RVHS H2 Mathematics 2023 Promo

1 An ellipse C has the equation $ax^2 + by^2 + cy = -3$ where a, b and c are constants. It is given that C passes through the points with coordinates $(0, 1.5)$, $(0.6, 0.6)$ and $(-1, 1)$.

(i) Find the values of a, b and c. [3]

(ii) Identify the centre of C. [2]

2 At a sit-down dinner, 2 adults, 4 boys and 4 girls are to be seated at a round table with ten identical chairs. Find the number of different ways that they can be seated if

(i) there are no restrictions; [1]

(ii) not all the 4 boys are seated together; [2]

(iii) the 2 adults are seated together, and a girl is on each side of them. [3]

3 The function f is given by $\mathrm{f}: x \mapsto \dfrac{1}{x-2} + 5,\ x > 2$.

(i) Explain why f^{-1} exists. [1]

(ii) Find f^{-1} in similar form. [3]

(iii) Find ff in a similar form and the exact range of ff. [3]

4 Given that $I = \displaystyle\int x^3 \left(1+x^2\right)^{-2} \mathrm{d}x$.

(i) By writing $I = \displaystyle\int x^2 \left[x\left(1+x^2\right)^{-2}\right] \mathrm{d}x$, use integration by parts to find an expression for I.

[3]

(ii) Use the substitution $u = 1 + x^2$ to find another expression for I. [3]

(iii) Show algebraically that your answers to parts **(i)** and **(ii)** differ by a constant. [1]

5 The curve C has equation $y = \dfrac{4x-32}{(3x+1)(x-5)},\ x \in \ ,\ x \neq -\dfrac{1}{3}, 5$.

(i) Show, algebraically, that $y \leq \dfrac{1}{16}$ or $y \geq 1$. [3]

(ii) Sketch the graph of curve C, stating the equations of the asymptotes, coordinates of its turning points and points where it crosses the axes. [3]

(iii) On the same diagram in part **(ii)**, sketch the line with equation $y = 1 - \dfrac{x}{8}$. Hence, or otherwise, solve the inequality $\dfrac{4x-32}{(3x+1)(x-5)} \geq 1 - \dfrac{x}{8}$. [2]

6 Let $f(n)=\dfrac{n-1}{n(n+1)}$.

(i) Express $f(n)$ in partial fractions. [1]

(ii) Hence find $\sum_{n=1}^{N} 2^n f(n)$. [3]

(iii) Find $\sum_{n=2}^{2N} 2^{n-1} f(n-1)$. [2]

(iv) Denote $S_N=\sum_{n=1}^{N} 2^n f(n)$. Find least integer k such that S_k is more than 5% of S_{10}. [2]

7 (i) Without the use of a calculator, show that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\tan x| \, dx = \ln 2$. [3]

(ii) Hence, find the exact volume of the solid formed from completely rotating region bounded by the graph of $y=|\tan x|+1$, the lines $x=-\dfrac{\pi}{4}$ and $x=\dfrac{\pi}{4}$, about the x-axis. [5]

8 Let S_n denote the sum of the first n terms of the sequence of numbers $u_1, u_2, u_3, \ldots$. It is given that $S_n = 2n^2+kn$, where k is a non-zero constant.

(i) Show that the sequence of numbers $u_1, u_2, u_3, \ldots$ is an arithmetic progression. [3]

A geometric progression has non-zero first term a and common ratio r. Given that the sum of the first 5 terms of the progression is $\dfrac{32}{33}$ of the sum of the first 10 terms of the progression.

(ii) Show that $r=\dfrac{1}{2}$ and find the infinite sum of the geometric progression in terms of a. [4]

(iii) Given further that u_{62}, u_{30}, u_t are three consecutive terms of the given geometric progression, find the value of t. [3]

9 A curve C has equation

$$y(x+1)^2+y^2-12x=0, \text{ where } x, y \geq 0.$$

(i) Show that $\dfrac{dy}{dx}=\dfrac{12-2y(x+1)}{(x+1)^2+2y}$. [2]

(ii) Find the equation of the normal L to the curve at the point $P(3, 2)$. [3]

(iii) Find the area of the triangle bounded by L and the axes. [1]

(iv) The curve C is transformed by a series of transformations resulting in a curve D, with equation

$$3y(2x+1)^2+9y^2-24x=0,$$

with point Q on D corresponding to the point P on C.
The normal L undergoes the same sequence of transformations as curve C, resulting in the line l.
Describe a possible sequence of transformations that maps C to D and hence deduce the exact area of the triangle bounded by l and the axes. [3]

10 A curve C which is symmetrical about the x-axis has parametric equations

$$x = \cos t + \cos 2t,\ y = \sin t + \sin 2t,\quad 0 \le t < 2\pi.$$

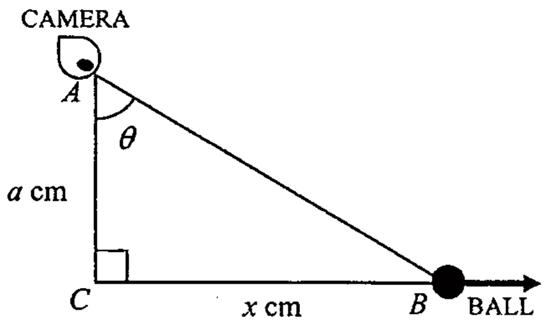
(i) Sketch C, indicating the coordinates of the x-intercepts. [2]

(ii) Find the equation of the tangent to C at the point where $t = \frac{2\pi}{3}$, giving your answer in exact form. [4]

(iii) Show that the point with coordinates $(-1, 0)$ lies on the line $y = mx + m$.

