Solutions – End of Year Exam 2020 Year 3 Express Additional Mathematics Paper 1

1 Solve the following pair of simultaneous equations [4]

$$4x^2 + 3xy + y^2 = 1$$

$$x + y = 1$$

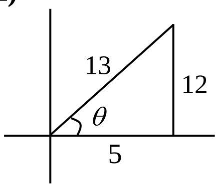
Solutions:	
1	$4x^2 + 3xy + y^2 = 1$ -- (1) $x + y = 1$ -- (2) From (2): $x = 1 - y$ -- (3) Sub (3) into (1): $4(1-y)^2 + 3(1-y)(y) + y^2 = 1$ $4 - 8y + 4y^2 + 3y - 3y^2 + y^2 = 1$ $2y^2 - 5y + 3 = 0$ $(y-1)(2y-3) = 0$ $y - 1 = 0$ or $2y - 3 = 0$ $y = 1$ or $y = \frac{3}{2}$ $x = 0$ or $x = -\frac{1}{2}$

2 Given that $\tan\theta = \dfrac{12}{5}$ and that θ is acute, find the exact value of

(i) $\cos(-\theta)$, [1]

(ii) $\cos(90° - \theta)$, [1]

(iii) $\tan(180° - \theta)$. [1]

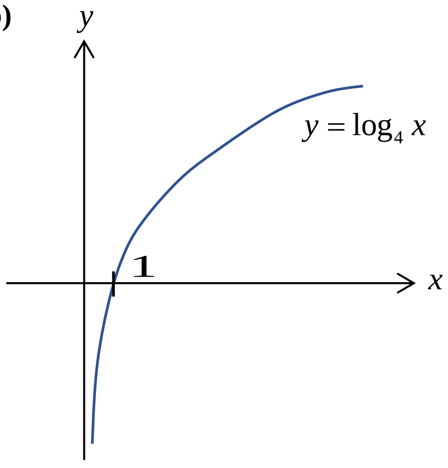
Solutions:	
2	**(i)**  $\cos(-\theta) = \cos\theta$ $= \dfrac{5}{13}$
	(ii) $\cos(90° - \theta) = \sin\theta$ $= \dfrac{12}{13}$
	(iii) $\tan(180° - \theta) = -\tan\theta$ $= -\dfrac{12}{5}$

3 **(a)** The graph of $y = \log_a(kx-1)$ passes through the points with coordinates $(1,0)$ and $(5,2)$.

(i) Determine the value of each of the constants a and k. [4]

(ii) Write down the range of values of x such that y is defined. [1]

(b) Sketch the graph of $y = \log_4 x$. [2]

Solutions:	
3	**(a) (i)** Sub $(1,0)$ into $y = \log_a(kx-1)$ $0 = \log_a(k-1)$ $k-1 = a^0$ $k = 2$ Sub $(5,2)$ into $y = \log_a(kx-1)$ $2 = \log_a(2\times 5-1)$ $2 = \log_a 9$ $9 = a^2$ $\therefore a = 3$ (rej $a = -3$)
	(a) (ii) $y = \log_a(2x-1)$ $2x-1>0$ $x > \frac{1}{2}$
	(b) 

4 It is given that $f(x) = 4x^3 - 16x^2 + 21x - 9$.

(a) Find the quotient when $f(x)$ is divided by $x^2 + 1$. [2]

(b) Prove that $x - 1$ is a factor of $f(x)$. [1]

(c) Hence, factorise $f(x)$ completely. [3]

(d) Express $\dfrac{x}{f(x)}$ in partial fractions. [5]

Solutions:

4 **(a)**

$$
\begin{array}{r|l}
 & \quad 4x - 16 \\
\hline
x^2+1 & 4x^3 - 16x^2 + 21x - 9 \\
 & -(4x^3 \qquad\qquad + 4x) \\
\hline
 & \qquad -16x^2 + 17x - 9 \\
 & \quad -(-16x^2 \qquad\quad - 16) \\
\hline
 & \qquad\qquad\quad 17x + 7 \\
\hline
\end{array}
$$

Quotient = $4x - 16$

(b) $f(1) = 4(1)^3 - 16(1)^2 + 21(1) - 9$

$= 0$

By factor theorem, $x - 1$ is a factor of $f(x)$.

