Question 1 (Connected Rates of Change)

(i)	$k = r\theta + 2r = r(\theta + 2) \Rightarrow \theta + 2 = \dfrac{k}{r} \Rightarrow \theta = \dfrac{k}{r} - 2$ $A = \dfrac{1}{2}r^2\theta = \dfrac{1}{2}r^2\left(\dfrac{k}{r} - 2\right) = \dfrac{kr}{2} - r^2$ $\dfrac{\mathrm{d}A}{\mathrm{d}r} = \dfrac{k}{2} - 2r$	Interestingly, many students could derive the formula $A = \dfrac{1}{2}r^2\theta$ by considering $A = \dfrac{\theta}{2\pi} \times \pi r^2$ but not many could do so similarly for the arc length formula $s = r\theta$. As a result, there are many students who could not prove the required result.
(ii)	$\dfrac{\mathrm{d}A}{\mathrm{d}t} = \left(\dfrac{k}{2} - 2r\right)\dfrac{\mathrm{d}r}{\mathrm{d}t}$ $= \left(\dfrac{k}{2} - 2 \times \dfrac{k}{3}\right)\dfrac{k}{10}$ $= -\dfrac{k^2}{60}$	This part of the question is better performed than the previous part as almost the whole cohort is able to relate the rates of change. However, many could not apply the information that “arc length is equal to the radius” in a useful manner.

Question 2 (Systems of Linear Equations)

$$(-2)^3 + a(-2)^2 + b(-2) + c = 0$$
$$\Rightarrow -8 + 4a - 2b + c = 0$$
$$\Rightarrow 4a - 2b + c = 8 \quad \text{--- (1)}$$

$$\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right)^3 + a\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right)^2 + b\left(3e^{i\left(-\frac{2}{3}\pi\right)}\right) + c = 0$$
$$27e^{i(-2\pi)} + 9ae^{i\left(-\frac{4}{3}\pi\right)} + 3be^{i\left(-\frac{2}{3}\pi\right)} + c = 0$$
$$27 + 9a\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + 3b\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + c = 0$$
$$\left(27 - \frac{9}{2}a - \frac{3}{2}b + c\right) + \left(\frac{9\sqrt{3}}{2}a - \frac{3\sqrt{3}}{2}b\right)i = 0$$

Comparing real parts, $27 - \frac{9}{2}a - \frac{3}{2}b + c = 0$ --- (2)

Comparing imaginary parts,

$$\frac{9\sqrt{3}}{2}a - \frac{3\sqrt{3}}{2}b = 0 \Rightarrow 3a - b = 0 \quad \text{--- (3)}$$

Solving (1), (2) and (3) with the GC, we get

$$a = 5, \quad b = 15, \quad c = 18.$$

<u>Alternative Method</u>

$$3e^{i\left(-\frac{2}{3}\pi\right)} = 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

Since all coefficients of the polynomial are real, then $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ is also a root of the equation by the Conjugate Root Theorem. Therefore,

$$z^3 + az^2 + bz + c$$
$$= (z+2)\left(z + \frac{3}{2} + \frac{3\sqrt{3}}{2}i\right)\left(z + \frac{3}{2} - \frac{3\sqrt{3}}{2}i\right)$$
$$= (z+2)\left(\left(z + \frac{3}{2}\right)^2 + \frac{9(3)}{4}\right)$$
$$= (z+2)(z^2 + 3z + 9)$$
$$= z^3 + 5z^2 + 15z + 18$$

Therefore $a = 5, \quad b = 15, \quad c = 18$

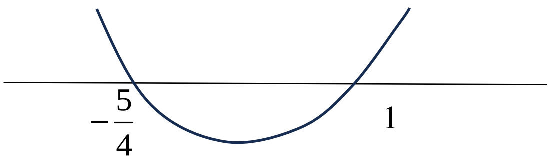
Both methods were fairly common. Many careless mistakes were made in calculations.

Common mistakes:

- ✗ Not stating "Since the coefficients of the polynomial are all real" when using the Conjugate Root Theorem.
- ✗ Incorrect conversion of complex numbers from exponential form to cartesian form. (Please identify the <u>quadrant</u> the complex number is in.)
- ✗ Not converting the complex numbers to cartesian form (in the first method).
- ✗ Writing " $z = 3e^{i\left(-\frac{2}{3}\pi\right)}$ is a root …".

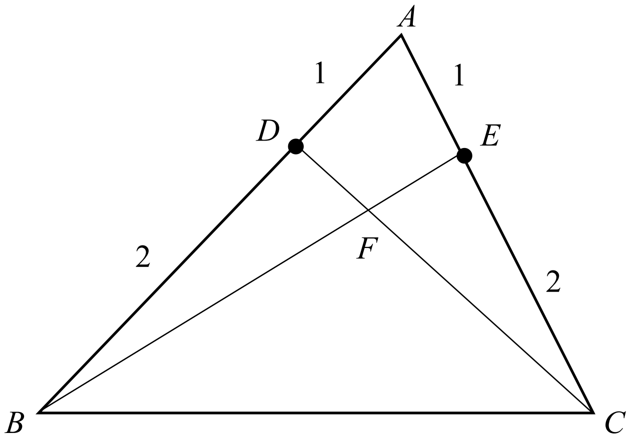
  (Should be " $3e^{i\left(-\frac{2}{3}\pi\right)}$ is a root …" or " $z = 3e^{i\left(-\frac{2}{3}\pi\right)}$ is a solution …").

Question 3 (Inequalities)

(i) **Method 1** $\dfrac{x-6}{4x^2+x-5} \geq 1$ $\Rightarrow \dfrac{x-6}{4x^2+x-5} - 1 \geq 0$ $\Rightarrow \dfrac{x-6-\left(4x^2+x-5\right)}{4x^2+x-5} \geq 0$ $\Rightarrow \dfrac{-4x^2-1}{4x^2+x-5} \geq 0$ $\Rightarrow \dfrac{4x^2+1}{4x^2+x-5} \leq 0$ $\Rightarrow \dfrac{4x^2+1}{(4x+5)(x-1)} \leq 0$ $\Rightarrow (4x+5)(x-1) \leq 0$ since $4x^2+1 \geq 1 > 0$ for all $x \in \mathbb{R}$ **Method 2** $\dfrac{x-6}{4x^2+x-5} \geq 1$ $\Rightarrow (x-6)\left(4x^2+x-5\right) \geq \left(4x^2+x-5\right)^2$ $\Rightarrow \left(4x^2+x-5\right)\left[(x-6)-\left(4x^2+x-5\right)\right] \geq 0$ $\Rightarrow \left(4x^2+x-5\right)\left(-4x^2-1\right) \geq 0$ $\Rightarrow (4x+5)(x-1)\left(-4x^2-1\right) \geq 0$ $\Rightarrow (4x+5)(x-1) \leq 0$ since $-4x^2-1 \leq 1 < 0$ for all $x \in \mathbb{R}$  Therefore, $-\dfrac{5}{4} < x < 1$ since $x \neq -\dfrac{5}{4}$ and $x \neq 1$	Many candidates correctly simplified the inequality to $\dfrac{-4x^2-1}{4x^2+x-5} \geq 0$. The correct factorisation of $4x^2+x-5$ into $(4x+5)(x-1)$ is often seen as well. Some candidates made mistakes in the algebraic manipulations e.g. writing the numerator as $4x^2-1$. There are some candidates who tried to factorise $4x^2+1$. Please note that complex roots are not considered for the determination of the critical values. Common mistakes: ✗ Inclusion of $x=-\dfrac{5}{4}$ and $x=1$ in the solution of the inequality. These values make the denominator in the original inequality 0, so they need to be excluded. ✗ Incorrect evaluation of signs in the number line, leading to incorrect range of solutions $x < -\dfrac{5}{4}$ or $1 < x$

