Suggested Solution to 2015 SH2 H2 Mathematics Preliminary Examination Paper 2

1 (a)	$x = 3\tan\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec^2\theta$ When $x=3$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$ When $x=\sqrt{3}$, $\tan\theta = \frac{\sqrt{3}}{3} \Rightarrow \theta = \frac{\pi}{6}$ $\int_{\sqrt{3}}^{3} \frac{1}{x^2\sqrt{x^2+9}}\,\mathrm{d}x$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{9\tan^2\theta\sqrt{9\tan^2\theta+9}}\left(3\sec^2\theta\right)\mathrm{d}\theta$ $= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\left(\tan^2\theta\right)\left(3\sec\theta\right)}\left(3\sec^2\theta\right)\mathrm{d}\theta$ $= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta$ $= \frac{1}{9}\left[\frac{-1}{\sin\theta}\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$ $= \frac{1}{9}\left(\frac{-1}{\sin\frac{\pi}{4}} + \frac{1}{\sin\frac{\pi}{6}}\right)$ $= \frac{2-\sqrt{2}}{9}$
1 (b)	$\int \ln\left(x^2+4\right)\mathrm{d}x$ $= \int 1\cdot\ln\left(x^2+4\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int x\left(\frac{2x}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int\left(\frac{2\left(x^2+4\right)-8}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - \int\left(2-\frac{8}{x^2+4}\right)\mathrm{d}x$ $= x\ln\left(x^2+4\right) - 2x + 4\tan^{-1}\left(\frac{x}{2}\right) + c$
2 (a)	Amount saved after n months > 20000 $1000 + 1000(1.05) + \cdots + 1000(1.05)^{n-1} > 20000$

$$\frac{1000(1.05^n-1)}{1.05-1} > 20000$$

$$20000(1.05^n-1) > 20000$$

$$1.05^n - 1 > 1$$

$$1.05^n > 2$$

$$n \ln 1.05 > \ln 2$$

$$n > \frac{\ln 2}{\ln 1.05} = 14.20669908$$

Using GC, smallest n = 15 i.e. it takes 15 months for Joanne's savings to exceed $20000.

Joanne's savings at the end of the Nth month

$= 20000(1.05^N - 1)$ (from part **(a)**).

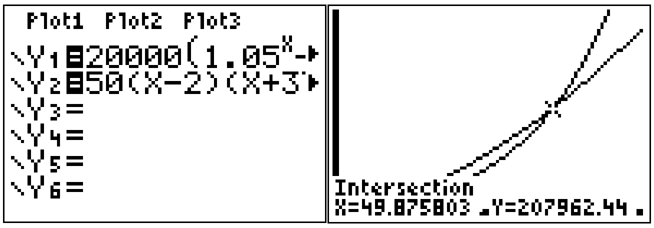
Jim's savings

= 0 + 0 + 2000 + (2000+100) + ….+ [2000 +100(N – 3)]

$= \frac{N-2}{2}\left[2\cdot 2000 + (N-3)(100)\right],\ \text{for } N \geq 3$

$= 50(N-2)(N+37),\ \text{for } N \geq 3$

$$20000(1.05^N - 1) > 50(N-2)(N+37)$$



By GC , we find that n = 50.

2 (b)

Let $x = 1 + r/100$.

For Joanne's total savings to be at least \$30000

$$1000x + 1000x^2 + \ldots\ldots + 1000x^{24} \geq 30000$$

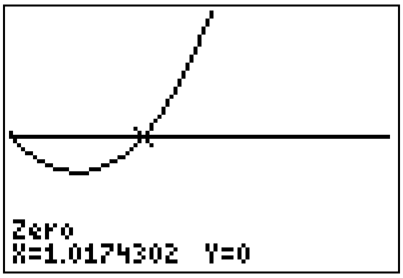
$$\frac{1000x\left(1-x^{24}\right)}{1-x} \geq 30000$$

