2023 SAJC H2 Math Promo Solutions

Q	Solution
1	Let $S_n = an^3 + bn^2 + cn + d$, a $\neq 1$ When $n=1$, $a+b+c+d=5---(1)$ When $n=2$, $8a+4b+2c+d=20---(2)$ When $n=3$, $27a+9b+3c+d=57---(3)$ When $n=4$, $64a+16b+4c+d=128---(4)$ Using GC to solve (1), (2), (3) and (4), $a=2, b=-1, c=4, d=0$. $\therefore S_n = 2n^3 - n^2 + 4n$
2(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin^{-1}(2x)}{1-4x^2}\right]$ $=\frac{\left(1-4x^2\right)\left[\frac{2}{\sqrt{1-(2x)^2}}\right]-\left[\sin^{-1}(2x)\right](-8x)}{\left(1-4x^2\right)^2}$ $=\frac{2\sqrt{1-4x^2}+8x\sin^{-1}(2x)}{\left(1-4x^2\right)^2}$
2(b)	$y^2 = 3\mathrm{e}^{4x} + 4$---(1) Differentiate with respect to x: $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12\mathrm{e}^{4x}$ $y\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{4x} --(1)$ Differentiate (1) with respect to x: $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 24\mathrm{e}^{4x}$ $=8\left(3\mathrm{e}^{4x}\right)$ $=8\left(y^2-4\right)$, from (1) $=8y^2-32$ $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-8y^2=-32$ (Shown) where $k=-32$

Q	Solution
3(a)	$u_n = \sum_{r=1}^{n} 6r\left(r+1\right)$ $= \sum_{r=1}^{n} 6r^2 + \sum_{r=1}^{n} 6r$ $= 6\left[\frac{1}{6}n\left(n+1\right)\left(2n+1\right)\right] + 6\left[\frac{n}{2}\left(n+1\right)\right]$ $= n\left(n+1\right)\left(2n+1+3\right)$ $= 2n\left(n+1\right)\left(n+2\right)$
(b) (i)	Let $\frac{1}{r\left(r+1\right)\left(r+2\right)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$ By cover-rule, $A = \frac{1}{\left(0+1\right)\left(0+2\right)} = \frac{1}{2}$ $B = \frac{1}{\left(-1\right)\left(-1+2\right)} = -1$ $C = \frac{1}{\left(-2\right)\left(-2+1\right)} = \frac{1}{2}$ $\frac{1}{r\left(r+1\right)\left(r+2\right)} = \frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}$

Q	**Solution**
(b) **(ii)**	$S_N=\sum_{r=1}^{N}\left(\frac{1}{r(r+1)(r+2)}\right)$ $=\sum_{r=1}^{N}\left(\frac{1}{r(r+1)(r+2)}\right)$ $=\sum_{r=1}^{N}\left(\frac{1}{2r}-\frac{1}{r+1}+\frac{1}{2(r+2)}\right)$ $=\left[\begin{array}{l}\frac{1}{2}-\frac{1}{2}+\cancel{\frac{1}{2(3)}}\\+\frac{1}{4}-\cancel{\frac{1}{3}}+\cancel{\frac{1}{2(4)}}\\+\cancel{\frac{1}{6}}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{2(5)}}\\+\cancel{\frac{1}{8}}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{2(6)}}\\+\cdots\\+\cancel{\frac{1}{2(N-2)}}-\cancel{\frac{1}{N-1}}+\cancel{\frac{1}{2(N)}}\\+\cancel{\frac{1}{2(N-1)}}-\cancel{\frac{1}{N}}+\frac{1}{2(N+1)}\\+\cancel{\frac{1}{2N}}-\frac{1}{N+1}+\frac{1}{2(N+2)}\end{array}\right]$ $=\frac{1}{4}+\frac{1}{2(N+1)}-\frac{1}{N+1}+\frac{1}{2(N+2)}$ $=\frac{1}{4}-\frac{1}{2(N+1)}+\frac{1}{2(N+2)}$ $=\frac{1}{4}+\frac{1}{2}\left(-\frac{1}{(N+1)(N+2)}\right)$ $=\frac{1}{4}-\frac{1}{2}\left[\frac{1}{(N+1)(N+2)}\right]$ Since $\frac{1}{(N+1)(N+2)}>0$, for $N\in\mathbb{Z}^{+}$, $S_N<\frac{1}{4}$.