(ii) $\dfrac{x-6x^2}{4+x-5x^2} \geq 1$ Replace x with $\dfrac{1}{y}$ and we get $\dfrac{\frac{1}{y}-6\left(\frac{1}{y}\right)^2}{4+\frac{1}{y}-5\left(\frac{1}{y}\right)^2} \geq 1 \Rightarrow \dfrac{\left(\frac{y-6}{y^2}\right)}{\left(\frac{4y^2+y-5}{y^2}\right)} \geq 1 \Rightarrow \dfrac{y-6}{4y^2+y-5} \geq 1$ Therefore, $-\dfrac{5}{4} < y < 1$, i.e. $-\dfrac{5}{4} < \dfrac{1}{x} < 1$. So $-\dfrac{5}{4} < \dfrac{1}{x} < 0$ or $0 < \dfrac{1}{x} < 1$ $x < -\dfrac{4}{5}$ or $x > 1$	Many students are able to identify that a suitable replacement is $\dfrac{1}{y}$. Common mistakes: ✗ Solving $-\dfrac{5}{4} < \dfrac{1}{x} < 1$ is not simply just taking the reciprocals of the terms and keeping the sign i.e., the inequality above is **not equivalent** to $-\dfrac{4}{5} < x < \dfrac{1}{1}$. A reliable way to solve reciprocals is to look at the graph $y = \dfrac{1}{x}$. ✗ Some students wrote '$x = \dfrac{1}{x}$' to mean 'replace x with $\dfrac{1}{x}$'. If the former holds, then $x = \pm 1$, which is not part of the solution set.

Question 4 (Vectors I)

	
Using Ratio Theorem, $\overrightarrow{BE}$ $=\frac{1}{3}\left(\overrightarrow{BC}+2\,\overrightarrow{BA}\right)$ $=\frac{1}{3}\left(\mathbf{c}-\mathbf{b}+2\left(\mathbf{a}-\mathbf{b}\right)\right)$ $=\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}$ $\overrightarrow{CD}$ $=\frac{1}{3}\left(\overrightarrow{CB}+2\,\overrightarrow{CA}\right)$ $=\frac{1}{3}\left(\mathbf{b}-\mathbf{c}+2\left(\mathbf{a}-\mathbf{c}\right)\right)$ $=\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}-\mathbf{c}$ $l_{BE}:\mathbf{r}=\mathbf{b}+\lambda\left(\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}\right)$ $=\frac{2}{3}\lambda\mathbf{a}+\left(1-\lambda\right)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c},\ \lambda\in\mathbb{R}$ (shown) $l_{CD}:\mathbf{r}=\mathbf{c}+\mu\left(\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}-\mathbf{c}\right)$ $=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+\left(1-\lambda\right)\mathbf{c}$, $\mu\in\mathbb{R}$ (shown) At F, $\frac{2}{3}\lambda\mathbf{a}+\left(1-\lambda\right)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c}=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+\left(1-\lambda\right)\mathbf{c}$ $\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)\mathbf{c}=\mathbf{0}$	Many students could find the vectors $\overrightarrow{BE}$ and $\overrightarrow{CD}$. However, some used vector addition to find the 2 vectors, which is very tedious. Common mistakes: × Some students do not know the difference between the direction vector $\overrightarrow{BE}$ and equation of the line BE. × Some students regarded **a**, **b**, **c** as vectors **i**, **j**, **k** and wrote $\frac{1}{3}\mathbf{c}+\frac{2}{3}\mathbf{a}-\mathbf{b}=\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ -1\end{pmatrix}$ × Many students did not understand the requirements of the question. They compared the coefficients of non-parallel vectors **a**, **b**, **c** at this stage: $\frac{2}{3}\lambda\mathbf{a}+\left(1-\lambda\right)\mathbf{b}+\frac{1}{3}\lambda\mathbf{c}$ $=\frac{2}{3}\mu\mathbf{a}+\frac{1}{3}\mu\mathbf{b}+\left(1-\lambda\right)\mathbf{c}$ As a result, they failed to obtain the required expression.

(ii) If *OACB* is a parallelogram, $\mathbf{c}=\mathbf{a}+\mathbf{b}$ $\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)\mathbf{c}=\mathbf{0}$ $\left(\frac{2}{3}\lambda-\frac{2}{3}\mu\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu\right)\mathbf{b}+\left(\frac{1}{3}\lambda+\mu-1\right)(\mathbf{a}+\mathbf{b})=\mathbf{0}$ $\left(\frac{2}{3}\lambda-\frac{2}{3}\mu+\frac{1}{3}\lambda+\mu-1\right)\mathbf{a}+\left(1-\lambda-\frac{1}{3}\mu+\frac{1}{3}\lambda+\mu-1\right)\mathbf{b}=\mathbf{0}$ $\left(\lambda+\frac{1}{3}\mu-1\right)\mathbf{a}+\left(\frac{2}{3}\mu-\frac{2}{3}\lambda\right)\mathbf{b}=\mathbf{0}$ Since **a** and **b** are not parallel to each other and non-zero vectors, $\lambda+\frac{1}{3}\mu-1=0 \quad \cdots(1)$ $\frac{2}{3}\mu-\frac{2}{3}\lambda=0 \quad \cdots(2)$ Solving, $\lambda=\mu=\frac{3}{4}$ Thus position vector of *F* is given by $=\mathbf{b}+\frac{3}{4}\left(\frac{1}{3}(\mathbf{a}+\mathbf{b})+\frac{2}{3}\mathbf{a}-\mathbf{b}\right)$ $=\frac{3}{4}\mathbf{a}+\frac{1}{2}\mathbf{b}$	This part was usually either not attempted or very badly done. Most students could not recognise that $\mathbf{c}=\mathbf{a}+\mathbf{b}$. Common mistakes: × Many did not read the question and stopped at the step $\left(\lambda+\frac{1}{3}\mu-1\right)\mathbf{a}+\left(\frac{2}{3}\mu-\frac{2}{3}\lambda\right)\mathbf{b}=\mathbf{0}$ × Many used $\mathbf{c}=\mathbf{a}-\mathbf{b}$ instead.

Question 5 (Maxima & Minima Problems)

Let V_h and v_h be the volumes of the original cone and cone that was removed. Since both cones are similar, $\frac{v_h}{V_h} = \left(\frac{ar}{r}\right)^3 = a^3$. Also, it is given that $h = \sqrt{4^2 - r^2}$. Hence $V = \pi(ar)^2 r + (V_h - v_h)$ $= \pi a^2 r^3 + (V_h - a^3 V_h)$ $= \pi a^2 r^3 + (1 - a^3)\left(\frac{1}{3}\pi r^2 h\right)$ $= \pi a^2 r^3 + \frac{(1 - a^3)}{3}\pi r^2 \sqrt{16 - r^2}$	This part was very well done. However, most students did not leave their answers in simplied form.