$$\frac{x\left(1-x^{24}\right)}{1-x} \geq 30$$

$$x\left(1-x^{24}\right) \leq 30(1-x)$$

$$(\because x = 1 + 100r > 1)$$

$$x^{25} - 31x + 30 \geq 0$$



Sketching $y = x^{25} - 31x + 30$, we see that $x \geq 1.0174302$,

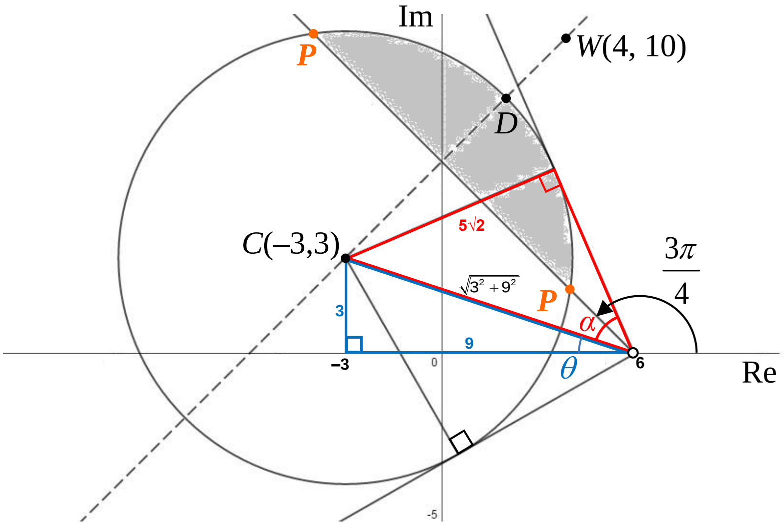
Hence the interest rate must be at least 1.74%

i.e. least value of r = 1.8 (correct to 1 decimal place)

3 (i)	Converting the equations of p_1 and p_2 to Cartesian form, $x + y - z = 0$ --- (1) $2x - y - 2z = 6$ --- (2) Using the PolySmlt2 app in the GC, $l : \mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \lambda \in \mathbf{R}$
3 (ii)	Equating the RHS of the two equations for l and l_1, we have: $\begin{pmatrix} 2+\lambda \\ -2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 2+\mu \\ \mu \\ 8+5\mu \end{pmatrix}$ $2+\lambda = 2+\mu$ ---- (3) $-2 = \mu$ ---- (4) $\lambda = 8+5\mu$ ---- (5) Substituting (4) into (3), we get $2+\lambda = 2-2 = 0 \Rightarrow \lambda = -2$. Substituting $\lambda = -2$ and $\mu = -2$ into (5), LHS = –2 = 8 + 5(−2) = RHS. Hence, l and l_1 intersect at exactly one point. Hence, p_1, p_2 and l_1 have exactly one common point of intersection.
3 part after (ii)	l lies on p_3. So l is perpendicular to the normal vector of p_3. $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0$ $1+b=0$ $b=-1$ (shown)

3 (iii)	<u>**Method 1: Finding angle between two planes**</u> Acute angle between p_1 and p_2 = acute angle between p_2 and p_3 $\cos^{-1}\frac{\left\lvert\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert}{\left(\sqrt{3}\right)\left(\sqrt{9}\right)}=\cos^{-1}\frac{\left\lvert\begin{pmatrix}1\\a\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert}{\left(\sqrt{2+a^2}\right)\left(\sqrt{9}\right)}$ $\frac{3}{\sqrt{3}}=\frac{\lvert 4-a\rvert}{\sqrt{2+a^2}}$ $3\sqrt{2+a^2}=\sqrt{3}\lvert 4-a\rvert$ $9\left(2+a^2\right)=3\left(16-8a+a^2\right)$ $6+3a^2=16-8a+a^2$ $2a^2+8a-10=0$ $a^2+4a-5=0$ $(a+5)(a-1)=0$ $a=-5$ or 1 $\left(\begin{array}{c}\text{rejected because } p_3 \text{ and } p_1\\ \text{are distinct and non-parallel}\end{array}\right)$ Since l lies on p_3, $d=\begin{pmatrix}2\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\-1\end{pmatrix}$ $=2-2(-5)$ $=12$ $\mathbf{r}\cdot\begin{pmatrix}1\\-5\\-1\end{pmatrix}=12$