<table>
<tr><th>Q</th><th>Solution</th></tr>
<tr><td>4(i)</td><td>

$\frac{w_n}{w_{n-1}} = \frac{e^{-u_n}}{e^{-u_{n-1}}}$
$= e^{-u_n+u_{n-1}}$
$= e^{-(u_n-u_{n-1})}$
$= e^{-\ln 3}$
$= \frac{1}{3}$, since $\{u_n\}$ is an arithmetic progression
Since $\frac{w_n}{w_{n-1}} = \frac{1}{3}$ is a constant (independent of *n*), the sequence of terms given by $w_n, n \in \mathbb{Z}^+$ is a geometric progression with a common ratio of $\frac{1}{3}$.

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<tr><td>(ii)</td><td>

$\frac{w_n}{w_{n-1}} = \frac{1}{3}$ is the common ratio of the geometric progression (given).
$|r| = \frac{1}{3}$
$\therefore |r| < 1$ ---(*)
Hence, $\sum_{r=1}^{\infty} w_r$ converges.

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<tr><td>(iii)</td><td>

$w_1 = e^{-\ln 3} = \frac{1}{3}$
$\left|\frac{w_1\left[1-\left(e^{-\ln 3}\right)^n\right]}{1-e^{-\ln 3}} - \frac{w_1}{1-e^{-\ln 3}}\right| < 0.005\left[\frac{w_1}{1-e^{-\ln 3}}\right]$
$\left|\frac{w_1}{1-e^{-\ln 3}}\right|\left|1-\left(e^{-\ln 3}\right)^n - 1\right| < 0.005\left(\frac{w_1}{1-e^{-\ln 3}}\right)$
$\left|\left(e^{\ln\left(\frac{1}{3}\right)}\right)^n\right| < 0.005$
$\left(\frac{1}{3}\right)^n < 0.005$

<table>
<tr><th><i>n</i></th><th>$\left(\frac{1}{3}\right)^n$</th></tr>
<tr><td>4</td><td>0.0123 > 0.005</td></tr>
<tr><td>5</td><td>0.0041 < 0.005</td></tr>
<tr><td>6</td><td>0.0014 < 0.005</td></tr>
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Smallest possible value of *n* = 5