(a)	Given that $a = 0.25$, $V = 0.0625\pi r^3 + 0.328125\pi r^2\sqrt{16-r^2}$ $\frac{\mathrm{d}V}{\mathrm{d}r} = 0.1875\pi r^2 + 0.328125\pi\left[2r\sqrt{16-r^2} + \frac{r^2(-2r)}{2\sqrt{16-r^2}}\right]$ $= 0.1875\pi r^2 + 0.328125\pi\left(\frac{2r(16-r^2)-r^3}{\sqrt{16-r^2}}\right)$ $= 0.1875\pi r^2 + 0.328125\pi r\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right)$ For stationary V, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$ $0.1875\pi r^2 + 0.328125\pi r\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right) = 0$ Since $0 < r < 4$, $r + \frac{7}{4}\left(\frac{32-3r^2}{\sqrt{16-r^2}}\right) = 0$ $\frac{7}{4}\left(\frac{3r^2-32}{\sqrt{16-r^2}}\right) = r$ $\frac{49}{16}\left(\frac{9r^4-192r^2+1024}{16-r^2}\right) = r^2$ $49(9r^4-192r^2+1024) = 256r^2-16r^4$ $457r^4 - 9664r^2 + 50176 = 0$	Most students could apply the Product rule but experienced great difficulty in simplifying the expression to show the required equation. Only a very small handful of students managed to show the expression correctly.
(b)	Using G.C, since $r > 0$, $r = 3.462322463$ or $r = 3.02637266$ When $r = 3.462322463$, $\frac{\mathrm{d}V}{\mathrm{d}r} = 0$. When $r = 3.02637266$, $\frac{\mathrm{d}V}{\mathrm{d}r} = 10.790112 \neq 0$ $h = \sqrt{16-3.462322463^2} = 2.00$ (to 3 s.f.)	Many did not attempt this part or tried to and obtained the correct value of r but did not check the corresponding value of the derivative.

Question 6 (Complex Numbers – Geometric Forms)

(i) $\dfrac{e^{i\theta}}{1-e^{i\theta}}=\dfrac{e^{i\theta}}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)}$ $=\dfrac{e^{i\frac{\theta}{2}}}{\left(-2\sin\frac{\theta}{2}\right)i}$ $=\dfrac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\left(-2\sin\frac{\theta}{2}\right)i}$ $=-\dfrac{1}{2i}\cot\dfrac{\theta}{2}-\dfrac{1}{2}$ $=\left(\dfrac{1}{2}\cot\dfrac{\theta}{2}\right)i-\dfrac{1}{2}$ $=\dfrac{1}{2}\left(i\cot\dfrac{\theta}{2}-1\right)$	Many students could not answer this part correctly. Most attempted to introduce $\frac{\theta}{2}$ by using half angle formula on $\cos\theta$ and $\sin\theta$ and could not proceed with meaningful simplification of the terms. **Learning points:** - Remember to try to factorise $e^{i\frac{\theta}{2}}$ first before other approaches when dealing with half angles in complex numbers setting.
(ii) $i=e^{i\frac{\pi}{2}}$ or $e^{i\left(-\frac{3\pi}{2}\right)}$ or $e^{i\left(\frac{5\pi}{2}\right)}$	Most students could give at least two different representations. $e^{i\left(-\frac{3\pi}{2}\right)}$ is the representation that are missed the most by the candidates.

(iii) $\left(\dfrac{w}{w+1}\right)^3 = \mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ or $\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{2}\right)}$ or $\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{2}\right)}$ by **(ii)** $\Rightarrow \dfrac{w}{w+1} = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ or $\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}$ or $\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}\right)}$ Now let $\dfrac{w}{w+1} = \mathrm{e}^{\mathrm{i}\theta}$, where $\theta = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$. Then we have $w = \mathrm{e}^{\mathrm{i}\theta} w + \mathrm{e}^{\mathrm{i}\theta}$ $\Rightarrow w\left(1-\mathrm{e}^{\mathrm{i}\theta}\right) = \mathrm{e}^{\mathrm{i}\theta}$ $\Rightarrow w = \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{2}\left(\mathrm{i}\cot\dfrac{\theta}{2} - 1\right)$ by **(i)** Therefore $w = \dfrac{1}{2}\left(\mathrm{i}\cot\left(-\dfrac{\pi}{4}\right) - 1\right),\ \dfrac{1}{2}\left(\mathrm{i}\cot\left(\dfrac{\pi}{12}\right) - 1\right)$ or $\dfrac{1}{2}\left(\mathrm{i}\cot\left(\dfrac{5\pi}{12}\right) - 1\right)$.	Most candidates did not write any useful working for this part. **Learning points:** - Exponential forms of complex number follows the exponential rules e.g., $z^3 = \mathrm{e}^{\mathrm{i}\alpha}$ implies $z = \mathrm{e}^{\mathrm{i}\frac{\alpha}{3}}$. Only a small number of candidates could make the connection with the earlier part and manipulated to obtain $w = \dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}} = \dfrac{1}{2}\left(\mathrm{i}\cot\dfrac{\theta}{2} - 1\right)$ Of which some did not divide the angle obtained $\theta = -\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ by 2 in the final solution.

Question 7 (Power Series)

(i)	Common errors:
Method 1 $y=\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow \frac{dy}{dx}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow \frac{d^2y}{dx^2}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\left(-2\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\right)$ $-2\cot\left(2x+\frac{\pi}{4}\right)\left(-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)\right)$ $=4y^3+4y\cot^2\left(2x+\frac{\pi}{4}\right)$ $=4y^3+4y\left(y^2-1\right)$ $=8y^3-4y$ (shown) **Method 2** $y=\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow \frac{dy}{dx}=-2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow \frac{dy}{dx}=-2y\cot\left(2x+\frac{\pi}{4}\right)$ $\Rightarrow \frac{d^2y}{dx^2}=-2\frac{dy}{dx}\cot\left(2x+\frac{\pi}{4}\right)-2y\left(-2\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)\right)$ $\Rightarrow \frac{d^2y}{dx^2}=4y\cot^2\left(2x+\frac{\pi}{4}\right)+4y^3$ $\Rightarrow \frac{d^2y}{dx^2}=4y\left(\operatorname{cosec}^2\left(2x+\frac{\pi}{4}\right)-1\right)+4y^3$ $\Rightarrow \frac{d^2y}{dx^2}=4y\left(y^2-1\right)+4y^3$ $\Rightarrow \frac{d^2y}{dx^2}=4y^3-4y+4y^3=8y^3-4y$ (shown)	- Many students did not apply the relevant formula in MF26 to differentiate $\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$. They resorted to converting the expression into ‘1/sin…’ and used the compound angle formula to unnecessarily expand the function and this resulted in tedious work for finding the second derivative in the later part and thus little success to prove the result. - Similarly, some students did not know how to differentiate $\cot\left(2x+\frac{\pi}{4}\right)$. They changed it to $\frac{1}{\tan\left(2x+\frac{\pi}{4}\right)}$ and differentiated the function using the Chain rule. A few **incorrectly** thought that $\frac{1}{\tan x}=\tan^{-1}x$. - did not apply Chain Rule fully, miss out the constant and negative sign - did not apply $1+\cot^2 x=\operatorname{cosec}^2 x$ **Learning points:** Please - familiarise yourselves with what formula are given in MF26. - memorise formulas for differentiating and integrating trigonometric functions and trigonometric identities that are not in MF26. - apply implicit differentiation wherever possible for this type of questions in Power Series

Method 3

$$y = \operatorname{cosec}\left(2x+\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2\operatorname{cosec}\left(2x+\frac{\pi}{4}\right)\cot\left(2x+\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2y\sqrt{y^2-1}$$

$$\Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y^2\left(y^2-1\right) = 4y^4-4y^2$$

$$\Rightarrow 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(16y^3-8y\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 8y^3-4y \quad \text{(shown)}$$