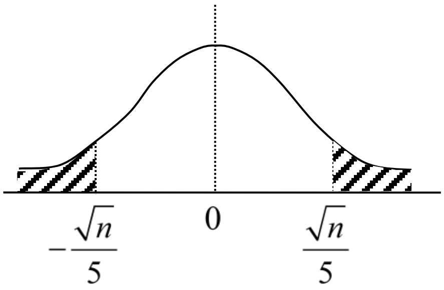
3 (iii)	**<u>Method 2: Finding mirror image of point in plane</u>** Alternatively, a point on p_1 is the origin, (0, 0, 0) and a point on p_2 is (3, 0, 0). Denote the foot of perpendicular of *O* onto p_2 by *P*. Then $\overrightarrow{OP}=\left(\frac{\begin{pmatrix}3-0\\0-0\\0-0\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}}{\sqrt{2^2+(-1)^2+(-2)^2}}\right)\left(\frac{\begin{pmatrix}2\\-1\\-2\end{pmatrix}}{\sqrt{2^2+(-1)^2+(-2)^2}}\right)$ $=\frac{6}{9}\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $=\begin{pmatrix}4/3\\-2/3\\-4/3\end{pmatrix}$ Let O' be the mirror image of *O* in p_2. Then $\overrightarrow{OO'}=2\overrightarrow{OP}$ $=\begin{pmatrix}8/3\\-4/3\\-8/3\end{pmatrix}$ Therefore a vector parallel to p_3 $=\begin{pmatrix}8/3\\-4/3\\-8/3\end{pmatrix}-\begin{pmatrix}2\\-2\\0\end{pmatrix}=\begin{pmatrix}2/3\\2/3\\-8/3\end{pmatrix}=\frac{2}{3}\begin{pmatrix}1\\1\\-4\end{pmatrix}$ Hence normal vector to $p_3=\begin{pmatrix}1\\1\\-4\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-5\\-1\end{pmatrix}$ Hence $a=-5$ and an equation for p_3 is $\mathbf{r}\cdot\begin{pmatrix}1\\-5\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\-1\end{pmatrix}=12$

4 (i), (ii)	
4 (iii)	Let W be the point representing $4+10\mathrm{i}$. Then $\min\lvert z-4-10\mathrm{i}\rvert = WD$ $= WC - CD$ $= \sqrt{(4-(-3))^2+(10-3)^2} - 5\sqrt{2}$ $= 7\sqrt{2} - 5\sqrt{2}$ $= 2\sqrt{2}$ i.e. $m = 2$ (shown)
4 (iv)	Minimum value of $\lvert\arg(w-6)\rvert$ $= \pi - \alpha - \theta$ $= \pi - \sin^{-1}\left(\frac{5\sqrt{2}}{\sqrt{3^2+9^2}}\right) - \tan^{-1}\left(\frac{3}{9}\right)$ $= \pi - \sin^{-1}\left(\frac{\sqrt{50}}{\sqrt{90}}\right) - \tan^{-1}\left(\frac{3}{9}\right)$ $= \pi - \sin^{-1}\left(\sqrt{\frac{5}{9}}\right) - \tan^{-1}\left(\frac{1}{3}\right)$ $= 1.978773429$ rad $= 1.979$ rad (to 3 d.p.)

5 (a)	Assign a number from 1 to 2500 to each of the 2500 students OR obtain a list of the students from the administration office in order of their identification numbers or registration numbers. Use a random number generator to generate 500 numbers randomly and select the members who are assigned those numbers (based on the ordering of the list).
5 (b)	Assign a number from 1 to 2500 to each of the 2500 students OR obtain a list of the students from the administration office in order of their identification numbers or registration numbers. Next, determine the sampling interval size $k = \frac{2500}{500} = 5$. Randomly select any student from the list, say the 1^{st} student. Select every 5^{th} student thereafter (i.e. 6^{th}, 11^{th}, 16^{th}…) until all 500 students are selected.
6 (i)	Let X be the number of students, out of 20 randomly chosen students, who wore their House T-shirts to school. Then $X \sim \mathrm{B}(20, 0.14)$. $\mathrm{P}(X = 3) = 0.24086$ $= 0.241$ (3 s.f.)
6 (ii)	Let Y be the number of students, out of 60 randomly chosen students, who wore their school uniforms to school. Then $Y \sim \mathrm{B}(60, 0.78)$. Since $n = 60 > 30$ is sufficiently large, $np = 46.8 > 5$ and $n(1-p) = 13.2 > 5$, therefore $Y \sim \mathrm{N}(np, np(1-p))$ approximately i.e. $Y \sim \mathrm{N}(46.8, 10.296)$ approximately. $\mathrm{P}(Y > 45) = \mathrm{P}(Y > 45.5)$ (by continuity correction) $= 0.6573140131$ $= 0.657$ (3 s.f.)