Hence state the range of values of m for which the line $y = mx + m$ intersects C at exactly 3 distinct points. [2]

11 Fixed vertical cameras have to rotate quickly to keep track on moving objects. A camera at A is attached on a vertical wall AC with fixed height a cm. A ball B is moving horizontally away from the wall AC. At time t seconds, the ball is x cm from the wall and the camera would have complete a rotation of θ radians from the wall to keep track of the ball. At this instant, the camera is rotating at a speed of $\frac{d\theta}{dt}$ radians per second. Assume that the triangle ABC lies on a vertical plane containing the path of the ball.



(i) By expressing $\tan\theta$ in terms of x and a, show that $\frac{d\theta}{dt} = \left(\frac{a}{a^2 + x^2}\right)\frac{dx}{dt}$. [4]

(ii) When the ball is 9 cm from the wall, it is travelling at 6 m per second. At this instant, find the speed the camera is rotating at in radians per second, leaving your answer in terms of a. [1]

(iii) Given that $a = 4$ and that the ball is travelling such that $x = t^2$ cm, find the exact maximum value of the rotating speed of the camera in radians per second. [7]

12 Two teams of scientists are investigating the population of wild hares, and the population of their predators, the wild lynxes, in the snowy mountains.

The number of hares in the snowy mountain is P thousand at a time of t years after the scientists begin observations. Initially, the first team of scientists observed that there were 50 thousand hares growing at the rate of 25 thousand hares per year. After several observations, they decided to model the growth rate of the hares, at any time t, to be $kP(100-P)$ hares per year, for some positive constant k.

(a) **(i)** Show that $\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{1}{100}P(100-P)$. [1]

(ii) Find P in terms of t. [5]

The number of lynxes in the snowy mountains is Q thousand at a time of t years after the scientists begin observations. Initially, the second team of scientists observed that there were 8 thousand lynxes decreasing at the rate of 0.8 thousand lynxes per year.

Both team of scientists collaborated and developed a refined model for the growth rate of the hares. The model assumed that, at any time t, the hares were growing at the rate of $P-\frac{1}{16}QP$ per year.

(b) **(i)** For the refined model, when $t=0$, show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=15$. [2]

(ii) Hence, find the Maclaurin series of P in the refined model, in ascending powers of t, up to and including the term in t^2. [2]

Two scientists, scientist A and B, both use the refined model to approximate the number of hares after some time. Scientist A approximates the number of hares one month after the scientists started observing, while scientist B approximates the number of hares ten years after the scientist started observing.

(iii) Given that both scientists used your answer in part **(b)(ii)**, state the approximation only provided by scientist A and suggest an appropriate reason why scientist A's approximation would be considered more appropriate as compared to scientist B's. [2]

2023 JC1 H2 MA Promo Solutions with Comments

1 An ellipse C has the equation $ax^2+by^2+cy=-3$ where a, b and c are constants. It is given that C passes through the points with coordinates $(0, 1.5)$, $(0.6, 0.6)$ and $(-1, 1)$.

(i) Find the values of a, b and c. [3]

(ii) Identify the centre of C. [2]

1	Solution [5] SoLE, Conic Section	
(i)	Sub $(0,1.5)$ into $ax^2+by^2+cy=-3$: $2.25b+1.5c=-3$ ----- (1) Sub $(0.6,0.6)$ into $ax^2+by^2+cy=-3$: $0.36a+0.36b+0.6c=-3$ ---- (2) Sub $(-1,1)$ into $ax^2+by^2+cy=-3$: $a+b+c=-3$ ----- (3) By GC, $a=1,\ b=4,\ c=-8$.	
(ii)	$x^2+4y^2-8y=-3$ $x^2+4\left(y^2-2y+1\right)-4=-3$ $x^2+4(y-1)^2=1$ Centre of C: (0, 1)	

2 At a sit-down dinner, 2 adults, 4 boys and 4 girls are to be seated at a round table with ten identical chairs. Find the number of different ways that they can be seated if

(i) there are no restrictions; [1]

(ii) not all the 4 boys are seated together; [2]

(iii) the 2 adults are seated together, and a girl is on each side of them. [3]

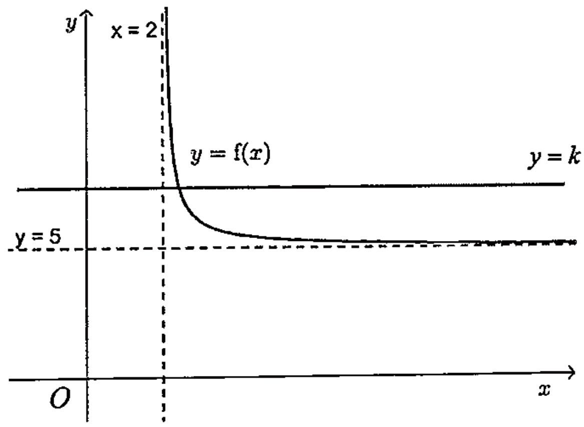
2	Solution [6] Permutation and Combination	
(i)	No. of ways $= (10-1)!$ $= 362\ 880$	
(ii)	No. of ways = No. of ways without restrictions − No. of ways for 4 boys to be seated together $= 362880 - (7-1)! \times 4!$ $= 345\ 600$	
(iii)	$2!$: to arrange the 2 adults within a unit. $\left({}^4C_2 \times 2!\right)$: choose 2 girls out of 4 to be seated on each side of the 2 adults and arrange them. $(7-1)!$: consider GAAG as a single superunit and arrange with the other 6 people. No. of ways $= (7-1)! \times \left({}^4C_2 \times 2!\right) \times 2!$ $= 17280$	

3 The function f is given by $\mathrm{f}: x \mapsto \dfrac{1}{x-2}+5, x>2$.

(i) Explain why f^{-1} exists. [1]

(ii) Find f^{-1} in similar form. [3]

(iii) Find ff in a similar form and the exact range of ff. [3]

3	Solution [7] Functions	
(i)	 For any horizontal line $y=k$ (for $k \in \mathbb{R}$), the line cuts the graph of $y=\mathrm{f}(x)$ at most once. Hence, f is one-to-one and therefore, f^{-1} exists.	
	Alternatively: $\mathrm{f}(b)=\mathrm{f}(c)$ $\dfrac{1}{b-2}+5=\dfrac{1}{c-2}+5$ $\dfrac{1}{b-2}=\dfrac{1}{c-2}$ $b-2=c-2$ $b=c$ Hence, f is one-to-one and therefore, f^{-1} exists.	

