of $\mathbf{a}$.

- If $\lambda > 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in the **same** direction.
- If $\lambda < 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in **opposite** directions.

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O, such that $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. We then say that the points O, A and B are **collinear**.

The **unit vector** in the **direction** of **a** denoted by $\hat{\mathbf{a}}$ is obtained by scaling **a** by $\frac{1}{|\mathbf{a}|}$, thus $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.

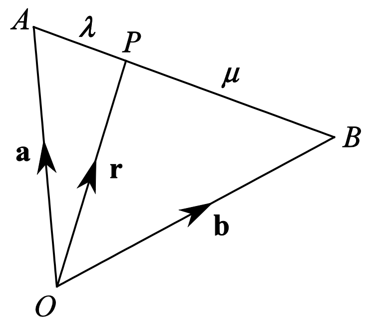
The vectors $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ are parallel since $\begin{pmatrix}-2\\-4\\4\end{pmatrix}$ is a scalar multiple of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ $(k=-2)$ but are in opposite directions since $k<0$. The points $(-2,-4,4)$, $(0,0,0)$ and $(1,2,-2)$ are also said to be collinear. The magnitude of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\sqrt{1^2+2^2+(-2)^2}=3$ so the unit vector in the direction of $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ is $\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero and **non-parallel** vectors:
If $\lambda\mathbf{a} = \mu\mathbf{b}$ for some $\lambda,\ \mu \in \mathbb{R}$, then $\lambda = \mu = 0$.
If $\alpha\mathbf{a} + \beta\mathbf{b} = s\mathbf{a} + t\mathbf{b}$ for some $\alpha, \beta, s, t \in \mathbb{R}$, then $\alpha = s,\ \beta = t$.

Note the importance of **non-parallel** vectors when comparing coefficients. Suppose $\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\0\\0\end{pmatrix}$ then $6\mathbf{a}+2\mathbf{b} = 4\mathbf{a}+3\mathbf{b}$ however we cannot "compare coefficients" of vectors **a** and **b** (note that **a** is parallel to **b**) as $6 \neq 4$ and $2 \neq 3$.

Ratio Theorem

Consider a triangle OAB with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
So $\mathbf{a}$ and $\mathbf{b}$ are non-zero and **non-parallel** vectors.
Let P be a point which divides AB in the ratio $\lambda : \mu$,
i.e. $\dfrac{AP}{PB} = \dfrac{\lambda}{\mu}$. If $\overrightarrow{OP} = \mathbf{p}$, then $\mathbf{p} = \dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$ (MF26)



Note that the Ratio Theorem is an immediate consequence of the addition of vectors.
From diagram, $\mathbf{p} - \mathbf{a} = \dfrac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a})$, rearranging we have $\mathbf{p} = \dfrac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}$. Sometimes, it is easier to use $\mathbf{p} - \mathbf{a} = \dfrac{\lambda}{\lambda + \mu}(\mathbf{b} - \mathbf{a})$ like the following example.

Points A, B and P have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ respectively, relative to the origin O. Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$, find $\mathbf{p}$ if P lies on AB produced such that $\dfrac{AB}{AP} = \dfrac{2}{5}$.

Solution: Easier to find directly, $\mathbf{p} - \mathbf{a} = \dfrac{5}{2}(\mathbf{b} - \mathbf{a})$ [DO NOT WRITE $\dfrac{\mathbf{b} - \mathbf{a}}{\mathbf{p} - \mathbf{a}} = \dfrac{2}{5}$. We cannot divide vectors]

$$\mathbf{p} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 2-2 \\ 6-2 \\ -3-5 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ -15 \end{pmatrix}$$

Scalar (Dot) Product

Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, relative to the origin O.

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where $\theta = \angle AOB$.

Re-arranging, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **same direction** then $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$, largest possible value.

If vectors $\mathbf{a}$ and $\mathbf{b}$ are in the **opposite direction** then $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$, smallest possible value.

Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, denoted by $\mathbf{a}\perp\mathbf{b}$, if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.
In particular, $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$ as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then $\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\cdot\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$.

Find the cosine of the angle between the vectors $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and determine if the angle is acute or obtuse.