Method 4

$$\frac{1}{y} = \sin\left(2x+\frac{\pi}{4}\right) \qquad (1)$$

$$-\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos\left(2x+\frac{\pi}{4}\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2y^2\cos\left(2x+\frac{\pi}{4}\right) \qquad (2)$$

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -4y\frac{\mathrm{d}y}{\mathrm{d}x}\cos\left(2x+\frac{\pi}{4}\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right)\\
&= 8y^3\cos^2\left(2x+\frac{\pi}{4}\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right) \text{ (from (2))}\\
&= 8y^3\left(1-\sin^2\left(2x+\frac{\pi}{4}\right)\right)+4y^2\sin\left(2x+\frac{\pi}{4}\right)\\
&= 8y^3\left(1-\frac{1}{y^2}\right)+4y^2\cdot\frac{1}{y} \text{ (from (1))}\\
&= 8y^3-8y+4y\\
&= 8y^3-4y \text{ (shown)}
\end{aligned}$$

(ii) $\dfrac{d^2y}{dx^2}=8y^3-4y \Rightarrow \dfrac{d^3y}{dx^3}=24y^2\dfrac{dy}{dx}-4\dfrac{dy}{dx}$ When $x=0$, $y=\operatorname{cosec}\left(\dfrac{\pi}{4}\right)=\sqrt{2}$ $\dfrac{dy}{dx}=-2\operatorname{cosec}\left(\dfrac{\pi}{4}\right)\cot\left(\dfrac{\pi}{4}\right)=-2\sqrt{2}$ $\dfrac{d^2y}{dx^2}=8\left(\sqrt{2}\right)^3-4\sqrt{2}=12\sqrt{2}$ $\dfrac{d^3y}{dx^3}=24(2)\left(-2\sqrt{2}\right)-4\left(-2\sqrt{2}\right)=-88\sqrt{2}$ Hence $\operatorname{cosec}\left(2x+\dfrac{\pi}{4}\right)\approx\sqrt{2}+\left(-2\sqrt{2}\right)\dfrac{x}{1!}+\left(12\sqrt{2}\right)\dfrac{x^2}{2!}+\left(-88\sqrt{2}\right)\dfrac{x^3}{3!}$ $=\sqrt{2}-2\sqrt{2}x+6\sqrt{2}x^2-\dfrac{44}{3}\sqrt{2}x^3$	**Common errors:** - obtained $\dfrac{d^3y}{dx^3}=24y^2-4$ or differentiated $8y^3$ w.r.t x wrongly. - did not simplify coefficient to $k\sqrt{2}$ - a few wrote the Maclaurin series as a sum of powers of $\left(x+\dfrac{\pi}{4}\right)$ **Learning points:** - Maclaurin's Theorem can be referenced from MF26 formula. - Show working to find the values of y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$, $\dfrac{d^3y}{dx^3}$ when $x=0$ **first**, **before** substituting the values into the Maclaurin series coefficients. Doing these simultaneously usually results in arithmetic errors.
(iii) Differentiating the expansion above, we get $-2\operatorname{cosec}\left(2x+\dfrac{\pi}{4}\right)\cot\left(2x+\dfrac{\pi}{4}\right)\approx-2\sqrt{2}+12\sqrt{2}x-44\sqrt{2}x^2.$ Therefore $\operatorname{cosec}\left(2x+\dfrac{\pi}{4}\right)\cot\left(2x+\dfrac{\pi}{4}\right)\approx\sqrt{2}-6\sqrt{2}x+22\sqrt{2}x^2.$ Let $2x+\dfrac{\pi}{4}=\dfrac{13\pi}{50}$. Then $x=\dfrac{\pi}{200}$. Hence $\operatorname{cosec}\left(\dfrac{13\pi}{50}\right)\cot\left(\dfrac{13\pi}{50}\right)\approx\sqrt{2}\left(1-6\left(\dfrac{\pi}{200}\right)+22\left(\dfrac{\pi}{200}\right)^2\right)$ $\approx\sqrt{2}\left(1-\dfrac{3}{100}\pi+\dfrac{11}{20000}\pi^2\right)$	Poorly attempted. Many students left this part blank or just substituted $x=\pi/200$ or $x=\pi/100$ into their answer in part **(ii)** and consequently were stuck with $\cot(x+\pi/4)$. **Learning points:** - 'Hence' implies there is a need to apply the results from the previous parts. Observe that $\operatorname{cosec}\left(2x+\dfrac{\pi}{4}\right)\cot\left(2x+\dfrac{\pi}{4}\right)\approx\left(-\dfrac{1}{2}\right)\dfrac{dy}{dx}$. - Differentiate the answer in part **(ii)** w.r.t x first before substituting $x=\pi/200$. Students who attempted to do these together in one step usually made arithmetic errors.

Question 8 (Sequences and Series)

(a) For $n \geq 2$, $u_n = An^2 + Bn + 2^{n+1} - \left(A(n-1)^2 + B(n-1) + 2^n\right)$ $= An^2 + Bn + 2^{n+1} - \left(An^2 - 2An + A + Bn - B + 2^n\right)$ $= 2(2^n) + 2An - A + B - 2^n = 2^n + (2n-1)A + B$ $u_3 = 2^3 + (2(3)-1)A + B = 21 \Rightarrow 5A + B = 13 \cdots (1)$ $u_5 = 2^5 + (2(5)-1)A + B = 53 \Rightarrow 9A + B = 21 \cdots (2)$ Solving (1) and (2). $A = 2$ and $B = 3$. Hence $u_n = \begin{cases} 2(1)^2 + 3(1) + 2^{1+1}, & n = 1 \\ 2^n + 4n + 1, & n \geq 2 \end{cases} = \begin{cases} 9, & n = 1 \\ 2^n + 4n + 1, & n \geq 2 \end{cases}$	**Common errors:** - take $S_3 = 21$ and $S_5 = 53$ - did not simplify $2^{n+1} - 2^n = 2\left(2^n\right) - 2^n = 2^n$ - tried to apply AP/GP formula - wrote $u_n = S_{n+1} - S_n$ - did not check for u_1 and S_1. Only a handful of students got the correct answer for u_n. **Learning points:** To find u_n from S_n, use $u_n = S_n - S_{n-1}$, $n \geq 2$ (*). Please check if the formula (*) is consistent with (i.e. is equal to) S_n for $n = 1$. If it is not, you need to write u_n as a piecewise function of *n*, and write $u_1 = S_1$ separately. If it is, the answer can be written as a single function $u_n = S_n - S_{n-1}$, $n \geq 1$.
(b)(i) $\sum_{r=1}^{n} \ln\left(\frac{r(r+2)}{(r+1)^2}\right) = \sum_{r=1}^{n}\left(\ln\left(\frac{r+2}{r+1}\right) - \ln\left(\frac{r+1}{r}\right)\right)$ $= \left(\cancel{\ln\left(\frac{3}{2}\right)} - \ln 2\right)$ $+ \left(\cancel{\ln\left(\frac{4}{3}\right)} - \cancel{\ln\left(\frac{3}{2}\right)}\right)$ $\vdots$ $+ \left(\cancel{\ln\left(\frac{n+1}{n}\right)} - \cancel{\ln\left(\frac{n}{n-1}\right)}\right)$ $+ \left(\ln\left(\frac{n+2}{n+1}\right) - \cancel{\ln\left(\frac{n+1}{n}\right)}\right)$ $= -\ln 2 + \ln\left(\frac{n+2}{n+1}\right)$ $= \ln\left(\frac{n+2}{n+1}\right) - \ln 2$ (shown)	**Common errors:** - could not apply laws of logarithm to simplify expression into the form $\ln r - 2\ln(r+1) + \ln(r+2)$ or $\ln\left(\frac{r+2}{r+1}\right) - \ln\left(\frac{r+1}{r}\right)$ or $\ln\left(\frac{r}{r+1}\right) - \ln\left(\frac{r+1}{r+2}\right)$ that would allow for the method of differences to be applied. - rewrote as $\ln(r^2 + 2r) - \ln(r+1)^2$ or attempted to use partial fractions, with little success. - did not put brackets properly, please note that $\sum_{r=1}^{n}\left(\ln\left(\frac{r+2}{r+1}\right) - \ln\left(\frac{r+1}{r}\right)\right) \neq \sum_{r=1}^{n} \ln\left(\frac{r+2}{r+1}\right) - \ln\left(\frac{r+1}{r}\right)$ - not cancelling the terms properly (or not cancelling any single term). Not showing at least one full cancellation above and at least one full cancellation below.