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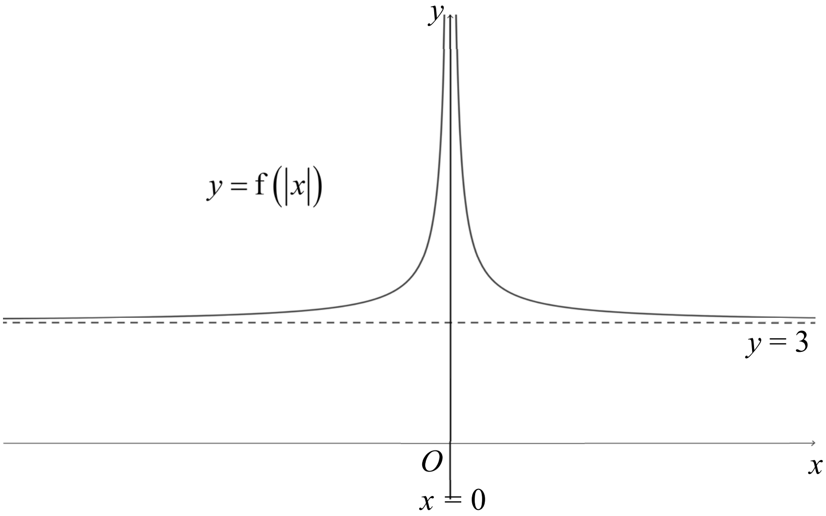
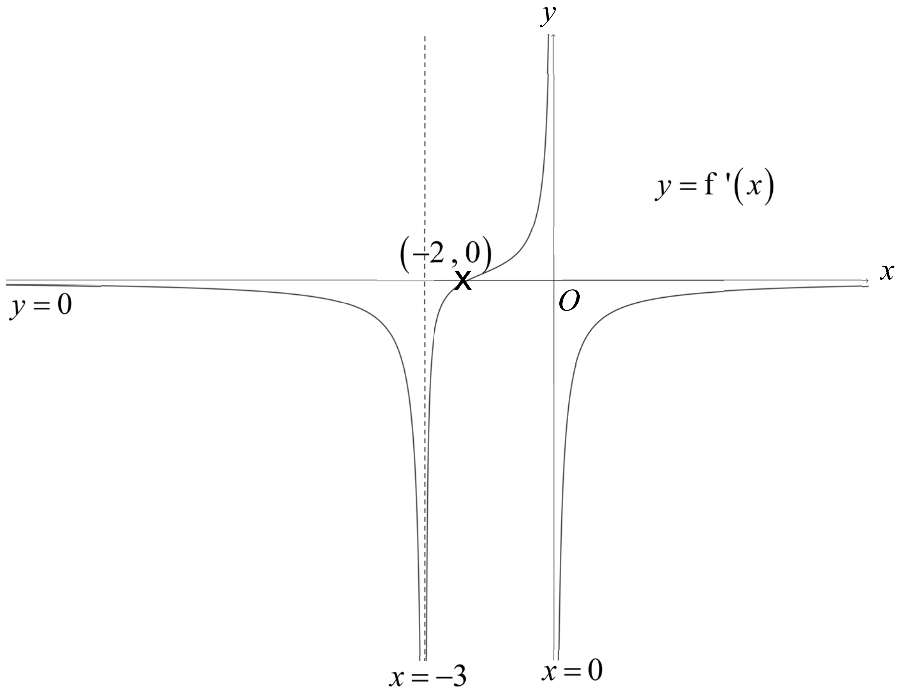
Q	Solution
5i	By Ratio Theorem, $\overrightarrow{OM}=\dfrac{2\mathbf{a}+\mathbf{b}}{3}$ Area of triangle $OBM=\dfrac{1}{2}\left\lvert\overrightarrow{OM}\times\overrightarrow{OB}\right\rvert$ $4=\dfrac{1}{2}\left\lvert\dfrac{1}{3}(2\mathbf{a}+\mathbf{b})\times\mathbf{b}\right\rvert$ $=\dfrac{1}{6}\lvert 2\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b}\rvert$ ----- (*) $=\dfrac{1}{6}\lvert 2\mathbf{a}\times\mathbf{b}+\mathbf{0}\rvert$ $=\dfrac{1}{3}\lvert\mathbf{a}\times\mathbf{b}\rvert$ $\lvert\mathbf{a}\times\mathbf{b}\rvert=12$ Alternative solution: Area of triangle $OBM=\dfrac{1}{2}\left\lvert\overrightarrow{OB}\times\overrightarrow{OM}\right\rvert$ $4=\dfrac{1}{2}\left\lvert\mathbf{b}\times\dfrac{1}{3}(2\mathbf{a}+\mathbf{b})\right\rvert$ $=\dfrac{1}{6}\lvert 2\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}\rvert$ $=\dfrac{1}{6}\lvert 2\mathbf{b}\times\mathbf{a}+\mathbf{0}\rvert$ $=\dfrac{1}{3}\lvert\mathbf{b}\times\mathbf{a}\rvert$ $=\dfrac{1}{3}\lvert\mathbf{a}\times\mathbf{b}\rvert$ since $\lvert\mathbf{b}\times\mathbf{a}\rvert=\lvert\mathbf{a}\times\mathbf{b}\rvert$ $\lvert\mathbf{a}\times\mathbf{b}\rvert=12$
ii	$(\mathbf{p}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})=\mathbf{0}$ $\overrightarrow{AP}\times\overrightarrow{AB}=\mathbf{0}$ $\overrightarrow{AP}$ is parallel to vector $\overrightarrow{AB}$ (Note : $\overrightarrow{AP}\neq\mathbf{0}$ and $\overrightarrow{AB}\neq\mathbf{0}$) Since line l that passes through point A and is parallel to vector $\overrightarrow{AB}$, $l_{AP}:\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a}),\lambda\in\mathbb{R}$
iii	$\mathbf{a}=(4\mathbf{a}\cdot\mathbf{b})\mathbf{b}$ Since $4\mathbf{a}\cdot\mathbf{b}$ is a scalar, **a** is a scalar multiple of **b**,