Solution

$$\cos\theta = \frac{(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})\cdot(\mathbf{i} + 2\mathbf{j} - \mathbf{k})}{\sqrt{2^2 + 3^2 + 2^2}\sqrt{1^2 + 2^2 + (-1)^2}}$$

$$\cos\theta = \frac{2 - 6 - 2}{\sqrt{17}\sqrt{6}} = -\sqrt{\frac{6}{17}} < 0$$

θ is an obtuse angle

Properties of Scalar Product

(i) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

(ii) $\mathbf{a}\cdot(\mathbf{b} \pm \mathbf{c}) = (\mathbf{a}\cdot\mathbf{b}) \pm (\mathbf{a}\cdot\mathbf{c})$

(iii) $\lambda(\mathbf{a}\cdot\mathbf{b}) = (\lambda\mathbf{a})\cdot\mathbf{b} = \mathbf{a}\cdot(\lambda\mathbf{b})$

(iv) $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 \Leftrightarrow |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ note the relationship between the scalar product and the modulus

<table>
<tr><td colspan="2">Given that $|\mathbf{a}| = 2,\ |\mathbf{b}| = 3,\ \mathbf{a}\cdot\mathbf{b} = -1$,</td></tr>
<tr><td>(i) $(\mathbf{a} - \mathbf{b})\cdot(2\mathbf{a} + \mathbf{b})$
$= 2\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} - 2\mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}$
$= 2|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2$
$= 2(2^2) - (-1) - 3^2$
$= 0$</td><td>(ii) $(2\mathbf{a} - \mathbf{b})\cdot(2\mathbf{a} + \mathbf{b})$
$= 4\mathbf{a}\cdot\mathbf{a} + 2\mathbf{a}\cdot\mathbf{b} - 2\mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}$
$= 4|\mathbf{a}|^2 - |\mathbf{b}|^2$
$= 4(2^2) - 3^2$
$= 7$</td></tr>
<tr><td colspan="2">(iii) $|\mathbf{a} - 2\mathbf{b}| = \sqrt{(\mathbf{a} - 2\mathbf{b})\cdot(\mathbf{a} - 2\mathbf{b})} = \sqrt{\mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} - 2\mathbf{b}\cdot\mathbf{a} + 4\mathbf{b}\cdot\mathbf{b}} = \sqrt{|\mathbf{a}|^2 - 4\mathbf{a}\cdot\mathbf{b} + 4|\mathbf{b}|^2} = \ldots = 2\sqrt{11}$</td></tr>
</table>

Applications of Scalar Product

Points A and B have position vectors **a** and **b** respectively, relative to the origin O.
Given that P is the point on the line OB such that $\overrightarrow{AP} \perp \overrightarrow{OB}$, it can be shown that

$$\overrightarrow{OP} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|} = \left(\mathbf{a} \cdot \hat{\mathbf{b}} \right) \hat{\mathbf{b}}$$

P is the foot of perpendicular from A to the line OB and P is also the point on the line OB nearest to A.
Thus, the **length of projection of vector a onto vector b** is given by $\left|\overrightarrow{OP}\right| = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right|$.

> Points A and B have position vectors **a** and **b** respectively, relative to the origin O.
>
> Given that $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, find the length of projection of **a onto vector b.**
> Find the position vector of the foot of the perpendicular from A to OB.
>
> **Solution** $$OP = \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| = \frac{\left|(2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\right|}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|2 - 6 - 2|}{\sqrt{6}} = \sqrt{6}$$
>
> $$\overrightarrow{OP} = \left(\mathbf{a} \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}} = -\left(\frac{6}{\sqrt{6}}\right)\frac{\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{6}} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

If α, β and γ are the angles between **a** and the component vectors **i, j** and **k** respectively, then $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \left(\frac{a_1}{|\mathbf{a}|}, \frac{a_2}{|\mathbf{a}|}, \frac{a_3}{|\mathbf{a}|}\right) = \left(\cos\alpha, \cos\beta, \cos\gamma\right)$. Component of $\hat{\mathbf{a}}$ are referred to as the direction cosines.

Vector (Cross) Product

Let **a** and **b** be two non-zero vectors that are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

$$\mathbf{a}\times\mathbf{b}=\left(|\,\mathbf{a}\,||\,\mathbf{b}\,|\sin\theta\right)\hat{\mathbf{n}}$$

where $\theta=\angle AOB$ is the angle between **a** and **b**, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to both **a** and **b**.

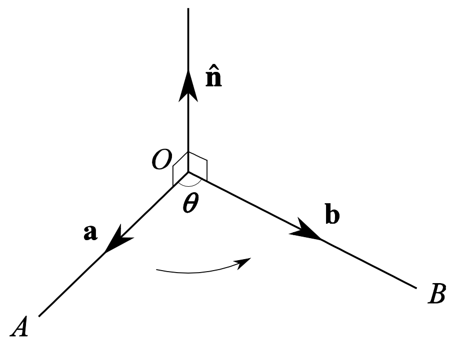


Figure 1

It follows that
Two non-zero vectors **a** and **b** are **perpendicular** if and only if $|\,\mathbf{a}\times\mathbf{b}\,|=|\,\mathbf{a}\,||\,\mathbf{b}\,|$. In particular, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$.