Alternative Solution $\sum_{r=1}^{n} \ln\left(\frac{r(r+2)}{(r+1)^2}\right) = \sum_{r=1}^{n}\left(\ln r - 2\ln(r+1) + \ln(r+2)\right)$ $= \left(\ln 1 - 2\ln(2) + \ln(3)\right)$ $+\left(\ln 2 - 2\ln(3) + \ln(4)\right)$ $+\left(\ln 3 - 2\ln(4) + \ln(5)\right)$ $\vdots$ $+(\ln(n-2) - 2\ln(n-1) + \ln(n))$ $+(\ln(n-1) - 2\ln(n) + \ln(n+1))$ $+(\ln(n) - 2\ln(n+1) + \ln(n+2))$ $= \ln 1 - \ln 2 - \ln(n+1) + \ln(n+2)$ $= \ln(n+2) - \ln(n+1) - \ln 2$ $= \ln\left(\frac{n+2}{n+1}\right) - \ln 2 \quad \text{(shown)}$	**Learning points:** - For the alternative solution, you need to write 1st 3 rows and last 3 rows so that one full cancellation above and one full cancellation below are shown. - For "Show" type of MOD question, you need to write out the terms after cancellation before the final shown answer.
(b)(ii) $\sum_{r=0}^{n} \ln\left(\frac{r^2+4r+3}{(r+2)^2}\right) = \sum_{r=0}^{n} \ln\left(\frac{(r+1)(r+3)}{(r+2)^2}\right)$ $= \sum_{q-1=0}^{q-1=n} \ln\left(\frac{q(q+2)}{(q+1)^2}\right)$ $= \sum_{q=1}^{q=n+1} \ln\left(\frac{q(q+2)}{(q+1)^2}\right)$ $= \ln\left(\frac{n+1+2}{n+1+1}\right) - \ln 2$ $= \ln\left(\frac{n+3}{n+2}\right) - \ln 2$	Common errors: - Many could not perform the appropriate substitution to **manipulate the current series to "appear" like the previous series**. - Many started with the series in part **(b)(i)** and could not proceed or could not get the correct answer. **Learning points:** - As the general term is different from the series in part **(b)(i)**, the correct technique is to use the substitution method. - Starting from the current series, replace r with $q-1$, change the general term to that in part **(b)(i)** and then replace the lower and upper limit with $q-1=0$ and $q-1=n$.

Question 9 (Curve Sketching)

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(a) $y=\left(x^2+cx\right)\mathrm{e}^{-x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(2x+c\right)\mathrm{e}^{-x}-\left(x^2+cx\right)\mathrm{e}^{-x}$

$=\left(-x^2-cx+2x+c\right)\mathrm{e}^{-x}$

$=-\left(x^2+\left(c-2\right)x-c\right)\mathrm{e}^{-x}$

At stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

$-\left(x^2+\left(c-2\right)x-c\right)\mathrm{e}^{-x}=0$

$x^2+\left(c-2\right)x-c=0$ ($\because \mathrm{e}^{-x}>0$ for all $x\in\mathbb{R}$).

Discriminant, $D=\left(c-2\right)^2-4\left(1\right)\left(-c\right)$

$=c^2-4c+4+4c$

$=c^2+4\geq 4>0$ for all $c\in\mathbb{R}$.

$\therefore$ The equation $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ has two real and distinct roots.

Therefore, the curve with equation $y=\left(x^2+cx\right)\mathrm{e}^{-x}$ has two stationary points for all real values of c.

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Please note that in questions that require a proof, it is important to demonstrate your reasoning and justify steps where necessary.

Common mistakes:

- not stating "$\mathrm{e}^{-x}>0$ (or $\mathrm{e}^{-x}\neq 0$) for all $x\in\mathbb{R}$" in the proof at the step of dividing throughout by e^{-x}.
- Solving for the roots and stopping short of explaining why both roots are **real**.
- computing the discriminant of $\left(x^2+\left(c-2\right)x-c\right)\mathrm{e}^{-x}$. This is incorrect because the expression above is **NOT** a quadratic expression in x, so it is meaningless to talk about the discriminant of $\left(x^2+\left(c-2\right)x-c\right)\mathrm{e}^{-x}$.
- writing the discriminant as "b^2-4ac" when c is already used in the question.
- writing $c^2>0$ for all real values of c. The correct inequality is $c^2\geq 0$ for all real values of c.
- assuming that "discriminant >0" straightaway. This is incorrect because you are assuming precisely what you need to prove.
- trying to show that "discriminant ≥ 0" instead of "discriminant >0". Recall your O-Level content on discriminants.
- It is appalling that a notable number of students expanded $\left(c-2\right)^2$ as c^2-2c+4.

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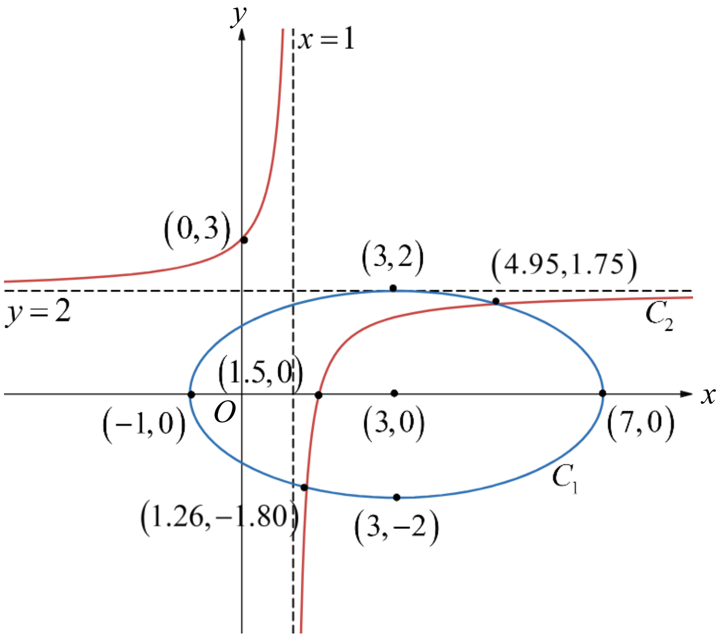
(b) For C_1:

$$x^2+4y^2-6x-7=0$$
$$x^2-6x+9-9+4y^2=7$$
$$(x-3)^2+4y^2=16$$
$$\frac{(x-3)^2}{16}+\frac{y^2}{4}=1$$