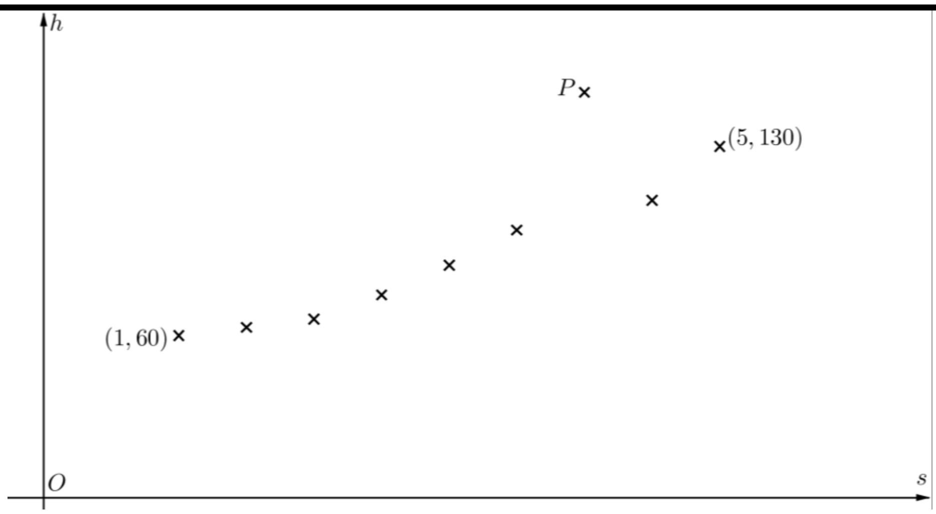
7 (i)	Number of ways = $\binom{5}{4} = 5$
7 (ii)	Task 1 : Select the colour to appear once → $\binom{5}{1}$ ways Task 2 : Select the colour to appear thrice → $\binom{4}{1}$ ways Hence total number of ways $= \binom{5}{1} \times \binom{4}{1} = 20$ or total number of ways $= \binom{5}{2} \times 2!$ or ${}^5\mathrm{P}_2 = 20$
7 (iii)	<u>Case 1 : All balls are distinct</u> Number of ways = 5 (from part **(i)**) <u>Case 2 : Exactly 3 identical balls are selected</u> Number of ways = 20 (from part **(ii)**). <u>Case 3: 2 pairs of identically-coloured balls</u> Number of ways $= \binom{5}{2} = 10$ <u>Case 4: 4 identically-coloured balls</u> Number of ways $= \binom{5}{1} = 5$ <u>Case 5: 1 pair of identically-coloured balls and 2 balls of different colours</u> Number of ways $= \binom{5}{3} \times \binom{3}{1} = 30$ or Number of ways $= \binom{5}{1} \times \binom{4}{2} = 30$ or Number of ways $= \binom{5}{2} \times \binom{3}{1} = 30$ or Hence total number of ways = 5 + 20 + 10 + 5 + 30 = 70