Two non-zero vectors **a** and **b** are **parallel** if and only if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.
In particular, $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$.

Given that $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\mathbf{0}$, what can be deduced about the vectors $\mathbf{u}$ and $\mathbf{v}$?

Solution:

$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=\mathbf{0}$

$\mathbf{u}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}+\mathbf{v}\times\mathbf{u}-\mathbf{v}\times\mathbf{v}=\mathbf{0}$

$\mathbf{0}+2\mathbf{v}\times\mathbf{u}-\mathbf{0}=\mathbf{0}$

$\mathbf{v}\times\mathbf{u}=\mathbf{0}$

$\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$ or $\mathbf{u}\parallel\mathbf{v}$

Properties of Vector Product

1. $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$.
2. $|\,\mathbf{a}\times\mathbf{b}\,|=|\,\mathbf{b}\times\mathbf{a}\,|=|\,\mathbf{a}\,||\,\mathbf{b}\,||\sin\theta\,|\le|\,\mathbf{a}\,||\,\mathbf{b}\,|$.
3. $\mathbf{a}\times(\mathbf{b}\pm\mathbf{c})=(\mathbf{a}\times\mathbf{b})\pm(\mathbf{a}\times\mathbf{c})$.
4. $\lambda(\mathbf{a}\times\mathbf{b})=(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})$.
5. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$ So $\mathbf{a}\times\mathbf{b}\perp\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}\perp\mathbf{b}$.

Examples of how the properties are used

$$\begin{aligned}|(\mathbf{a}+2\mathbf{b})\times(3\mathbf{a}-\mathbf{b})|&=|3\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+6\mathbf{b}\times\mathbf{a}-2\mathbf{b}\times\mathbf{b}|\\&=|\mathbf{0}+\mathbf{b}\times\mathbf{a}+6\mathbf{b}\times\mathbf{a}+\mathbf{0}|\\&=7|\mathbf{b}\times\mathbf{a}|\end{aligned}$$

$$\begin{aligned}|(\alpha\mathbf{a}+\beta\mathbf{b})\times(\alpha\mathbf{a}-\beta\mathbf{b})|&=|\alpha^2\mathbf{a}\times\mathbf{a}-\alpha\beta\mathbf{a}\times\mathbf{b}+\alpha\beta\mathbf{b}\times\mathbf{a}-\beta^2\mathbf{b}\times\mathbf{b}|\\&=\alpha\beta|\mathbf{0}+\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}+\mathbf{0}|\\&=2\alpha\beta|\mathbf{b}\times\mathbf{a}|\end{aligned}$$

Let $A(a_1,a_2,a_3)$ and $B(b_1,b_2,b_3)$ be two points in three-dimensional space and let the position vectors of A and B with respect to the origin O be **a** and **b** respectively.

Then vector (cross) product $\mathbf{a}\times\mathbf{b}$, is the vector given by

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix} \quad \text{(MF26)}$$

Applications of Vector Product

Let the points A and B have position vectors **a** and **b** with respect to the origin O, and let θ be the angle between **a** and **b**. Let C be the point such that $OACB$ is a parallelogram.

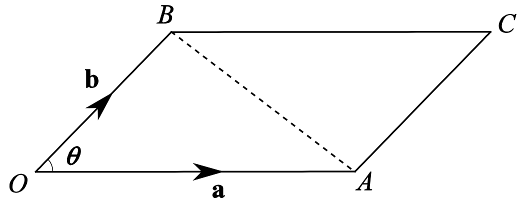
Then we have

(1) Area of triangle OAB is $\frac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|$.

(2) Area of parallelogram $OACB$ is $\left|\mathbf{a}\times\mathbf{b}\right|$.

(3) Distance from B to OA is $\left|\mathbf{b}\times\hat{\mathbf{a}}\right|$.