(ii)

Let $y=\frac{1}{x-2}+5$

$$y-5=\frac{1}{x-2}$$

$$x-2=\frac{1}{y-5}$$

$$x=\frac{1}{y-5}+2$$

From graph, $D_{f^{-1}}=R_f=(5,\infty)$

$\therefore f^{-1}:x\mapsto\frac{1}{x-5}+2,\ x>5$.

(iii)

$$ff(x)=\frac{1}{\frac{1}{x-2}+5-2}+5$$

$$=\frac{1}{\frac{1+3(x-2)}{x-2}}+5$$

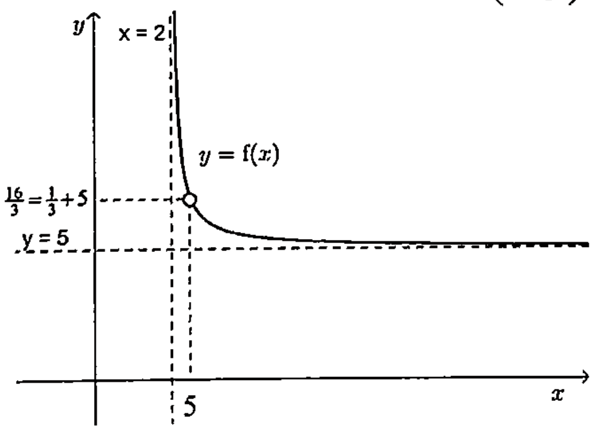
$$=\frac{x-2}{3x-5}+5$$

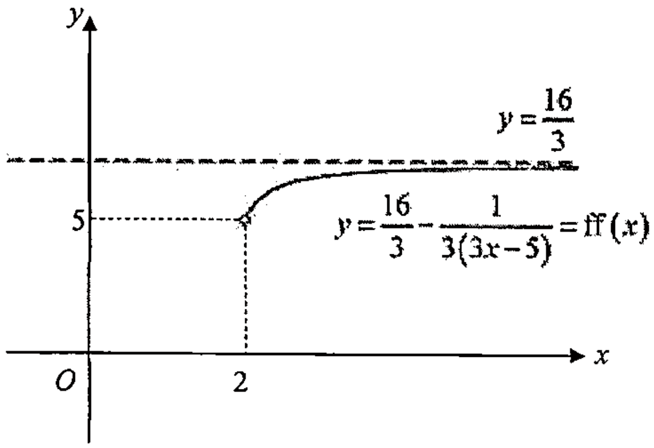
(or $\frac{x-2}{3x-5}+5=\frac{16}{3}-\frac{1}{3(3x-5)}=\frac{16x-27}{3x-5}$)

$ff:x\mapsto\frac{x-2}{3x-5}+5,\ x>2$

Method 1

$D_f=(2,\infty)\xrightarrow{f}R_f=(5,\infty)\xrightarrow{f}R_{ff}=\left(5,\frac{16}{3}\right)$



	Method 2  $R_{ff} = \left(5, \frac{16}{3}\right)$.	

4 Given that $I=\int x^3\left(1+x^2\right)^{-2}\,\mathrm{d}x$.

(i) By writing $I=\int x^2\left[x\left(1+x^2\right)^{-2}\right]\mathrm{d}x$, use integration by parts to find an expression for I. [3]

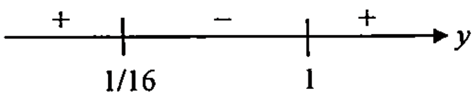
(ii) Use the substitution $u=1+x^2$ to find another expression for I. [3]

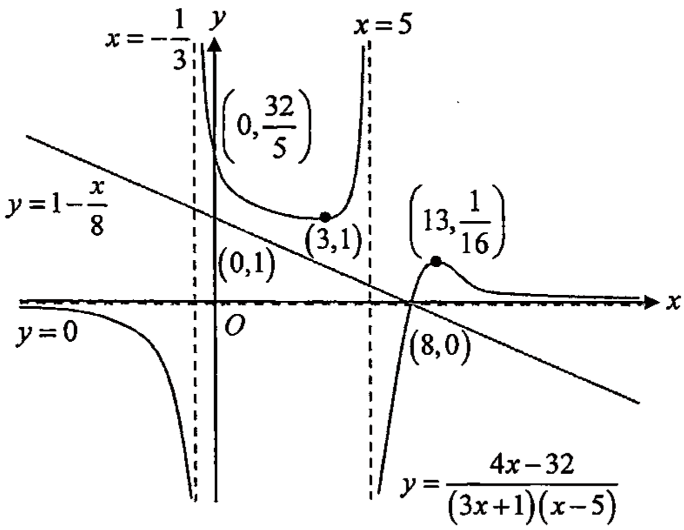
(iii) Show algebraically that your answers to parts (i) and (ii) differ by a constant. [1]