Q	Solution
	a and **b** are parallel vectors. Since θ, the angle between **a** and **b**, is either $0°$ or $180°$, $\cos\theta = \pm 1$. Given $\mathbf{a} = (4\mathbf{a}\cdot\mathbf{b})\mathbf{b}$, $\mathbf{a} = (4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta)\mathbf{b}$ $\lvert\mathbf{a}\rvert = 4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\lvert\cos\theta\rvert\lvert\mathbf{b}\rvert$ $\lvert\mathbf{a}\rvert = 4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert^2\lvert\cos\theta\rvert,\ \lvert\cos\theta\rvert = 1$ Since $\lvert\mathbf{a}\rvert \neq 0$ $\div\lvert\mathbf{a}\rvert, \quad \lvert\mathbf{b}\rvert^2 = \frac{1}{4}$ $\lvert\mathbf{b}\rvert = \frac{1}{2}$ since $\lvert\mathbf{b}\rvert > 0$ Alternative method: Given $\mathbf{a} = (4\mathbf{a}\cdot\mathbf{b})\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b} = (4\mathbf{a}\cdot\mathbf{b})\mathbf{b}\cdot\mathbf{b}$ $\mathbf{a}\cdot\mathbf{b} = (4\mathbf{a}\cdot\mathbf{b})\lvert\mathbf{b}\rvert^2$ Since $\mathbf{a}\cdot\mathbf{b} \neq 0$ as $\mathbf{a}//\mathbf{b}$ $\div\mathbf{a}\cdot\mathbf{b},\ \lvert\mathbf{b}\rvert^2 = \frac{1}{4}$ $\lvert\mathbf{b}\rvert = \frac{1}{2}$ since $\lvert\mathbf{b}\rvert > 0$
6(a) **(i)**	

(a)
(ii)

$y=\mathrm{f}(x)$	$y=\mathrm{f}'(x)$
$(-2,0)$	$(-2,0)$
$x=-3$	$x=-3$
$x=0$	$x=0$
$y=3$	$y=0$



(b)

$$y=\frac{1}{3}(x+1)^2$$

$\downarrow$ C': Scaling parallel to the *x*-axis by a scale factor ½**:** Replace *x* with 2*x*

$$y=\frac{1}{3}(2x+1)^2$$

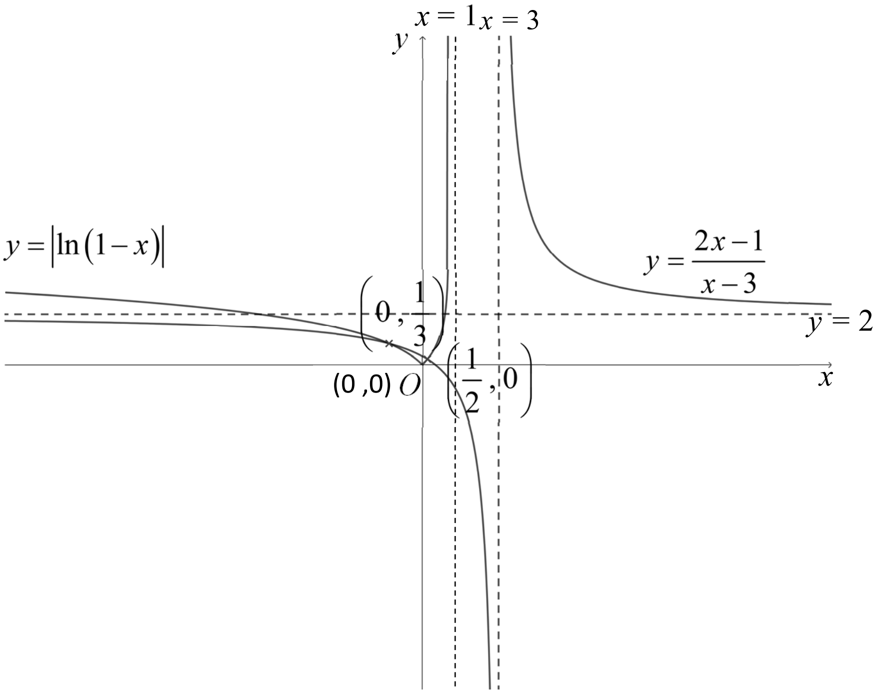
$\downarrow$ B': Translation of 4 units in the negative *x*-direction**:** Replace *x* with *x* + 4