(Recall the **length of projection** of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$ is $\left|\mathbf{b}\cdot\hat{\mathbf{a}}\right|$)



<u>Revision Tutorial Questions</u>

1 For each question, indicate the letter corresponding to the correct answer.

		Answer
I	The modulus of the vector $6\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ is **(a)** $\sqrt{23}$ **(b)** $\sqrt{11}$ **(c)** 1 **(d)** 49 **(e)** 7	
II	The unit vector in the direction of the vector $-\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ is **(a)** $-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$ **(b)** $\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$ **(c)** $\pm\left(\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)$ **(d)** $-\frac{1}{5}\mathbf{i}+\frac{2}{5}\mathbf{j}-\frac{2}{5}\mathbf{k}$ **(e)** $-\frac{1}{9}\mathbf{i}+\frac{2}{9}\mathbf{j}-\frac{2}{9}\mathbf{k}$	
III	If $\mathbf{a}=2\mathbf{i}-\mathbf{j}$ and $\mathbf{b}=2\mathbf{i}+\mathbf{j}$ then $\mathbf{a}\bullet\mathbf{b}$ is **(a)** $4\mathbf{i}-\mathbf{j}$ **(b)** 3 **(c)** 0 **(d)** $4\mathbf{i}^2-\mathbf{j}^2$ **(e)** 5	
IV	If $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and $\mathbf{b}=2\mathbf{i}-\mathbf{j}$ then $\mathbf{a}\times\mathbf{b}$ is **(a)** $2\mathbf{i}-\mathbf{j}$ **(b)** $\mathbf{0}$ **(c)** $3\mathbf{k}$ **(d)** $-3\mathbf{k}$ **(e)** 1	
V	It is given that the points *A*, *B* and *C* are collinear. Referred to the origin *O*, $\overrightarrow{OA}=\mathbf{i}+\mathbf{j}$, $\overrightarrow{OB}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ *and* $\overrightarrow{OC}=3\mathbf{i}+a\mathbf{j}+b\mathbf{k}$. Find *a* and *b*. **(a)** $a=-3,\ b=2$ **(b)** $a=3,\ b=-2$ **(c)** $a=0,\ b=1$ **(d)** $a=-1,\ b=0$ **(e)** $a=6,\ b=-1$	
VI	It is given that the points *A*, *B* and *C* are collinear. If $\mathbf{a}=-2\mathbf{b}+3\mathbf{c}$ then *C* divides *AB* in the ratio **(a)** 1:2 **(b)** 2:3 **(c)** 3:2 **(d)** -2:3 **(e)** 2:1	
VII	The angle between the vectors $\begin{pmatrix}-4\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ is given by **(a)** $\cos^{-1}(0)$ **(b)** $\cos^{-1}(1)$ **(c)** $\cos^{-1}\left(\frac{\sqrt{2}}{10}\right)$ **(d)** $\sin^{-1}\left(\frac{-1}{5\sqrt{2}}\right)$ **(e)** $\cos^{-1}\left(\frac{-1}{5\sqrt{2}}\right)$	
VIII	**a**, **b** and **c** are the position vectors of the vertices of a triangle. The area of the triangle can be expressed as **(a)** $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ **(b)** $\frac{1}{2}\lvert\mathbf{b}\times\mathbf{c}\rvert$ **(c)** $\frac{1}{2}\lvert(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{b})\rvert$ **(d)** $\frac{1}{2}\lvert\mathbf{b}-\mathbf{a}\rvert\times\lvert\mathbf{c}-\mathbf{a}\rvert$ **(e)** $\lvert(\mathbf{b}-\mathbf{a})\bullet(\mathbf{c}-\mathbf{a})\rvert$	
IX	Given vectors **a** and **b**, and scalar λ, which statement is false in general. **(a)** $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ **(b)** $\lvert\lambda\mathbf{a}\bullet\mathbf{b}\rvert=\lambda\lvert\mathbf{a}\bullet\mathbf{b}\rvert$ **(c)** $\mathbf{a}\bullet(\mathbf{b}\times\mathbf{a})=0$ **(d)** $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2\mathbf{a}\times\mathbf{b}$ **(e)** $(\mathbf{a}+\mathbf{b})\bullet(\mathbf{a}-\mathbf{b})=\lvert\mathbf{a}\rvert^2-\lvert\mathbf{b}\rvert^2$	

X	Referred to the origin O, the points A and B have position vectors **a** and **b** respectively. Which of the following conditions below imply that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=\mathbf{0}$? **(1)** **a** and **b** are perpendicular **(2)** **a** and **b** are parallel **(3)** Either **a** or **b** (or both) are the zero vector. **(4)** The points O, A and B are collinear. **(5)** $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert$ **(a)** All of the above **(b)** (1) and (5) only **(c)** (2), (3) and (4) only **(d)** (2) and (3) only **(e)** None of the above	

Answer Key

I	II	III	IV	V	VI	VII	VIII	IX	X
(e)	(a)	(b)	(d)	(a)	(e)	(e)	(c)	(b)	(c)