4	Solution [7] Integration by parts, by Substitution	
(i)	$\int x^2\left[x\left(1+x^2\right)^{-2}\right]\mathrm{d}x$ $=-\frac{x^2}{2}\left(1+x^2\right)^{-1}-\int\frac{-x}{1+x^2}\,\mathrm{d}x$ $=-\frac{x^2}{2\left(1+x^2\right)}+\frac{1}{2}\int\frac{2x}{1+x^2}\,\mathrm{d}x$ $=-\frac{x^2}{2\left(1+x^2\right)}+\frac{1}{2}\ln\left\|1+x^2\right\|+c_1$ $=-\frac{x^2}{2\left(1+x^2\right)}+\frac{1}{2}\ln\left(1+x^2\right)+c_1.$ (as $1+x^2>0$) $u=x^2,\ \frac{\mathrm{d}v}{\mathrm{d}x}=x\left(1+x^2\right)^{-2}=\frac{1}{2}\left[2x\left(1+x^2\right)^{-2}\right]$ $\frac{\mathrm{d}u}{\mathrm{d}x}=2x,\ v=\frac{1}{2}\left(\frac{\left(1+x^2\right)^{-1}}{-1}\right)$	
(ii)	$u=1+x^2\Rightarrow 1=2x\frac{\mathrm{d}x}{\mathrm{d}u}$ $\int x^3\left(1+x^2\right)^{-2}\,\mathrm{d}x$ $=\frac{1}{2}\int x^2\left(1+x^2\right)^{-2}\,2x\mathrm{d}x$ $=\frac{1}{2}\int(u-1)u^{-2}\,\mathrm{d}u$ $=\frac{1}{2}\int\frac{1}{u}-u^{-2}\,\mathrm{d}u$	

5 The curve C has equation $y=\frac{4x-32}{(3x+1)(x-5)}$, $x\in\mathbb{R}$, $x\neq-\frac{1}{3},5$.

	(i) Show, algebraically, that $y \le \frac{1}{16}$ or $y \ge 1$. [3] (ii) Sketch the graph of curve C, stating the equations of the asymptotes, coordinates of its turning points and points where it crosses the axes. [3] (iii) On the same diagram in part (ii), sketch the line with equation $y=1-\frac{x}{8}$. Hence, or otherwise, solve the inequality $\frac{4x+32}{(3x+1)(x-5)} \ge 1-\frac{x}{8}$. [2] $=\frac{1}{2}\left(\ln\lvert u\rvert-\frac{u^{-1}}{-1}\right)+c_2$ $=\frac{1}{2}\ln\lvert 1+x^2\rvert+\frac{1}{2(1+x^2)}+c_2.$ $=\frac{1}{2}\ln(1+x^2)+\frac{1}{2(1+x^2)}+c_2.$ (as $1+x^2>0$) **OR** $x=\sqrt{u-1} \Rightarrow \frac{dx}{du}=\frac{1}{2\sqrt{u-1}}$ $\int x^3(1+x^2)^{-2}\,dx$ $=\int (u-1)\sqrt{u-1}(u^{-2})\,\frac{1}{2\sqrt{u-1}}\,du$ $=\frac{1}{2}\int (u-1)u^{-2}\,du$ $=\frac{1}{2}\int \frac{1}{u}-u^{-2}\,du$ $=\frac{1}{2}\left(\ln\lvert u\rvert-\frac{u^{-1}}{-1}\right)+c_2$ $=\frac{1}{2}\ln\lvert 1+x^2\rvert+\frac{1}{2(1+x^2)}+c_2.$ $=\frac{1}{2}\ln(1+x^2)+\frac{1}{2(1+x^2)}+c_2.$ (as $1+x^2>0$)	
(iii)	$-\frac{x^2}{2(1+x^2)}+\frac{1}{2}\ln(1+x^2)+c_1-\left(\frac{1}{2}\ln(1+x^2)+\frac{1}{2(1+x^2)}+c_2\right)$ $=-\frac{x^2}{2(1+x^2)}-\frac{1}{2(1+x^2)}+c_1-c_2$ $=-\frac{x^2+1}{2(1+x^2)}+c_1-c_2$ $=-\frac{1}{2}+c_1-c_2.$ (Shown)	

5	Solution [8] Graphing Techniques, Inequalities	
(i)	y can take value if the equation $y=\dfrac{4x-32}{(3x+1)(x-5)}$ has real solutions. $y=\dfrac{4x-32}{(3x+1)(x-5)}$ $\left(3x^2-14x-5\right)y=4x-32$ $3yx^2+(-14y-4)x+(32-5y)=0$ Taking discriminant ≥ 0, $(-14y-4)^2-4(3y)(32-5y)\geq 0$ $(7y+2)^2-\left(96y-15y^2\right)\geq 0$ $49y^2+28y+4-96y+15y^2\geq 0$ $64y^2-68y+4\geq 0$ $16y^2-17y+1\geq 0$ $(16y-1)(y-1)\geq 0$  $y\leq\dfrac{1}{16}$ or $y\geq 1$	

(ii)		
(iii)	From the graph in **(ii)**, $-\frac{1}{3}<x<5$ or $x\geq 8$.	

6 Let $f(n)=\dfrac{n-1}{n(n+1)}$.

(i) Express $f(n)$ in partial fractions. [1]

(ii) Hence find $\sum_{n=1}^{N} 2^n f(n)$. [3]

(iii) Find $\sum_{n=2}^{2N} 2^{n-1} f(n-1)$. [2]

(iv) Denote $S_N=\sum_{n=1}^{N} 2^n f(n)$. Find least integer k such that S_k is more than 5% of S_{10}. [2]

6	Solution [8] Sigma Notation, Method of difference	
(i)	$\dfrac{n-1}{n(n+1)}=\dfrac{2}{n+1}-\dfrac{1}{n}$	
(ii)	$\sum_{n=1}^{N} 2^n f(n)$ $=\sum_{n=1}^{N} 2^n\left(\dfrac{2}{n+1}-\dfrac{1}{n}\right)$ $=\sum_{n=1}^{N}\left(\dfrac{2^{n+1}}{n+1}-\dfrac{2^n}{n}\right)$ $=\dfrac{2^2}{2}-2$ $+\dfrac{2^3}{3}-\dfrac{2^2}{2}$ $+\dfrac{2^4}{4}-\dfrac{2^3}{3}$ $\vdots$ $+\dfrac{2^N}{N}-\dfrac{2^{N-1}}{N-1}$ $+\dfrac{2^{N+1}}{N+1}-\dfrac{2^N}{N}$ $=\dfrac{2^{N+1}}{N+1}-2$	