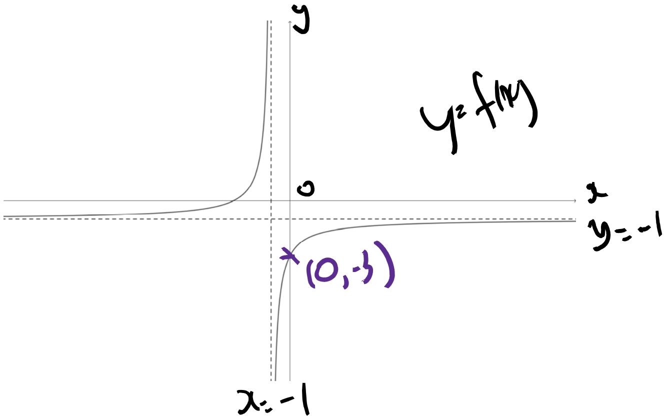
$$y=\frac{1}{3}(2(x+4)+1)^2$$

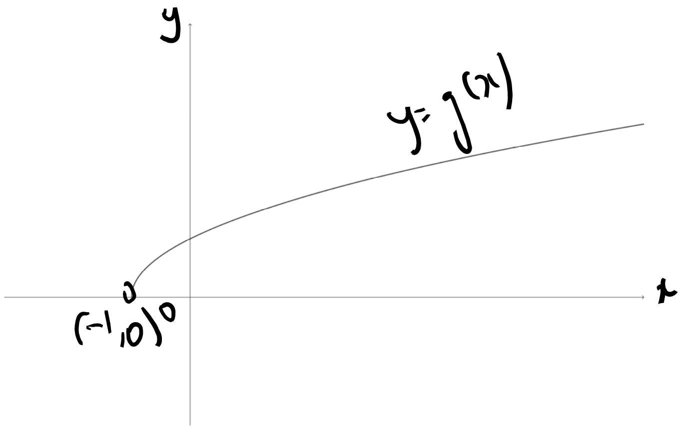
$$y=\frac{1}{3}(2x+9)^2$$

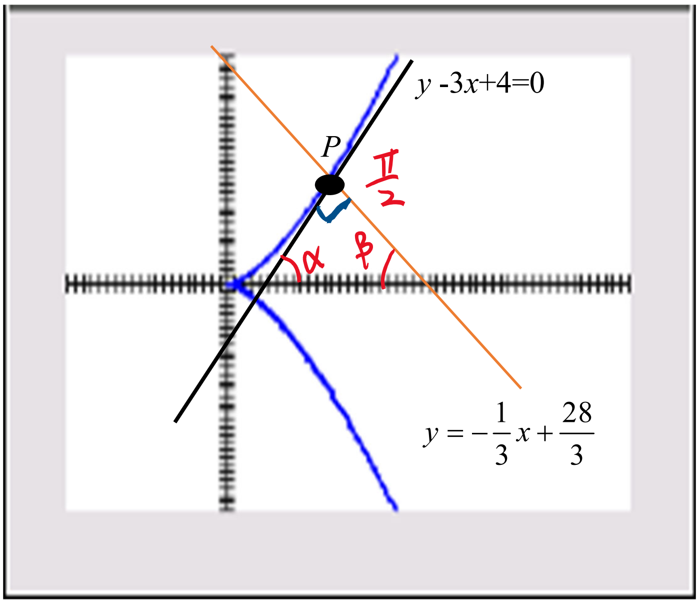
$\downarrow$ A': Reflection about the *x*-axis**:** Replace *y* with $-y$

$$y=-\frac{1}{3}(2x+9)^2$$

7(i)	$y=\frac{2x-1}{x-3}=2+\frac{5}{x-3}$ Intersection with axes: $\left(0,\frac{1}{3}\right)$ and $\left(\frac{1}{2},0\right)$ Asymptotes: $x=3$, $y=2$ For $y=\ln\left(1-x\right)$, Intersection with axes: When $y=0$, $x=0$. $\left(0,0\right)$ Asymptote: $x=1$  From the graph, the points of intersection are (-1.33 , 0.844) and (0.195 , 0.217). Hence, solving $\frac{2x-1}{x-3}=\left\|\ln x-1\right\|$, from the graph, x = -1.33 (to 3 sf) or 0.195 (to 3 sf)
(ii)	Solving $\frac{2x-1}{x-3}\le\left\|\ln\left(1-x\right)\right\|$, from the graph in (i), we have $x\le-1.33$ or $0.195\le x<1$
(iii)	$\frac{2x+1}{x-2}\le\left\|\ln\left(-x\right)\right\|$ Let $y = x + 1$