Source: ACJC Prelim 9758/2018/01/Q3(b)

2 Referred to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{a}$ is a unit vector, $|\mathbf{b}|=3$, $|\mathbf{c}|=\sqrt{3}$ and angle AOC is $\frac{\pi}{6}$ radians. Given that $3\mathbf{a}+\mathbf{c}=k\mathbf{b}$ where $k\neq 0$, by considering $(3\mathbf{a}+\mathbf{c})\cdot(3\mathbf{a}+\mathbf{c})$, find the exact values of k. [4]

	Solution
2	$(3\mathbf{a}+\mathbf{c})\cdot(3\mathbf{a}+\mathbf{c})=9\mathbf{a}\cdot\mathbf{a}+3\mathbf{a}\cdot\mathbf{c}+3\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{c}$ $(3\mathbf{a}+\mathbf{c})\cdot(3\mathbf{a}+\mathbf{c})=9\mathbf{a}\cdot\mathbf{a}+6\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}$ $k\mathbf{b}\cdot k\mathbf{b}=9\mathbf{a}\cdot\mathbf{a}+6\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c}$ $k^2\lvert\mathbf{b}\rvert^2=9\lvert\mathbf{a}\rvert^2+6\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert\cos\frac{\pi}{6}+\lvert\mathbf{c}\rvert^2$ $k^2\lvert\mathbf{b}\rvert^2=9(1)+6(1)(\sqrt{3})\cos\frac{\pi}{6}+(\sqrt{3})^2$ $k^2(3)^2=21$ $k=\pm\frac{\sqrt{21}}{3}$

Source : PJC Prelim 9758/2018/02/Q1

3 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point C lies on OA produced and is such that $OC = \lambda OA$, where $\lambda > 1$.
The point D lies on OB, between O and B, such that $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{OB}$. It is given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 8$ and $\angle AOB = 60°$.

(i) Show that $\overrightarrow{OD} = \frac{1}{4}\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \lambda\mu\mathbf{a} + (1-\mu)\mathbf{b}$, where μ is a parameter. [2]

The point E lies on the line BC.

(iii) Find the values of μ, in terms of λ, such that the area of triangle ODE is $\sqrt{300}$. [4]

<table>
<tr><td></td><td>Solution</td></tr>
<tr><td>3(i)</td><td><u>Method 1:</u>
$\overrightarrow{OD} = \left(4\cos 60°\right)\hat{\mathbf{b}}$
$= \left(4\cos 60°\right)\frac{\mathbf{b}}{|\mathbf{b}|}$
$= (2)\frac{\mathbf{b}}{8}$
$= \frac{1}{4}\mathbf{b}$ (shown)
<u>Method 2:</u>
$\overrightarrow{OD} = \left|\mathbf{a}\bullet\hat{\mathbf{b}}\right|\hat{\mathbf{b}}$
$= \left|\mathbf{a}\bullet\frac{\mathbf{b}}{|\mathbf{b}|}\right|\frac{\mathbf{b}}{|\mathbf{b}|}$
$= \frac{1}{|\mathbf{b}|^2}\left|\mathbf{a}\bullet\mathbf{b}\right|\mathbf{b}$
$= \frac{1}{|\mathbf{b}|^2}|\mathbf{a}||\mathbf{b}|\left|\cos 60°\right|\mathbf{b}$
$= \frac{1}{8}(4)\left(\frac{1}{2}\right)\mathbf{b}$
$= \frac{1}{4}\mathbf{b}$ (shown)
<u>Method 3:</u></td></tr>
</table>