(iii)	$\sum_{n=2}^{2N} 2^{n-1}\mathrm{f}(n-1)$ $= \sum_{n+1=2}^{n+1=2N} 2^{n}\mathrm{f}(n)$ (replace n by $n+1$) $= \sum_{n=1}^{2N-1} 2^{n}\mathrm{f}(n)$ $= \frac{2^{(2N-1)+1}}{(2N-1)+1} - 2 = \frac{2^{2N-1}}{N} - 2.$	
(iv)	$S_k > 0.05S_{10}$ $\Rightarrow \frac{2^{k+1}}{k+1} - 2 > 0.05\left(\frac{2^{11}}{11} - 2\right)$ i.e. $\frac{2^{k+1}}{k+1} > 11.2091$ Solving using GC,	

k	$\frac{2^{k+1}}{k+1}$
5	$\frac{64}{6} = 10.67 \quad < 11.2091$
6	$\frac{128}{7} = 18.29 \quad > 11.2091$
7	$32 > 11.2091$

$\therefore k \geq 6$

Least $k = 6$.

Alternatively,

$S_{10} = \frac{2026}{11} = 184.182$

$0.05S_{10} = 9.20909$

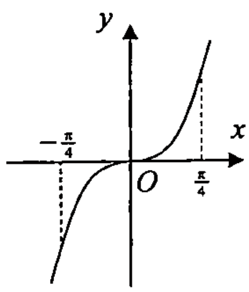
k	S_k
5	$\frac{26}{3} = 8.67 < 9.20909$
6	$\frac{114}{7} = 16.29 > 9.20909$
7	$30 > 9.20909$

For $S_k > 0.05S_{10}$, $k \geq 6$

Least $k = 6$.

7 **(i)** Without the use of a calculator, show that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\tan x| \,dx = \ln 2$. [3]

(ii) Hence, find the exact volume of the solid formed from completely rotating region bounded by the graph of $y = |\tan x| + 1$, the lines $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$, about the x-axis. [5]

7	Solution [8] Applications of Integration	
(i)	$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\tan x\| \,dx$ $= \int_{-\frac{\pi}{4}}^{0} -\tan x \,dx + \int_{0}^{\frac{\pi}{4}} \tan x \,dx$ $= 2\int_{0}^{\frac{\pi}{4}} \tan x \,dx$ $= 2\left[\ln(\sec x)\right]_{0}^{\frac{\pi}{4}}$ $= 2\left(\ln\left(\frac{1}{\cos\frac{\pi}{4}}\right) - \ln\left(\frac{1}{\cos 0}\right)\right)$ $= 2\left(\ln\left(\sqrt{2}\right) - \ln(1)\right)$ $= \ln 2.$  	
(ii)	Required volume $= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\|\tan x\| + 1\right)^2 \,dx$ $= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\tan x\|^2 + 2\|\tan x\| + 1 \,dx$ $= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x + 1 \,dx + \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\|\tan x\| \,dx$ $= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x \,dx + 2\pi \ln 2$ $= \pi\left[\tan x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + 2\pi \ln 2$ $= \pi\left(\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right) + 2\pi \ln 2$ $= 2\pi(1 + \ln 2)$ units³.	

8 Let S_n denote the sum of the first n terms of the sequence of numbers $u_1, u_2, u_3, \ldots$. It is given that $S_n = 2n^2 + kn$, where k is a non-zero constant.

(i) Show that the sequence of numbers $u_1, u_2, u_3, \ldots$ is an arithmetic progression. [3]

A geometric progression has non-zero first term a and common ratio r. Given that the sum of the first 5 terms of the progression is $\frac{32}{33}$ of the sum of the first 10 terms of the progression.

(ii) Show that $r = \frac{1}{2}$ and find the infinite sum of the geometric progression in terms of a. [4]

(iii) Given further that u_{62}, u_{30}, u_t are three consecutive terms of the given geometric progression, find the value of t. [3]

8	Solution [10] APGP	
(i)	$u_n = S_n - S_{n-1}$ $= 2n^2 + kn - \left(2(n-1)^2 + k(n-1)\right)$ $= 2\left(n^2 - n^2 + 2n - 1\right) + k(n - n + 1)$ $= 4n + k - 2$ <u>Alternative 1</u>: $u_n - u_{n-1} = 4n + k - 2 - \left(4(n-1) + k - 2\right)$ $= 4(n - n + 1) + 0$ $= 4$ As the difference between two consecutive terms is a constant, the sequence $u_1, u_2, u_3, \ldots$ is an arithmetic progression. <u>Alternative 2</u>: $u_n = 4n + k - 2$ $= (k+2) + 4(n-1)$ Hence, $u_1, u_2, u_3, \ldots$ is an arithmetic progression, where $u_1 = k + 2$ and its common difference is 4, a constant.	

(ii)	$\frac{a(r^5-1)}{r-1}=\frac{32}{33}\left(\frac{a(r^{10}-1)}{r-1}\right)$ $r^5-1=\left(\frac{32}{33}\right)(r^{10}-1)$ $r^5-1=\left(\frac{32}{33}\right)(r^5-1)(r^5+1)$ $1=\frac{32}{33}r^5+\frac{32}{33}$ (as $r\neq1$ due to $a\neq0$) $r^5=\frac{1}{32}$ $r=\frac{1}{2}$ (Shown) Infinite Sum $=\frac{a}{1-\frac{1}{2}}=2a$.	
(iii)	$\frac{u_{30}}{u_{62}}=r=\frac{1}{2}$ $\frac{4(30)+k-2}{4(62)+k-2}=\frac{1}{2}$ $238+2k=248+k$ $k=10$ $\frac{u_t}{u_{30}}=r=\frac{1}{2}$ $\frac{4t+10-2}{4(30)+10-2}=\frac{1}{2}$ $8t+16=128$ $t=14$.	