	$\frac{2(x+1)-1}{(x+1)-3} \le \lvert \ln(1-(x+1)) \rvert$ $\frac{2y-1}{y-3} \le \lvert \ln(1-y) \rvert$ $y \le -1.33$ or $0.195 \le y < 1$ $x+1 \le -1.33$ or $0.195 \le x+1 < 1$ $x \le -2.33$ or $-0.805 \le x < 0$
8(i)	$k=-1$ This is because there is no image for $x = -1$ under f. (or, $\mathrm{f}(-1)$ is undefined)
(ii)	Let $y=\frac{-x-3}{x+1}$, for $x \in \mathbb{R}, x \ne -1$ $y=\frac{-x-3}{x+1}$ $y(x+1)=-x-3$ $xy+y=-x-3$ $xy+x=-y-3$ $x(y+1)=-y-3$ $x=\frac{-y-3}{y+1}$ Since $x=\mathrm{f}^{-1}(y)=\frac{-y-3}{y+1}$, $\mathrm{f}^{-1}(x)=\frac{-x-3}{x+1}$ Since $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, $\forall x \in \mathbb{R}, x \ne -1$, $\mathrm{f}^2(x)=\mathrm{ff}^{-1}(x)=x$
(iii)	

	 $D_g = \left(-1\,,\infty\right) \xrightarrow{g} R_g = \left(0,\infty\right) \xrightarrow{f} R_{fg} = \left(-3,-1\right)$ Alternative method: Using fg to find range.
9 **(i)**	$x = t^2, \quad y = t^3$ --- (1) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\left(\frac{dx}{dt}\right)} = \frac{3t^2}{2t} = \frac{3}{2}t$ The equation of the tangent at the point with parameter *t* is $y - t^3 = \frac{3}{2}t\left(x - t^2\right)$ $\Rightarrow 2y - 2t^3 = 3t(x - t^2)$ $\Rightarrow 2y - 2t^3 = 3tx - 3t^3$ $\therefore 2y - 3tx + t^3 = 0$ (Proved).
(ii)	A cubic equation has at most 3 real roots. Given that ($\underline{a}$, *b*) is a fixed point, the equation $2b - 3at + t^3 = 0$ is a cubic equation in terms of *t*. Hence there are at most 3 real values of *t* for a fixed value of *x* and *y* and therefore at most 3 tangents can pass through the fixed point (*a* , *b*).
(iii)	When $t = 2$, $2y - 3tx + t^3 = 0$ $\Rightarrow 2y - 6x + 8 = 0$ $\Rightarrow y - 3x + 4 = 0 - - - (2)$ Since the tangent at *P* meets the curve again at $Q\ \left(k^2\ , k^3\right)$, substituting equation (1) into (2):

	$k^3-3k^2+4=0$ Solving using GC, $k=-1$ or 2 (rejected since the $t=k$ value at P is 2). Hence the tangent will meet the curve again at $k=-1$.
(iv)	When $t=2,\ x=4, y=8$. $P\ (4\ ,\ 8)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=3$ Gradient of tangent is 3 Hence gradient of normal is $-\frac{1}{3}$ Equation of normal is $\Rightarrow y-8=-\frac{1}{3}(x-4)$ $\Rightarrow y=-\frac{1}{3}x+\frac{28}{3}$
(v)	
(vi)	From the graph, $\alpha+\beta=\frac{\pi}{2}$ $\alpha=\tan^{-1}(3)$ $\beta=\tan^{-1}(\frac{1}{3})$ since β is acute $\therefore \tan^{-1}(3)+\tan^{-1}(\frac{1}{3})\ =\frac{\pi}{2}$ (Shown)

10 **(i)**	Since $A\left(-5,-7,7\right)$ lies on plane π_1, $\begin{pmatrix}-5\\-7\\7\end{pmatrix}\bullet\begin{pmatrix}2\\-5\\p\end{pmatrix}=4$ $-10+35+7p=4$ $p=-3$
(ii)	Let the acute angle between line l_1 and the plane π_1 be θ . $\sin\theta=\frac{\left\lvert\begin{pmatrix}3\\2\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\-5\\-3\end{pmatrix}\right\rvert}{\left\lVert\begin{pmatrix}3\\2\\-2\end{pmatrix}\right\rVert\left\lVert\begin{pmatrix}2\\-5\\-3\end{pmatrix}\right\rVert}=\left\lvert\frac{2}{\sqrt{17}\sqrt{38}}\right\rvert=\frac{2}{\sqrt{646}}$ $\theta=0.0788$ rad (to 3 sig fig) or 4.5° (to 1 dec pl)
(iii)	Given that $\lambda=1$, $\overrightarrow{OB}=\begin{pmatrix}1+3\\-3+2\\3-2\end{pmatrix}=\begin{pmatrix}4\\-1\\1\end{pmatrix}$ Hence, $B\left(4,-1,1\right)$ $l_{BF}:\mathbf{r}=\begin{pmatrix}4\\-1\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-5\\-3\end{pmatrix},\alpha\in\mathbb{R}$ Since F is on l_{BF} , $\overrightarrow{OF}=\begin{pmatrix}4\\-1\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-5\\-3\end{pmatrix}$ for some $\alpha\in\mathbb{R}$ Since F is on π_1 , $\left[\begin{pmatrix}4\\-1\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\-5\\-3\end{pmatrix}\right]\bullet\begin{pmatrix}2\\-5\\-3\end{pmatrix}=4$ $8+4\alpha+5+25\alpha-3+9\alpha=4$ $38\alpha=-6$ $\alpha=-\frac{3}{19}$