	$\overrightarrow{AD}\cdot\overrightarrow{OB}=0$ $\left(\overrightarrow{OD}-\overrightarrow{OA}\right)\cdot\overrightarrow{OB}=0$ $\overrightarrow{OD}\cdot\overrightarrow{OB}-\overrightarrow{OA}\cdot\overrightarrow{OB}=0$ $\left\lvert\overrightarrow{OD}\right\rvert\left\lvert\overrightarrow{OB}\right\rvert=\left\lvert\overrightarrow{OA}\right\rvert\left\lvert\overrightarrow{OB}\right\rvert\cos 60^{\circ}$ $\left\lvert\overrightarrow{OD}\right\rvert=4\cos 60^{\circ}=2$ Hence, $\overrightarrow{OD}=\frac{2}{8}\overrightarrow{OB}=\frac{1}{4}\mathbf{b}$ (Shown)
(ii)	$\overrightarrow{BC}=\lambda\mathbf{a}-\mathbf{b}$ Equation of line *BC*: $\mathbf{r}=\mathbf{b}+\mu\left(\lambda\mathbf{a}-\mathbf{b}\right)$ $\mathbf{r}=\lambda\mu\mathbf{a}+\left(1-\mu\right)\mathbf{b}$ (shown)
(iii)	Area of $\Delta ODE=\frac{1}{2}\left\lvert\overrightarrow{OD}\times\overrightarrow{OE}\right\rvert=\sqrt{300}$ $\frac{1}{2}\left\lvert\frac{1}{4}\mathbf{b}\times\left[\lambda\mu\mathbf{a}+\left(1-\mu\right)\mathbf{b}\right]\right\rvert=\sqrt{300}$ $\frac{1}{8}\left\lvert\mathbf{b}\times\lambda\mu\mathbf{a}\right\rvert=\sqrt{300}$ $\lambda\left\lvert\mathbf{b}\times\mu\mathbf{a}\right\rvert=8\sqrt{300}$ (since $\lambda>0$) $\lambda\lvert\mu\rvert\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\left\lvert\sin 60^{\circ}\right\rvert=8\sqrt{300}$ $\lambda\lvert\mu\rvert(8)(4)\left(\frac{\sqrt{3}}{2}\right)=8\sqrt{300}$ $\lvert\mu\rvert=\frac{5}{\lambda}$ $\mu=\frac{5}{\lambda}$ or $\mu=-\frac{5}{\lambda}$

Source: EJC Prelim 9758/2018/01/Q5

4 Referred to the origin *O*, the points *P* and *Q* have position vectors $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel, non-zero vectors. Point *R* is on *PQ* produced such that $PQ:QR=1:\lambda$. Point *M* is the mid-point of *OR*.

(i) Find the position vector of *R* in terms of λ, $\mathbf{p}$ and $\mathbf{q}$. [1]

F is a point on *OQ* such that *F*, *P* and *M* are collinear.

(ii) Find the ratio *OF*:*FQ*, in terms of λ. [4]

	Solution
4(i)	By the ratio theorem,

	$\overrightarrow{OQ} = \dfrac{\overrightarrow{OR} + \lambda\overrightarrow{OP}}{1+\lambda}$ $\overrightarrow{OR} = (1+\lambda)\mathbf{q} - \lambda\mathbf{p}$
(ii)	Since the point F lies on line OQ, $\overrightarrow{OF} = t\mathbf{q}$, for some $t \in \mathbb{R}$. $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$ $= \dfrac{1}{2}\overrightarrow{OR} - \mathbf{p}$ $= \dfrac{(1+\lambda)}{2}\mathbf{q} - \dfrac{\lambda}{2}\mathbf{p} - \mathbf{p}$ $= \dfrac{(1+\lambda)}{2}\mathbf{q} - \left(\dfrac{\lambda}{2}+1\right)\mathbf{p}$ Since the point F also lies on line PM, $\overrightarrow{OF} = \mathbf{p} + s\overrightarrow{PM}$, for some $s \in \mathbb{R}$. $= \mathbf{p} + s\left[\dfrac{(1+\lambda)}{2}\mathbf{q} - \left(\dfrac{\lambda}{2}+1\right)\mathbf{p}\right]$ $= \left(1 - \dfrac{s\lambda}{2} - s\right)\mathbf{p} + \dfrac{s(1+\lambda)}{2}\mathbf{q}$ Since $\mathbf{p}$ and $\mathbf{q}$ are non-parallel & non-zero vectors, comparing coefficients of $\mathbf{p}$ and $\mathbf{q}$ against $\overrightarrow{OF} = t\mathbf{q}$, we have $1 - \dfrac{s\lambda}{2} - s = 0$ $s\left(\dfrac{\lambda}{2}+1\right) = 1$ $s = \dfrac{1}{\dfrac{\lambda}{2}+1} = \dfrac{2}{\lambda+2}$ $\overrightarrow{OF} = \dfrac{s(1+\lambda)}{2}\mathbf{q} = \dfrac{2}{\lambda+2}\left(\dfrac{1+\lambda}{2}\right)\mathbf{q}$ $= \dfrac{1+\lambda}{2+\lambda}\mathbf{q} = \dfrac{1+\lambda}{2+\lambda}\overrightarrow{OQ}$ Thus, $OF : FQ = 1+\lambda : 1$

Source: MI Prelim 9758/2018/02/Q4

5 Relative to the origin O, the position vectors of points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\mathbf{a}$ and $\mathbf{b} - \mathbf{a}$ are perpendicular and C lies on AB produced such that $AC : AB = 4:3$.