For C_2:

$$y=\frac{2x-3}{x-1}$$
$$=2-\frac{1}{x-1}$$



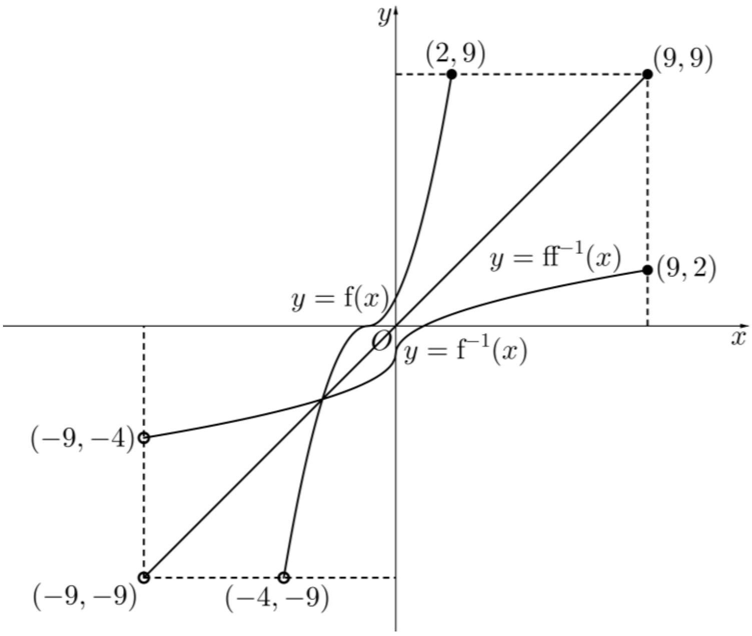
C_2 was generally well-sketched, but many mistakes were made in the sketch of C_1.

Common mistakes:

- ✗ completing the square incorrectly. Please note that there is no y term in the equation of the ellipse.
- ✗ missing labels such as coordinates of centre and vertices of the ellipse, points of intersection between the curves, equations of the curves (or at least C_1 and C_2), and the origin.
- ✗ The positions of C_1 and C_2 relative to each other are not accurate. For example, the top of the ellipse not touching the asymptote $y=2$ at one point, or the point $(1.5, 0)$ is too near or to the right of the point $(3, 0)$.
- ✗ sketching a hyperbola instead of an ellipse.

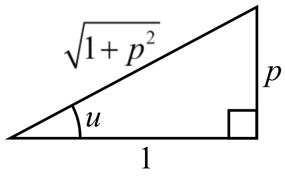
Question 10 (Functions)

(i) For $-4<x<-1$, $\mathrm{f}(x)=-(x+1)^2$ Let $y=-(x+1)^2$. $(x+1)^2=-y$ $x+1=\pm\sqrt{-y}$ $x=-1\pm\sqrt{-y}$ Since $-4<x<-1$, $x=-1-\sqrt{-y}$ For $-1\le x\le 2$, $\mathrm{f}(x)=(x+1)^2$ Let $y=(x+1)^2$. $(x+1)^2=y$ $x+1=\pm\sqrt{y}$ $x=-1\pm\sqrt{y}$ Since $-1\le x\le 2$, $x=-1+\sqrt{y}$ Thus, $\mathrm{f}^{-1}(x)=\begin{cases}-1-\sqrt{-x}, & \text{for } x\in\mathbb{R}, -9<x<0,\\ -1+\sqrt{x}, & \text{for } x\in\mathbb{R}, 0\le x\le 9.\end{cases}$	Many assumed that $\sqrt{-y}$ should be rejected, not realising that ***y* can be negative**. Hence many omitted the rule $-1-\sqrt{-x}$. Unfortunately, there are still a lot of students who did not consider $\pm\sqrt{\ }$ and did not use the domain of the function to determine the correct square root. **The skill of choosing the correct square root has been tested so many times in many topics** and thus it is expected that candidates should have a good grasp of this skill by now. Another group of students recalled the relevance of $\pm$ but were unsure where it should be placed and hence modulus signs were used everywhere without proper consideration. Examples include $-1\pm\sqrt{\lvert x\rvert}$, $\sqrt{\lvert x\rvert}\pm 1$, $1+\lvert\sqrt{x}\rvert$ etc. Domains of the inverse were NOT properly thought through, with many thinking that the value -1 was relevant since the rule was $-1\pm\sqrt{\ldots}$.

(ii) 	For such questions, details are necessary, including **domains, nature of turning points, inclusion and exclusion of points and symmetrical properties**. Many did not sketch the graphs to scale and hence made the diagram awkward-looking. The composite function $y = \mathrm{ff}^{-1}(x)$ was poorly sketched with many not paying attention to the domain. This type of questions has appeared many times in practice questions and revision. Thus, it is expected that candidates should not be scoring 2 marks or less for such questions.
(iii) For $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}(x) = x$. From the graph, intersection occurs for $-4 < x \le -1$. Thus $-(x+1)^2 = x$ $x^2 + 2x + 1 = -x$ $x^2 + 3x + 1 = 0$ $x = \frac{-3 \pm \sqrt{3^2 - 4}}{2}$ $x = \frac{-3 \pm \sqrt{5}}{2}$ $x = \frac{-3 - \sqrt{5}}{2}$ $(\because x < -1)$ Thus, for $\mathrm{f}(x) \le \mathrm{f}^{-1}(x)$, $-4 < x \le \frac{-3-\sqrt{5}}{2}$.	The use of calculator is not allowed since there is an 'exact' requirement. Unfortunately, candidates still left their answers in non-exact form. The skill of equating $\mathrm{f}(x)$ to x to solve the inequality is also available in practice questions. Candidates should use the graph to decide the correct rule of $\mathrm{f}(x)$ to equate to x. If so, at least 2 marks would have been easily obtained.

Question 11 (Differential Equations)