8 (i)	Any two of the following three: 1. The demand for each single-room occurs **singly** i.e. does not occur at the same instant. 2. Each demand for a room in a day is **independent** of (or unaffected by) a demand occurring in any other part of the same day. 3. The **mean** number of demands over any time interval of the same duration within the day is **constant**.
8 (ii)	Let X denote the random variable for the number of demands for a single-bed room in a day. Then $X \sim \text{Po}(5.8)$. $\text{P}(X \geq 10)$ $= 1 - \text{P}(X \leq 9)$ $= 0.0708$
8 (iii)	Let Y denote the random variable for the number of demands for a double-bed room in a day. Then $Y \sim \text{Po}(37.1)$. $X + Y \sim \text{Po}(5.8 + 37.1)$ $\Rightarrow X + Y \sim \text{Po}(42.9)$ $\text{P}(X \geq 1 \text{ and } Y \geq 36 \mid X + Y = 38)$ $\text{P}(X \geq 1 \text{ and } Y \geq 36 \mid X + Y = 38)$ $= \dfrac{\text{P}(X \geq 1 \text{ and } Y \geq 36 \text{ and } X + Y = 38)}{\text{P}(X + Y = 38)}$ $= \dfrac{\text{P}(X = 1 \text{ \& } Y = 37) + \text{P}(X = 2 \text{ \& } Y = 36)}{\text{P}(X + Y = 38)}$ $= \dfrac{\text{P}(X = 1)\text{P}(Y = 37) + \text{P}(X = 2)\text{P}(Y = 36)}{\text{P}(X + Y = 38)}$ $= 0.092653$ $= 0.0927$ (to 3 s.f.)

9 (a)	Let X and Y be the weight in, grams, of a randomly chosen handphone and a randomly chosen tablet respectively. $X \sim \mathrm{N}\left(130, 5^2\right)$ and $Y \sim \mathrm{N}\left(330, 6^2\right)$ Note that $\mathrm{E}\left(X_1 + ... + X_5 - 2Y\right) = 5\mathrm{E}(X) - 2\mathrm{E}(Y) = -10$ $\mathrm{Var}\left(X_1 + ... + X_5 - 2Y\right) = 5\,\mathrm{Var}(X) + 4\,\mathrm{Var}(Y) = 269$ Hence, $X_1 + X_2 + X_3 + X_4 + X_5 - 2Y \sim \mathrm{N}\left(-10, 269\right)$ $\mathrm{P}\left(X_1 + X_2 + X_3 + X_4 + X_5 - 2Y < 0\right) = 0.729$ (to 3.s.f)
9 (b)	Let $\bar{X} = \dfrac{X_1 + X_2 + ... + X_n}{n}$ denote the sample mean mass of the n handphones. Then, $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{5^2}{n}\right)$. $\mathrm{P}\left(\left\lvert \bar{X} - \mu \right\rvert > 1\right) \leq 0.04$. Standardising, $Z = \dfrac{\bar{X} - \mu}{\frac{5}{\sqrt{n}}} \sim \mathrm{N}\left(0,1\right)$. Thus the probability becomes $\mathrm{P}\left(\lvert Z \rvert > \dfrac{1}{5/\sqrt{n}}\right) \leq 0.04$  By symmetry, $\mathrm{P}\left(Z < -\dfrac{\sqrt{n}}{5}\right) \leq 0.02$ $-\dfrac{\sqrt{n}}{5} < -2.053748911$ $n \geq 105.4471146.$ Thus, least value of n is 106.

10 (a) (i)	Let E be the event that a randomly selected bottle of cultured milk is grape flavoured $\mathrm{P}(E)=0.6\times0.3+0.4\times0.1$ $=0.22$
10 (a) (ii)	Let F be the event that a bottle of cultured milk is manufactured by Yacoat $\mathrm{P}(F\mid E)=\dfrac{\mathrm{P}(F\cap E)}{\mathrm{P}(E)}$ $=\dfrac{0.6\times0.3}{0.22}$ $=\dfrac{9}{11}$ or 0.818
10 (b)	$0.4q-0.4p=0.1 \Rightarrow q-p=0.25$ --- (1) Also, we have $p+q+0.1=1 \Rightarrow p+q=0.9$ ---(2) Solving (1) and (2), $p=0.325$ and $q=0.575$
10 (c)	$\mathrm{P}(A)=0.4\times0.6\times2!=0.48$ $\mathrm{P}(B)$ $=(0.6\times0.5+0.4\times0.4)^2$ or $\underbrace{(0.6\times0.5)^2}_{\substack{\text{Case 1:}\\ \text{Both are Yacoat \& Apple}}}+\underbrace{(0.4\times0.4)^2}_{\substack{\text{Case 2:}\\ \text{Both are Vitergent \& Apple}}}+\underbrace{(0.6\times0.5)(0.4\times0.4)\times2!}_{\substack{\text{Case 3:}\\ \text{One Yacoat \& Apple, the}\\ \text{other Vitergent \& Apple}}}$ $=0.2116$ $\mathrm{P}(A\cap B)=\underbrace{(0.6\times0.5)}_{\substack{\text{1st bottle}\\ \text{(Yacoat \& Apple)}}}\times\underbrace{(0.4\times0.4)}_{\substack{\text{2nd bottle}\\ \text{(Vitergent \& Apple)}}}\times\underset{\text{Permutation}}{2!}=0.096$ Therefore, $\mathrm{P}(A\cup B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $=0.48+0.2116-0.096$ $=\dfrac{1489}{2500}$ or 0.5956