(i) If $\mathbf{a}$ is a unit vector, show that $|\mathbf{b}| > 1$. [3]

Given further that $|\mathbf{b}| = 2$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. [1]

(ii) The direction cosines of **b** are 0.6, λ, μ and **b** is perpendicular to the y-axis. Find

(a) the angle **b** makes with the x-axis, [1]

(b) λ and μ. [2]

(iii) By expressing $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, show that $|\mathbf{c}\cdot\mathbf{a}| = \mathbf{a}\cdot\mathbf{a}$. [3]

Hence state the length of projection of $\mathbf{c}$ onto $\mathbf{a}$ in terms of $\mathbf{a}$. [1]

(iv) Give a geometrical interpretation of $|\mathbf{c}\times\mathbf{a}|$ and hence evaluate $\dfrac{|\mathbf{c}\times\mathbf{a}|}{|\mathbf{b}\times\mathbf{a}|}$. [2]

	Solution
5(i)	$\mathbf{a} \perp \mathbf{b}-\mathbf{a} \Rightarrow \mathbf{a}\cdot(\mathbf{b}-\mathbf{a})=0$ $\Rightarrow \mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2=1$ (since $\lvert\mathbf{a}\rvert=1$) $\Rightarrow \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta=1$ where θ is the angle between **a** and **b** $\Rightarrow \cos\theta=\dfrac{1}{\lvert\mathbf{b}\rvert}<1\Rightarrow\lvert\mathbf{b}\rvert>1$. (shown) $\lvert\mathbf{b}\rvert=2\Rightarrow\cos\theta=\dfrac{1}{2}\Rightarrow\theta=60^\circ$.
(ii) (a)	Let α be the required angle. $\cos\alpha=0.6\Rightarrow\alpha=\cos^{-1}0.6=53.1^\circ$. (1 d.p.)
(ii) (b)	$\lambda=\cos 90^\circ=0$. $0.6^2+0^2+\mu^2=1\Rightarrow\mu^2=1-0.36=0.64\Rightarrow\mu=\pm0.8$.
(iii)	By Ratio Theorem, $\mathbf{b}=\dfrac{\mathbf{a}+3\mathbf{c}}{4}\Rightarrow\mathbf{c}=\dfrac{4\mathbf{b}-\mathbf{a}}{3}$. $\lvert\mathbf{c}\cdot\mathbf{a}\rvert=\left\lvert\left(\dfrac{4\mathbf{b}-\mathbf{a}}{3}\right)\cdot\mathbf{a}\right\rvert=\left\lvert\dfrac{4}{3}(\mathbf{b}\cdot\mathbf{a})-\dfrac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ $=\left\lvert\dfrac{4}{3}(\mathbf{a}\cdot\mathbf{a})-\dfrac{1}{3}(\mathbf{a}\cdot\mathbf{a})\right\rvert$ (from (**i**)) $=\lvert\mathbf{a}\cdot\mathbf{a}\rvert=\left\lvert\lvert\mathbf{a}\rvert^2\right\rvert=\lvert\mathbf{a}\rvert^2=\mathbf{a}\cdot\mathbf{a}$. (shown) Required length $=\dfrac{\lvert\mathbf{c}\cdot\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}=\dfrac{\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert}=\dfrac{\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert}=\lvert\mathbf{a}\rvert$.
(iv)	$\lvert\mathbf{c}\times\mathbf{a}\rvert$ = Twice the area of triangle OAC $\lvert\mathbf{b}\times\mathbf{a}\rvert$ = Twice the area of triangle OAB = $(AB)h$, where h is the height of triangle OAB with AB as the base. Since A, B and C are collinear, $\lvert\mathbf{c}\times\mathbf{a}\rvert=(AC)h$. Hence, $\dfrac{\lvert\mathbf{c}\times\mathbf{a}\rvert}{\lvert\mathbf{b}\times\mathbf{a}\rvert}=\dfrac{(AC)h}{(AB)h}=\dfrac{AC}{AB}=\dfrac{4}{3}$.

Source: SAJC Prelim 9758/2018/01/Q6

6 Relative to the origin O, the position vectors of points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.
The point C with position vector $\mathbf{c}$ lies on the line segment AB such that $AC : CB$ is $\lambda : 1-\lambda$. It is given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}| = \frac{4}{3}$ and the angle formed between OA and OB is $120°$.