(i)	Since $p=\frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=kp$. $\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=k\frac{\mathrm{d}p}{\mathrm{d}x}$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=ak\sqrt{1+\left(\frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$ $\Rightarrow k\frac{\mathrm{d}p}{\mathrm{d}x}=ak\sqrt{1+p^2}$ $\Rightarrow \frac{\mathrm{d}p}{\mathrm{d}x}=a\sqrt{1+p^2}$ (shown)	Some students were confused about what to do here. A number of students skipped critical steps.
(ii)	$p=\tan u \Rightarrow \frac{\mathrm{d}p}{\mathrm{d}x}=\frac{\mathrm{d}p}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\left(\sec^2 u\right)\frac{\mathrm{d}u}{\mathrm{d}x}$. Thus, $\frac{\mathrm{d}p}{\mathrm{d}x}=a\sqrt{1+p^2}$ $\Rightarrow \left(\sec^2 u\right)\frac{\mathrm{d}u}{\mathrm{d}x}=a\sqrt{1+\tan^2 u}$ $\Rightarrow \left(\sec^2 u\right)\frac{\mathrm{d}u}{\mathrm{d}x}=a\sec u$ **<u>Remember</u> that $\sec^2 u=1+\tan^2 u$.** **Since the DE is of the form $\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{g}(u)$, simplify as $\frac{1}{\mathrm{g}(u)}\frac{\mathrm{d}u}{\mathrm{d}x}=1\Rightarrow\int\frac{1}{\mathrm{g}(u)}\,\mathrm{d}u=\int 1\,\mathrm{d}x$.** $\Rightarrow \sec u\frac{\mathrm{d}u}{\mathrm{d}x}=a$ $\Rightarrow \int\sec u\,\mathrm{d}u=\int a\,\mathrm{d}x$ **Integral formula for the secant function is in MF26.** $\Rightarrow \ln\left(\sec u+\tan u\right)=ax+c \quad \because \lvert u\rvert<\frac{\pi}{2}$ $\Rightarrow \ln\left(\sqrt{1+p^2}+p\right)=ax+c$ (Note: $p=\tan u\Rightarrow\sec^2 u=1+p^2$) $\Rightarrow \sqrt{1+p^2}+p=\mathrm{e}^{ax+c}$ $\Rightarrow \sqrt{1+p^2}+p=A\mathrm{e}^{ax}$, $A=\mathrm{e}^c$	Many students were not familiar with the procedure to solve a differential equation via a given substitution. For some, their reductions even led to an equation that was void of any derivative. Please revise. Some students could not apply the technique needed to solve this DE (method of separation). Workings such as $p=\int\sqrt{1+p^2}\,\mathrm{d}x$ and $u=\int\sec u\,\mathrm{d}x$ were fairly common. Some students approached the question as one on integration by substitution, which is acceptable. However, many of these students simplified $\int\frac{1}{\sqrt{1+p^2}}\,\mathrm{d}p$ incorrectly to $\int\frac{1}{\sqrt{1+\tan^2 u}}\times\frac{1}{\sec^2 u}\,\mathrm{d}u$. From MF26, if $\lvert u\rvert<\frac{\pi}{2}$, $\int\sec u\,\mathrm{d}u$ is equals to $\ln\left(\sec u+\tan u\right)+c$ aka no modulus is required. Working with modulus should be as follows: $\int\sec u\,\mathrm{d}u=\int a\,\mathrm{d}x$ $\Rightarrow \ln\lvert\sec u+\tan u\rvert=ax+c$ $\Rightarrow \lvert\sec u+\tan u\rvert=\mathrm{e}^{ax+c}$ $\Rightarrow \sec u+\tan u=\pm\mathrm{e}^{ax+c}=\pm\mathrm{e}^c\mathrm{e}^{ax}$ $\Rightarrow \sec u+\tan u=A\mathrm{e}^{ax}$, $A=\pm\mathrm{e}^c$ Some students had no modulus but still had a $\pm$ later. This is incorrect.

(ii)	Since the turning point lies on the y-axis, when $x=0$, $p=\frac{1}{k}\frac{\mathrm{d}y}{\mathrm{d}x}=0$. Hence, $\sqrt{1+0^2}+0=A\mathrm{e}^0 \Rightarrow A=1$. So, $\sqrt{1+p^2}+p=\mathrm{e}^{ax} \Rightarrow \sqrt{1+p^2}=\mathrm{e}^{ax}-p$ $\Rightarrow 1+p^2=\left(\mathrm{e}^{ax}-p\right)^2=\mathrm{e}^{2ax}-2p\mathrm{e}^{ax}+p^2$ $\Rightarrow 2p\mathrm{e}^{ax}=\mathrm{e}^{2ax}-1$ $\Rightarrow p=\frac{\mathrm{e}^{2ax}-1}{2\mathrm{e}^{ax}}=\frac{\mathrm{e}^{ax}-\mathrm{e}^{-ax}}{2}$ (shown)	Some students could not convert $\sec u$ to $\sqrt{1+p^2}$. Please use trigonometric identities or the right-angled triangle method:  $\boxed{\sec u=\frac{\text{hyp}}{\text{adj}}=\sqrt{1+p^2}}$ To simplify an equation comprising a surd, isolate the surd term on its own on one side of the equation first, **before** squaring both sides.
(iii)	$\frac{1}{0.701}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{-0.0329x}-\mathrm{e}^{0.0329x}}{2}$ $y=0.701\int\frac{\mathrm{e}^{-0.0329x}-\mathrm{e}^{0.0329x}}{2}\,\mathrm{d}x$ $=\frac{0.701}{2}\left(\frac{\mathrm{e}^{-0.0329x}}{-0.0329}-\frac{\mathrm{e}^{0.0329x}}{0.0329}\right)+d$ $=d-10.653\left(\mathrm{e}^{-0.0329x}+\mathrm{e}^{0.0329x}\right)$ Since $y=192$ when $x=0$, $192=d-10.653\left(\mathrm{e}^{-0.0329(0)}+\mathrm{e}^{0.0329(0)}\right)$ $\Rightarrow d=213$ (to 3 s.f.) **OR** Since $y=0$ when $x=91$, $0=d-10.653\left(\mathrm{e}^{-0.0329(91)}+\mathrm{e}^{0.0329(91)}\right)$ $\Rightarrow d=213$ (to 3 s.f.) Thus, $y=213-10.7\left(\mathrm{e}^{-0.0329x}+\mathrm{e}^{0.0329x}\right)$ (to 3 s.f.)	Common errors: ➢ Considering $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ (which is pointless for this question part; please integrate part **(ii)** result) ➢ $\mathrm{e}^{0.0329x}=\mathrm{e}^{0.0329}\times\mathrm{e}^{x}$ (demonstrating very poor grasp of the laws of indices, a *Secondary Math* concept) ➢ $\mathrm{e}^{-0.0329x}-\mathrm{e}^{0.0329x}=2\mathrm{e}^{(\text{something})x}$ (thinking that the exponents can somehow be combined even though the powers are unequal) ➢ $\int\mathrm{e}^{0.0329x}\,\mathrm{d}x=0.0329\mathrm{e}^{0.0329x}+c$ (differentiating instead of integrating) ➢ forgetting to include the arbitrary constant (which is a lethal mistake for a differential equation problem) ➢ leaving the final solution with an arbitrary constant (The curve has a fixed position in the x-y plane; thus clearly this question requires a *particular* solution.) ➢ "Since $y=192$ when $x=0$, d (or c) $=192$." (being very flippant and not paying due diligence in evaluating the arbitrary constant, thus failing to realise that $\mathrm{e}^0=1$ and not 0) ➢ not simplifying the final answer

Question 12 (Vectors II)

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(i) $\pi_1 : \mathbf{r}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=25$ and $\pi_2 : \mathbf{r}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-15$

Substituting $(x,0,z)$.

$\begin{pmatrix}x\\0\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=5 \Rightarrow 2x+z=25 \cdots\cdots (1)$

$\begin{pmatrix}x\\0\\z\end{pmatrix}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-3 \Rightarrow x-2z=-15 \cdots\cdots (2)$

Solving (1) and (2), we get $x=7$ and $z=11$.

Thus the position vector of such a point on l_1 is $\begin{pmatrix}7\\0\\11\end{pmatrix}$.

Direction vector of $l_1 = \begin{pmatrix}2\\-3\\1\end{pmatrix}\times\begin{pmatrix}1\\k\\-2\end{pmatrix}=\begin{pmatrix}6-k\\1-(-4)\\2k+3\end{pmatrix}=\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}$

Hence a vector equation of l_1 is

$$\mathbf{r}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix}, \quad \lambda\in\mathbb{R}$$

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<u>Skills/Concepts Tested</u>

- Find the equation of the line of intersection between two planes **without** the use of GC by
  - solving two equations with three unknowns through expressing two variables in terms of the third, or
  - finding the point on the line and the direction of the line (i.e. cross product of the normal vectors of the two planes).
- Write an equation of a line

<u>Common Mistakes</u>

- Interpreted "independent of *k*" as letting $k=0$
- Did erroneous workings to find *k* before proceeding to use GC to find the equation of the line of intersection
- Did not know what to get out from "solving the 2 equations with 3 unknowns" hence the workings appeared aimless
- Careless algebraic manipulation
- Missing "$\mathbf{r}=$" and/or "$\lambda\in\mathbb{R}$" when writing an equation of a line