9 A curve C has equation

$$y(x+1)^2 + y^2 - 12x = 0, \text{ where } x, y \geq 0.$$

(i) Show that $\dfrac{dy}{dx} = \dfrac{12-2y(x+1)}{(x+1)^2+2y}$. [2]

(ii) Find the equation of the normal L to the curve at the point $P(3, 2)$. [3]

(iii) Find the area of the triangle bounded by L and the axes. [1]

(iv) The curve C is transformed by a series of transformations resulting in a curve D, with equation

$$3y(2x+1)^2 + 9y^2 - 24x = 0,$$

with point Q on D corresponding to the point P on C.
The normal L undergoes the same sequence of transformations as curve C, resulting in the line l.
Describe a possible sequence of transformations that maps C to D and hence deduce the exact area of the triangle bounded by l and the axes. [3]

9	Solution [9 marks] Implicit differentiation, Graphing Transformations	
(i)	$y(x+1)^2 + y^2 - 12x = 0$ Diff wrt x: $y(2)(x+1) + (x+1)^2 \dfrac{dy}{dx} + 2y\dfrac{dy}{dx} - 12 = 0$ --- (*) $\left((x+1)^2 + 2y\right)\dfrac{dy}{dx} = 12 - 2y(x+1)$ $\dfrac{dy}{dx} = \dfrac{12-2y(x+1)}{(x+1)^2+2y}$ (shown)	
(ii)	$\dfrac{dy}{dx} = \dfrac{12-2(2)(3+1)}{(3+1)^2+2(2)}$ $= \dfrac{-4}{20}$ $= -\dfrac{1}{5}$ Gradient of normal $= -\dfrac{1}{\left(-\frac{1}{5}\right)} = 5$ Thus, equation of normal is	

	$y-2=5(x-3)$ $y=5x-13$	
(iii)	When $y=0$, $x=\dfrac{13}{5}$ When $x=0$, $y=-13$ Area of triangle $=\dfrac{1}{2}\left(\dfrac{13}{5}\right)(13)=\dfrac{169}{10}$ units^2.	
(iv)	$3y(2x+1)^2+9y^2-24x=0$ $3y(2x+1)^2+(3y)^2-12(2x)=0$ Since x has been replaced by $2x$, (1) scaling of scale factor $\dfrac{1}{2}$ along x-direction Since y has been replaced by $3y$, (2) scaling of scale factor $\dfrac{1}{3}$ along y-direction Thus, area of new triangle $=16.9\times\dfrac{1}{2}\times\dfrac{1}{3}=\dfrac{169}{60}$ units^2.	

10 A curve C which is symmetrical about the x-axis has parametric equations

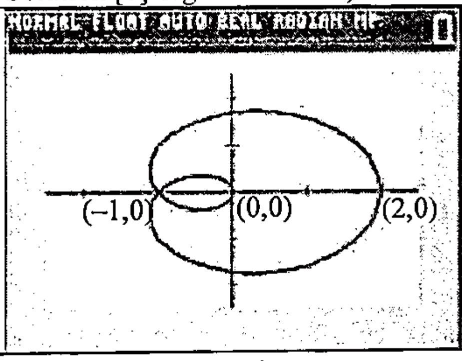
$$x=\cos t+\cos 2t,\ y=\sin t+\sin 2t,\ 0\le t<2\pi.$$

(i) Sketch C, indicating the coordinates of the x-intercepts. [2]

(ii) Find the equation of the tangent to C at the point where $t=\frac{2\pi}{3}$, giving your answer in exact form. [4]

(iii) Show that the point with coordinates $(-1,0)$ lies on the line $y=mx+m$.

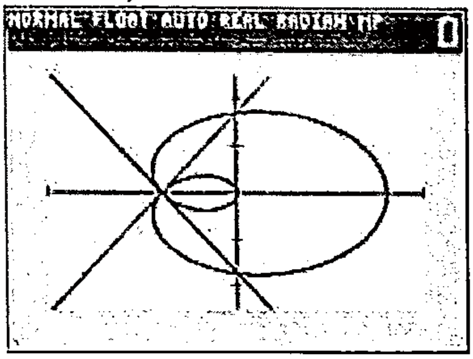
Hence state the range of values of m for which the line $y=mx+m$ intersects C at exactly 3 distinct points. [2]

<table>
<tr><td>10</td><td>Solution [8] Sigma Notation, Method of difference</td><td></td></tr>
<tr><td>(i)</td><td>
</td><td></td></tr>
<tr><td>(ii)</td><td>
$x=\cos t+\cos 2t \Rightarrow \frac{dx}{dt}=-\sin t-2\sin 2t$
$y=\sin t+\sin 2t \Rightarrow \frac{dy}{dt}=\cos t+2\cos 2t$
$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$
$=-\frac{\cos t+2\cos 2t}{\sin t+2\sin 2t}$