	$\overrightarrow{OF} = \begin{pmatrix}4\\-1\\1\end{pmatrix} - \frac{3}{19}\begin{pmatrix}2\\-5\\-3\end{pmatrix} = \frac{2}{19}\begin{pmatrix}35\\-2\\14\end{pmatrix} = \begin{pmatrix}\frac{70}{19}\\-\frac{4}{19}\\\frac{28}{19}\end{pmatrix}$
(iv)	Let B' be the point of reflection of B in the plane π_1. Since F is midpoint of B and B', $\overrightarrow{OF} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OB'}\right)$ $\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB} = 2\begin{pmatrix}\frac{70}{19}\\-\frac{4}{19}\\\frac{28}{19}\end{pmatrix} - \begin{pmatrix}4\\-1\\1\end{pmatrix} = \begin{pmatrix}\frac{64}{19}\\\frac{11}{19}\\\frac{37}{19}\end{pmatrix}$ $\overrightarrow{AB'} = \overrightarrow{OB'} - \overrightarrow{OA} = \begin{pmatrix}\frac{64}{19}\\\frac{11}{19}\\\frac{37}{19}\end{pmatrix} - \begin{pmatrix}-5\\-7\\7\end{pmatrix} = \begin{pmatrix}\frac{159}{19}\\\frac{144}{19}\\-\frac{96}{19}\end{pmatrix} = \frac{1}{19}\begin{pmatrix}159\\144\\-96\end{pmatrix}$ Vector equation of line AB': $\mathbf{r} = \begin{pmatrix}-5\\-7\\7\end{pmatrix} + \mu\begin{pmatrix}159\\144\\-96\end{pmatrix},\ \mu \in \mathbb{R}.$
(v)	The line is parallel to the vector : $\begin{pmatrix}3\\2\\-2\end{pmatrix} \times \begin{pmatrix}2\\-5\\-3\end{pmatrix} = \begin{pmatrix}-16\\5\\-19\end{pmatrix}$ Vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}4\\-1\\1\end{pmatrix} + s\begin{pmatrix}-16\\5\\-19\end{pmatrix}, s \in \mathbb{R}.$

11(i)	$ZY = WX = 10\sin\theta,$ $OY = 10\cos\theta,$ $\tan\frac{\pi}{3} = \frac{WX}{OX} \Rightarrow OX = \frac{10\sqrt{3}}{3}\sin\theta$ $A = ZY \times XY$ $= 10\sin\theta \times \left(10\cos\theta - \frac{10\sqrt{3}}{3}\sin\theta\right)$ $= 100\sin\theta\cos\theta - \frac{100\sqrt{3}}{3}\sin^2\theta$ $= 50\sin 2\theta - \frac{100\sqrt{3}}{3}\sin^2\theta$ $= 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)$
(ii)	$A = 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)$ $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 50\left(2\cos 2\theta - \frac{2\sqrt{3}}{3}\left[2\sin\theta\cos\theta\right]\right) = 50\left(2\cos 2\theta - \frac{2\sqrt{3}}{3}\sin 2\theta\right) = 100\left(\cos 2\theta - \frac{\sqrt{3}}{3}\sin 2\theta\right)$ For stationary values of A, let $\frac{\mathrm{d}A}{\mathrm{d}\theta} = 0$ $\Rightarrow 100\left(\cos 2\theta - \frac{\sqrt{3}}{3}\sin 2\theta\right) = 0$ $\cos 2\theta - \frac{\sqrt{3}}{3}\sin 2\theta = 0$ $\tan 2\theta = \frac{3}{\sqrt{3}} = \sqrt{3}$, since $\cos 2\theta \neq 0$ Since $\theta < \frac{\pi}{3}$, $\therefore\ 0 < 2\theta < \frac{2\pi}{3} < \pi.$ Therefore, $2\theta = \frac{\pi}{3}$ $\Rightarrow \theta = \frac{\pi}{6}$ **Second Derivative Test** $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2} = 100\left(-2\sin 2\theta - \frac{2\sqrt{3}}{3}\cos 2\theta\right) = -200\left(\sin 2\theta + \frac{\sqrt{3}}{3}\cos 2\theta\right)$