<u>Overall Comments</u>

- Poorly attempted

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(i) Alternatively, $\pi_1\ :\ \mathbf{r}\cdot\begin{pmatrix}2\\-3\\1\end{pmatrix}=5\Rightarrow 2x-3y+z=25\ \cdots\cdots\ (1)$ and $\pi_2\ :\ \mathbf{r}\cdot\begin{pmatrix}1\\k\\-2\end{pmatrix}=-3\Rightarrow x+ky-2z=-15\ \cdots\cdots\ (2)$ Take $y=t$. Then $(1): 2x-3t+z=25\Rightarrow z=25+3t-2x \quad (3)$ $(2): x+kt-2z=-15 \quad (4)$ Substituting equation (3) into (4), $x+kt-2(25+3t-2x)=-15$ $x+kt-50-6t+4x=-15$ $5x+(k-6)t=35$ $x=7+\left(\frac{6-k}{5}\right)t$ Substituting into equation (3), $z=25+3t-2\left[7+\left(\frac{6-k}{5}\right)t\right]$ $=25+3t-14-\left(\frac{12-2k}{5}\right)t$ $=11+\left(\frac{3+2k}{5}\right)t$ Thus, l_1 has equation $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}7\\0\\11\end{pmatrix}+t\begin{pmatrix}\left(\frac{6-k}{5}\right)t\\t\\\left(\frac{3+2k}{5}\right)t\end{pmatrix},\ t\in\mathbb{R}$ or $\mathbf{r}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}6-k\\5\\2k+3\end{pmatrix},\ \lambda=\frac{t}{5}\in\mathbb{R}$	

(ii) $l_1 : \mathbf{r} = \begin{pmatrix} 7 \\ 0 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 6-k \\ 5 \\ 2k+3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ $l_2 : \mathbf{r} = \begin{pmatrix} -15 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \quad \beta \in \mathbb{R}$ Since both lines are skew, they are not parallel and not intersecting. Suppose both lines are parallel, we have $\begin{pmatrix} 6-k \\ 5 \\ 2k+3 \end{pmatrix} = t \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$. Then $t = 5$. For *x*-coordinate: $6-k = 3(5) \Rightarrow k = -9$ For *z*-coordinate: $2k+3 = -5 \Rightarrow k = -4$ Hence we conclude lines are not parallel regardless of *k*. Suppose both lines are intersecting, we have $\begin{pmatrix} 7 \\ 0 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 6-k \\ 5 \\ 2k+3 \end{pmatrix} = \begin{pmatrix} -15 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ Then $\beta = 5\lambda$ and $7 + \lambda(6-k) = -15 + 3(5\lambda) \Rightarrow \lambda(k+9) = 22$ and $11 + \lambda(2k+3) = -5\lambda \Rightarrow \lambda(k+4) = -\frac{11}{2}$ Hence $\frac{k+9}{k+4} = -4 \Rightarrow k+9 = -4k-16 \Rightarrow k = -5$ Therefore for both lines to be skew, *k* can be any real number except −5.	Skills/Concepts Tested - Skew lines are **NOT** parallel and **NOT** intersecting - For such questions, students are expected to assume the lines **ARE** parallel and intersecting to find the value(s) of *k* before concluding that *k* **CANNOT** take those values found Common Mistakes - Students did not attempt to find the value(s) of *k* when the lines are parallel. For the few who did, most did not carry out correct workings for this or did not interpret the "solutions" correctly - When assuming that the 2 lines intersect, students left *k* in terms of unknowns and did not know that they need to find the value(s) explicitly (which is expected in the question) - Students attempted to directly "solve" equations like $\begin{pmatrix} \\ \\ \end{pmatrix} \neq \begin{pmatrix} \\ \\ \end{pmatrix}$ Overall Comments - Most students did not attempt this part - For the few who did, it was very poorly attempted.

(iii)
$\overrightarrow{OA}=\begin{pmatrix}7\\0\\11\end{pmatrix}+\lambda\begin{pmatrix}2\\5\\11\end{pmatrix}$ for some $\lambda\in\mathbb{R}$
$\overrightarrow{OB}=\begin{pmatrix}-15\\0\\0\end{pmatrix}+\beta\begin{pmatrix}3\\1\\-1\end{pmatrix}$ for some $\beta\in\mathbb{R}$
Then we have $\overrightarrow{AB}\cdot\begin{pmatrix}2\\5\\11\end{pmatrix}=0$ and $\overrightarrow{AB}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix}=0$
i.e. $\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}.\begin{pmatrix}2\\5\\11\end{pmatrix}=0\Rightarrow-150\lambda=165\Rightarrow\lambda=-\frac{11}{10}$
$\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}.\begin{pmatrix}3\\1\\-1\end{pmatrix}=0\Rightarrow11\beta=55\Rightarrow\beta=5$
Therefore $\overrightarrow{AB}=\begin{pmatrix}-22+3\beta-2\lambda\\\beta-5\lambda\\-11-\beta-11\lambda\end{pmatrix}=\begin{pmatrix}-4.8\\10.5\\-3.9\end{pmatrix}$.
Hence $\left\|\overrightarrow{AB}\right\|=\sqrt{(-4.8)^2+(10.5)^2+(-3.9)^2}=\sqrt{\frac{297}{2}}$
	<u>Skills/Concepts Tested</u>
	- Find distance between two skew lines by
	  - identifying two points A (on one line) and B (on the other line) such that A and B are nearest to each other (i.e. $\overrightarrow{AB}$ is perpendicular to both lines). The distance between A and B is the distance between the skew lines, or
	  - identifying a point on each line, say C on one line and D on the other, and finding the length of projection of $\overrightarrow{CD}$ on the direction vector that is perpendicular to both lines with the use of an appropriate formula
	
	<u>Common Mistakes</u>
	- Use of wrong formula/vectors to compute the distance
	- Let $\overrightarrow{AB}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ and attempt to solve for the 3 unknowns with only 2 equations instead of expressing $\overrightarrow{AB}$ using the equations of l_1 and l_2 which yields 2 unknowns
	
	<u>Overall Comments</u>
	- Poorly attempted
	- A handful of students were not successful with this part because they did not get the correct answer in **(i)**.
	- It is commendable that there were students who could not get any answer in **(i)** made use of k = 4 and GC to find the equation of l_1 to make some progress in this part.

(iv) Let the acute angle between l_2 and π_1 be θ. $$\sin\theta = \frac{\left\lvert \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} . \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{11}\sqrt{14}} = \frac{2}{\sqrt{154}}$$	<u>Skills/Concepts Tested</u> - Use an appropriate formula to find the acute angle between a line and a plane directly <u>Common Mistakes</u> - Many students gave the acute angle instead of the sine of the acute angle hence losing 1 mark <u>Overall Comments</u> - More students could get at least 1 mark for this part - Many students did not include the modulus function in the formula but still get the correct answer since the dot product in the numerator is positive.
(v) Let that shortest distance be d. $$\sin\theta = \frac{\sqrt{\frac{297}{2}}}{d} = \frac{2}{\sqrt{154}}$$ $\Rightarrow d = 75.6$ (to 3 s.f.)	<u>Skills/Concepts Tested</u> - Visualisation and linkage to the earlier parts <u>Common Mistakes</u> - Students used the formula for length of projection to find the shortest distance <u>Overall Comments</u> - Most students did not attempt this part - For those who did, none did this part by using answers from the earlier parts and a simple trigo ratio. Instead they used an onerous method of finding $\overrightarrow{OP}$ followed by a formula to find the shortest distance (which most of these students applied the wrong formula)