11(i)	$\bar{x} = 30 + \dfrac{360}{40} = 39$ $s^2 = \dfrac{1}{39}\left[4050 - \dfrac{360^2}{40}\right] = 20.76923 = 20.8$
11(ii) 1st part	$\mathrm{H}_0 : \mu = 40 \quad \mathrm{H}_1 : \mu < 40$ Under H_0, $Z = \dfrac{\bar{X} - 40}{\sqrt{\dfrac{20.76923}{40}}} \sim \mathrm{N}(0,1)$ or $\bar{X} \sim \mathrm{N}\left(40, \dfrac{20.76923}{40}\right)$ Reject H_0 if p-value < 0.04. From GC, p-value = 0.0826 > 0.04 or observed test statistic value, $z_{\text{calc}} = -1.3878 > -1.7506$, so we do not reject H_0. There insufficient evidence at the 4% significance level, to conclude that the manufacturer has overstated the mean weight of his products.
11 (ii) 2nd part	There is a probability of 0.04 that we wrongly conclude that the manufacturer has overstated the mean weight of his products. **OR** There is a probability of 0.04 that we conclude that the mean weight of the manufacturer's products is less than 40 grams when the mean weight is actually 40 grams. **OR** There is a probability of 0.04 that we reject the claim that the mean weight of the manufacturer's products is 40 grams when this claim is actually correct.
11(iii)	$\mathrm{H}_0 : \mu = \mu_0 \quad \mathrm{H}_1 : \mu < \mu_0$ Under H_0, $T = \dfrac{\bar{X} - \mu_0}{\sqrt{\left(\dfrac{20}{19}\right)(0.25)/20}} \sim t(19)$ Do not reject H_0 if observed value of test statistic > critical value: $\dfrac{41 - \mu_0}{\sqrt{\left(\dfrac{20}{19}\right)(0.25)/20}} > -1.72913$ $\dfrac{41 - \mu_0}{\sqrt{0.25/19}} > -1.72913$ $41 - \mu_0 > -1.72913\sqrt{0.25/19}$ $\mu_0 < 41 + 1.72913\sqrt{0.25/19}$ $\mu_0 < 41.19834$ $\mu_0 < 41.1$ or 41.2 (3 s.f.) or $\mu_0 \le 41.1$

12(i), (ii)	
12(iii)	r-value = 0.9812; We use the h on s line since h is dependent on s OR We use the h on s line since s is the independent variable. h on s line: h = 35.851 + 17.486s When h = 100, s = 3.669 = 3.67 (3 s.f.)
12(iv)	For model (I), r-value = 0.9898 For model (II), r-value = 0.9380 Hence model (II) is more suitable since the correlation coefficient is nearer to 1, which suggests a stronger linear relationship between h and s^2 as compared to that between h and e^s.
12(v)	Since the data point $(\bar{s}, \bar{h})$ lies on the regression line, the answer in part **(iii)** will not change as the equation of the regression line is not affected. **OR** The addition of the data point $(\bar{s}, \bar{h})$ does not add further to the sum of squares of the h-errors between the current regression line and the data points, hence the h-errors remain minimised with the same line.