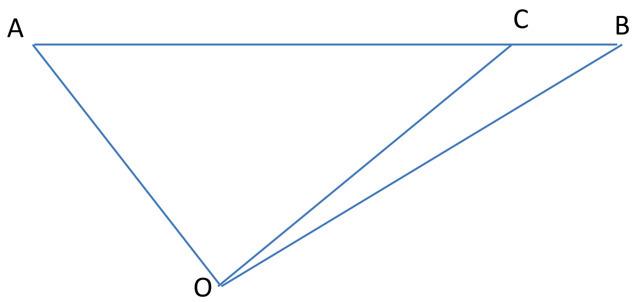
(i) Find the value of λ such that the points O, A and C form a right angle AOC. [4]

(ii) Find $\mathbf{m}$, the position vector of M, the midpoint of AC, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

A circle is drawn with AC as its diameter and O is a point on the circumference of the circle drawn.

(iii) Determine if OB is a tangent to the circle described above. [3]

(iv) Give a geometrical interpretation of $|\mathbf{b} \bullet \mathbf{m}|$. Hence, explain $(\mathbf{b} \bullet \mathbf{m})\mathbf{b}$ in terms of its magnitude and direction. [2]

	Solution
6(i)	Since $AC : CB$ is $\lambda : 1-\lambda$,  By Ratio Theorem, $\mathbf{c} = \dfrac{\lambda\mathbf{b} + (1-\lambda)\mathbf{a}}{\lambda + 1 - \lambda}$ $= \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ Since OC is perpendicular to OA,

	$\mathbf{c} \bullet \mathbf{a} = 0$ $\left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] \bullet \mathbf{a} = 0$ $\lambda\mathbf{b} \bullet \mathbf{a} + (1-\lambda)\mathbf{a} \bullet \mathbf{a} = 0$ $\lambda\left[\frac{4}{3} \times 1 \times \cos 120°\right] + (1-\lambda)\lvert\mathbf{a}\rvert^2 = 0, \text{ since } \lvert\mathbf{a}\rvert = \frac{4}{3}$ $\lambda\left[\frac{4}{3} \times \left(-\frac{1}{2}\right)\right] + (1-\lambda)\left(\frac{4}{3}\right)^2 = 0$ $\left(-\frac{2}{3}\right)\lambda + \frac{16}{9}(1-\lambda) = 0$ $-\frac{2}{3}\lambda + \frac{16}{9} - \frac{16}{9}\lambda = 0$ $\frac{22}{9}\lambda = \frac{16}{9}$ $\lambda = \frac{16}{22} = \frac{8}{11}$
(ii)	$\mathbf{c} = \frac{3}{11}\mathbf{a} + \frac{8}{11}\mathbf{b}$ By Mid-point Theorem, find $\overrightarrow{OM} = \mathbf{m}$, where M is the mid-point of AC. $\overrightarrow{OM} = \mathbf{m}$ $= \frac{\mathbf{c} + \mathbf{a}}{2}$ $= \frac{1}{2}\left[\frac{3}{11}\mathbf{a} + \frac{8}{11}\mathbf{b} + \mathbf{a}\right]$ $= \frac{1}{2}\left(\frac{14}{11}\mathbf{a} + \frac{8}{11}\mathbf{b}\right)$ $= \frac{7}{11}\mathbf{a} + \frac{4}{11}\mathbf{b}$
(iii)	$\mathbf{m} \bullet \mathbf{b} = \left(\frac{7}{11}\mathbf{a} + \frac{4}{11}\mathbf{b}\right) \bullet \mathbf{b}$ $= \frac{7}{11}\mathbf{a} \bullet \mathbf{b} + \frac{4}{11}\mathbf{b} \bullet \mathbf{b}$ $= \frac{7}{11}\mathbf{a} \bullet \mathbf{b} + \frac{4}{11}\lvert\mathbf{b}\rvert^2$ $= \frac{7}{11}\left(-\frac{2}{3}\right) + \frac{4}{11}(1)^2$ $= -\frac{2}{33} \neq 0$ Since the vector **b** is not perpendicular to **m**, where *OM* is the radius of the circle, *OB* is not a tangent to the circle.
(iv)	$\lvert\mathbf{b} \bullet \mathbf{m}\rvert$ is the length of projection of **m** on **b**.

	$(\mathbf{b} \bullet \mathbf{m})\mathbf{b}$ is a vector with magnitude $\lvert \mathbf{b} \bullet \mathbf{m} \rvert$, which is the length of projection of **m** on **b**. Moreover, it is in the opposite direction of **b** as $\mathbf{m} \bullet \mathbf{b} = -\frac{2}{33} < 0$.