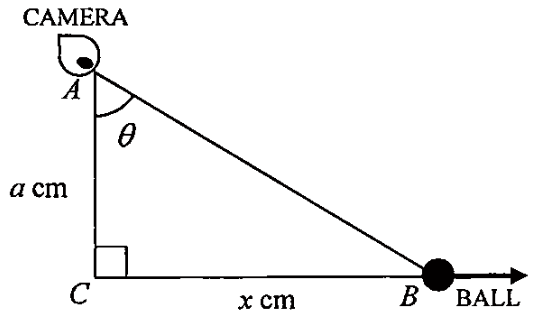
At $t=\frac{2\pi}{3}$: $x=-1,\ y=0,$
$\frac{dy}{dx}=-\frac{\cos\frac{2\pi}{3}+2\cos\frac{4\pi}{3}}{\sin\frac{2\pi}{3}+2\sin\frac{4\pi}{3}}=-\frac{-\frac{1}{2}-1}{\frac{\sqrt{3}}{2}-\sqrt{3}}=-\sqrt{3}$
$\therefore$ Equation of tangent at $t=\frac{2\pi}{3}$: $y=-\sqrt{3}(x-(-1))$
i.e. $y=-\sqrt{3}x-\sqrt{3}$
</td><td></td></tr>
</table>

<table>
<tr><td></td><td></td><td></td></tr>
<tr><td>(iii)</td><td>For $y = mx + m$, when $x = -1$, $y = m(-1) + m = 0$.
Therefore, $(-1, 0)$ is on the line $y = mx + m$ for all real values of m. (shown)

Hence, for the line $y = mx + m$ to cut the curve C at 3 points, $m \in \mathbb{R} \setminus \{\sqrt{3}, -\sqrt{3}\}$. (OR $m \in \mathbb{R}, m \neq \pm\sqrt{3}$)

(Note: the line $y = mx + m$ cuts the curve C at 2 points when $m = \pm\sqrt{3}$.)

</td><td></td></tr>
</table>

11 Fixed vertical cameras have to rotate quickly to keep track on moving objects. A camera at A is attached on a vertical wall AC with fixed height a cm. A ball B is moving horizontally away from the wall AC. At time t seconds, the ball is x cm from the wall and the camera would have complete a rotation of θ radians from the wall to keep track of the ball. At this instant, the camera is rotating at a speed of $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ radians per second. Assume that the triangle ABC lies on a vertical plane containing the path of the ball.



(i) By expressing $\tan\theta$ in terms of x and a, show that $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\left(\frac{a}{a^2+x^2}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$. [4]

(ii) When the ball is 9 cm from the wall, it is travelling at 6 m per second. At this instant, find the speed the camera is rotating at in radians per second, leaving your answer in terms of a. [1]

(iii) Given that $a=4$ and that the ball is travelling such that $x=t^2$ cm, find the **exact** maximum value of the rotating speed of the camera in radians per second. [7]

11	Solution [12] Rates of Change, Max and Min	
(i)	$\tan\theta=\frac{x}{a}$. <u>Alternative 1</u> $\tan\theta=\frac{x}{a}$ $\theta=\tan^{-1}\left(\frac{x}{a}\right)$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=\left(\frac{1}{1+\left(\frac{x}{a}\right)^2}\right)\left(\frac{1}{a}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$	

	$\frac{d\theta}{dt}=\left(\frac{a^2}{a^2\left(1+\left(\frac{x}{a}\right)^2\right)}\right)\left(\frac{1}{a}\right)\frac{dx}{dt}$ $\frac{d\theta}{dt}=\left(\frac{a}{a^2+x^2}\right)\frac{dx}{dt}$ (Shown) <u>Alternative 2</u> $\tan\theta=\frac{x}{a}$ Differentiate with respect to t. $\left(\sec^2\theta\right)\left(\frac{d\theta}{dt}\right)=\left(\frac{1}{a}\right)\left(\frac{dx}{dt}\right)$ $\frac{d\theta}{dt}=\left(\frac{1}{a\sec^2\theta}\right)\left(\frac{dx}{dt}\right)$ $\frac{d\theta}{dt}=\left(\frac{1}{a\left(\tan^2\theta+1\right)}\right)\left(\frac{dx}{dt}\right)$ $\frac{d\theta}{dt}=\left(\frac{1}{a\left((x/a)^2+1\right)}\right)\left(\frac{dx}{dt}\right)$ $\frac{d\theta}{dt}=\left(\frac{a}{a^2\left((x/a)^2+1\right)}\right)\left(\frac{dx}{dt}\right)$ $\frac{d\theta}{dt}=\left(\frac{a}{a^2+x^2}\right)\left(\frac{dx}{dt}\right)$ (Shown)	
(ii)	When $x=9, \frac{dx}{dt}=600$, $\frac{d\theta}{dt}=\left(\frac{a}{a^2+9^2}\right)(600)=\frac{600a}{a^2+81}$ radians per second.	
(iii)	$\frac{dx}{dt}=2t$ $\frac{d\theta}{dt}=\left(\frac{4}{16+t^4}\right)(2t)=\frac{8t}{16+t^4}$	

	$\frac{d^2\theta}{dt^2}=\frac{\left(16+t^4\right)(8)-(8t)\left(4t^3\right)}{\left(16+t^4\right)^2}$ $=\frac{128-24t^4}{\left(16+t^4\right)^2}.$ For stationary values, $\frac{d^2\theta}{dt^2}=0$. $\frac{128-24t^4}{\left(16+t^4\right)^2}=0$ $128-24t^4=0$ $t^4=\frac{16}{3}$ $t=\frac{2}{\sqrt[4]{3}}$ (rej −ve root as $t\geq 0$) <u>Method 1(Second Derivative Test)</u> By GC, when $t=\frac{2}{\sqrt[4]{3}}$, $\frac{d^3\theta}{dt^3}=-0.74029<0$ By second derivative test, $\frac{d\theta}{dt}$ attains a maximum at $t=\frac{2}{\sqrt[4]{3}}$. <u>Alternatively, one could do it algebraically</u>: $\frac{d^3\theta}{dt^3}=\frac{\left(16+t^4\right)^2\left(-96t^3\right)-2\left(16+t^4\right)\left(16+4t^3\right)\left(128-24t^4\right)}{\left(16+t^4\right)^4}$ At $t=\frac{2}{\sqrt[4]{3}}$, $128-24t^4=0$, so, $\frac{d^3\theta}{dt^3}=\frac{\left(16+t^4\right)^2\left(-96at^3\right)-0}{\left(16+t^4\right)^4}=-\frac{96t^3\left(16+t^4\right)^2}{\left(16+t^4\right)^4}<0$	

12 Two teams of scientists are investigating the population of wild hares, and the population of their predators, the wild lynxes, in the snowy mountains.