When $\theta = \frac{\pi}{6} \Rightarrow 2\theta = \frac{\pi}{3}$ is acute

$$\Rightarrow \frac{\mathrm{d}^2 A}{\mathrm{d}\theta^2} = -200\left(\sin\frac{\pi}{3} + \frac{\sqrt{3}}{3}\cos\frac{\pi}{3}\right) = -200\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}\left(\frac{1}{2}\right)\right) = -\frac{400}{3}\sqrt{3} < 0$$

$\therefore A$ achieves maximum value when $\theta = \frac{\pi}{6}$

Alternatively, **First Derivative Test**

$$\frac{\mathrm{d}A}{\mathrm{d}\theta} = 100\left(\cos 2\theta - \frac{\sqrt{3}}{3}\sin 2\theta\right) = \frac{200}{\sqrt{3}}\cos\left(2\theta + \frac{\pi}{6}\right)$$, using R-formula

θ	$\left(\frac{\pi}{6}\right)^-$	$\frac{\pi}{6}$	$\left(\frac{\pi}{6}\right)^+$
$\frac{\mathrm{d}A}{\mathrm{d}\theta}$	positive	0	negative

At $\theta = \frac{\pi}{6}$,

$$A = 50\left(\sin\frac{\pi}{3} - \frac{2\sqrt{3}}{3}\sin^2\frac{\pi}{6}\right)$$

$$= 50\left[\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{3}\left(\frac{1}{2}\right)^2\right]$$

$$= 50\left(\frac{2\sqrt{3}}{6}\right)$$

$$= \frac{50\sqrt{3}}{3}\text{ units}^2$$

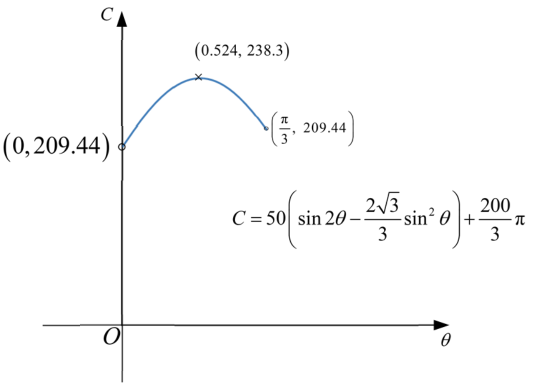
The maximum value of A is $\frac{50\sqrt{3}}{3}\text{ m}^2$

(iii)

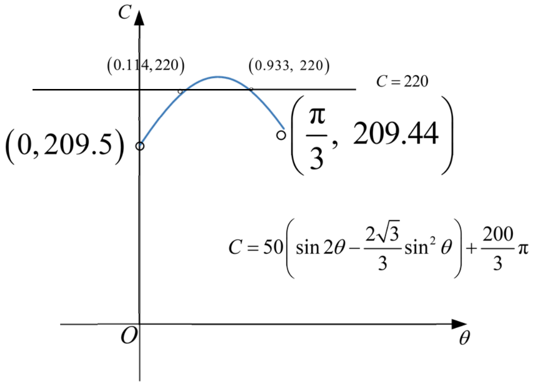
$$A = 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)$$

Remaining Areas, $A_1 = \frac{1}{2}(10)^2\left(\frac{\pi}{3}\right) - (A) = \frac{50}{3}\pi - 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)$

$$\begin{aligned} C &= 5A + 4A_1 \\ &= 5\left[50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)\right] + 4\left[\frac{50}{3}\pi - 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right)\right] \\ &= 50\left(\sin 2\theta - \frac{2\sqrt{3}}{3}\sin^2\theta\right) + \frac{200}{3}\pi \end{aligned}$$



(iv) Using a graphical solution, by adding the line $C = 220$,



$0 < \theta < 0.114$ or $0.933 < \theta < \frac{\pi}{3}$