The number of hares in the snowy mountain is P thousand at a time of t years after the scientists begin observations. Initially, the first team of scientists observed that there were 50 thousand hares, growing at the rate of 25 thousand hares per year. After several observations, they decided to model the growth rate of the hares, at any time t, to be $kP(100-P)$ hares per year, for some positive constant k.

(a) (i) Show that $\frac{dP}{dt}=\frac{1}{100}P(100-P)$. [1]

(ii) Find P in terms of t. [5]

The number of lynxes in the snowy mountains is Q thousand at a time of t years after the scientists begin observations. Initially, the second team of scientists observed that there were 8 thousand lynxes decreasing at the rate of 0.8 thousand lynxes per year.

Both team of scientists collaborated and developed a refined model for the growth rate of the hares. The model assumed that, at any time t, the hares were growing at the rate of $P-\frac{1}{16}QP$ per year.

(b) (i) For the refined model, when $t=0$, show that $\frac{d^2P}{dt^2}=15$. [2]

(ii) Hence, find the Maclaurin series of P in the refined model, in ascending powers of t, up to and including the term in t^2. [2]

Two scientists, scientist A and B, both use the refined model to approximate the number of hares after some time. Scientist A approximates the number of hares one month after the scientists started observing, while scientist B approximates the number of hares ten years after the scientist started observing.

(iii) Given that both scientists used your answer in part **(b)(ii)**, state the approximation **only** provided by scientist A and suggest a reason why scientist A's approximation would be considered more appropriate as compared to scientist B's. [2]

12	Solution [12] Differential Equation, Maclaurin Series	
(ai)	$\frac{dP}{dt}=kP(100-P)$ When $t=0, P=50, \frac{dP}{dt}=25$	

	$25=k(50)(50)$ $k=\dfrac{1}{100}$ Hence, $\dfrac{dP}{dt}=\dfrac{1}{100}P(100-P)$ (Shown).	
(ii)	$\dfrac{dP}{dt}=\dfrac{1}{100}P(100-P)$ $\dfrac{100}{P(100-P)}\dfrac{dP}{dt}=1$ $\int \dfrac{1}{P}+\dfrac{1}{100-P}\ dP=\int 1\ dt$ $\ln\lvert P\rvert-\ln\lvert 100-P\rvert=t+c$ $\ln\left\lvert\dfrac{P}{100-P}\right\rvert=t+c$ $\dfrac{P}{100-P}=\pm e^{t+c}=Ae^t$, where $A=\pm e^c$ **<u>OR</u>** $\dfrac{dP}{dt}=\dfrac{1}{100}P(100-P)$ $\dfrac{100}{100P-P^2}\dfrac{dP}{dt}=1$ $\int\dfrac{100}{50^2-(P-50)^2}\ dP=\int 1\ dt$ $\dfrac{100}{2(50)}\ln\left\lvert\dfrac{50+P-50}{50-P+50}\right\rvert=t+c$ $\ln\left\lvert\dfrac{P}{100-P}\right\rvert=t+c$ When $t=0, P=50$, $\dfrac{50}{100-50}=Ae^0$ $A=1$ $\dfrac{P}{100-P}=e^t$ $P=100e^t-Pe^t$ $P+Pe^t=100e^t$ $P(1+e^t)=100e^t$ $P=\dfrac{100e^t}{1+e^t}=\dfrac{100}{1+e^{-t}}$.	
(bi)	$\dfrac{dP}{dt}=P-\dfrac{1}{16}QP$ $\dfrac{d^2P}{dt^2}=\dfrac{dP}{dt}-\dfrac{1}{16}Q\dfrac{dP}{dt}-\dfrac{1}{16}P\dfrac{dQ}{dt}$	

	When $t=0$, $P=50, Q=8, \frac{dP}{dt}=25, \frac{dQ}{dt}=-0.8$. $\frac{d^2P}{dt^2}=25-\frac{1}{16}(8)(25)-\frac{1}{16}(50)(-0.8)=15$. (Shown)	
(bii)	When $t=0$, $P=50, \frac{dP}{dt}=25, \frac{d^2P}{dt^2}=15$. $P=50+25t+\frac{15}{2!}t^2+...$ $P=50+25t+7.5t^2+...$	
(biii)	When $t=\frac{1}{12}$, $P\approx 50+25\left(\frac{1}{12}\right)+7.5\left(\frac{1}{12}\right)^2=52.135$ thousand Scientist A would approximate 52.1 thousand hares. Acceptable reasons It is more appropriate as $t=\frac{1}{12}$ **is closer to 0 as compared to** $t=10$, making the approximation at $t=\frac{1}{12}$ more accurate. **OR** When $0<t<1$, when $t=\frac{1}{12}$, the terms t^n for $n\geq 3$ are **a lot smaller and are negligible**. While, when $t>1$ when $t=10$, the terms t^n for $n\geq 3$ **are significantly large and not negligible**. So, the Maclaurin Series in (bii) is inappropriate $t=10$ as it ignores these values. **OR** As $t=\frac{1}{12}$ is **close to the initial conditions**, the **assumptions made** by the scientist for the refined model are likely still **applicable**. However, when $t=10$, it **is far too long after the assumptions are made** and are unlikely to be applicable.	