(c) $f(x) = 4x^3 - 16x^2 + 21x - 9$

$= (x-1)(Ax^2 + Bx + C)$

$$
\begin{array}{r|rrrr}
 & 4 & -16 & 21 & -9 \\
1 & & 4 & -12 & 9 \\
\hline
 & 4 & -12 & 9 & 0
\end{array}
$$

By synthetic division,

$f(x) = (x-1)(4x^2 - 12x + 9)$

$= (x-1)(2x-3)^2$

(d) $\dfrac{x}{f(x)} = \dfrac{x}{(x-1)(2x-3)^2}$

$= \dfrac{A}{x-1} + \dfrac{B}{2x-3} + \dfrac{C}{(2x-3)^2}$

$\therefore x = A(2x-3)^2 + B(x-1)(2x-3) + C(x-1)$

Sub $x = 1$,

$\therefore A = 1$

Compare coefficients of x^2:

	$0 = 4(1) + 2B$ $\therefore B = -2$ Compare constants: $0 = 9(1) + 3(-2) - C$ $\therefore C = 3$ $\therefore \dfrac{x}{(x-1)(2x-3)^2} = \dfrac{1}{x-1} - \dfrac{2}{2x-3} + \dfrac{3}{(2x-3)^2}$

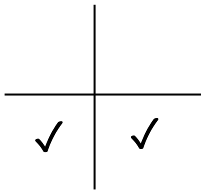
5 The equation of a graph is $y = 2\sin 2x + 1$ for $0 \le x \le \pi$.

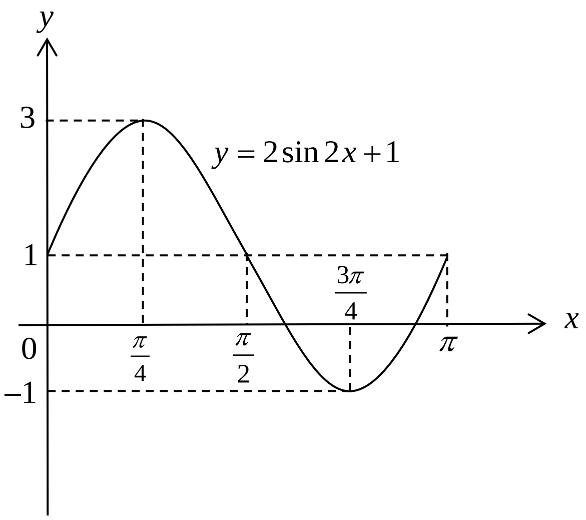
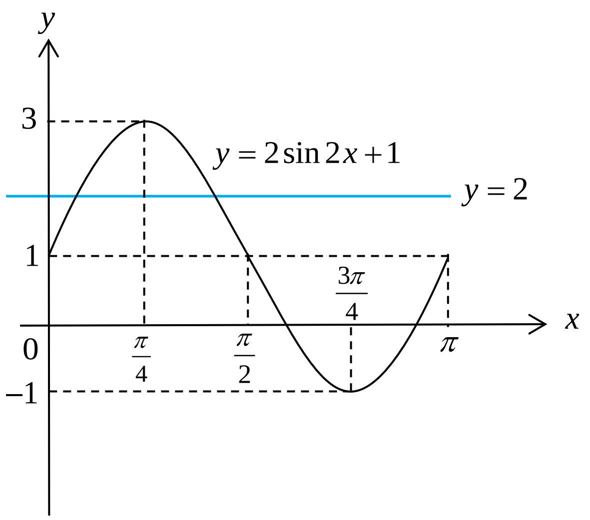
(i) State the period and amplitude of y. [2]

(ii) Solve $y = 0$ for $0 \le x \le \pi$, giving your answer in exact form. [3]

(iii) Sketch the graph of $y = 2\sin 2x + 1$ for $0 \le x \le \pi$. [3]

(iv) By drawing a suitable straight line on the same axis in **(iii)**, find the number of solutions to the equation $2\sin 2x = 1$. [3]

Solutions:	
5	**(i)** period $= \frac{2\pi}{2}$ $= \pi$ amplitude $= 2$
	(ii) $y = 0$ $2\sin 2x + 1 = 0$ $\sin 2x = -\frac{1}{2}$  $\alpha = \sin^{-1}\left(\frac{1}{2}\right)$ $= \frac{\pi}{6}$ $0 \le x \le \pi$ $0 \le 2x \le 2\pi$ $2x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$ $= \frac{7\pi}{6}, \frac{11\pi}{6}$ $x = \frac{7\pi}{12}, \frac{11\pi}{12}$

	(iii) 
	(iv) $2\sin 2x = 1$ $2\sin 2x + 1 = 1 + 1$ $\therefore y = 2$  $\therefore 2$ solutions

6 **(i)** Given that $3\lg(xy) = 2 + 2\lg x - \lg y$, express x in terms of y. [4]

(ii) Solve the equation $\log_4(x+2) - 4\log_{16}(x-1) = 1$. [4]

Solutions:	
6	**(i)** $3\lg(xy) = 2 + 2\lg x - \lg y$ $\lg(xy)^3 - 2\lg x + \lg y = 2$ $\lg(xy)^3 - \lg x^2 + \lg y = 2$ $\lg\left(\frac{x^3y^3}{x^2} \times y\right) = 2$ $\lg(xy^4) = 2$ $xy^4 = 10^2$ $x = \frac{100}{y^4}$ OR $3\lg xy = 2 + 2\lg x - \lg y$ $\lg(xy)^3 = \lg 10^2 + \lg x^2 - \lg y$ $\lg(xy)^3 = \lg\left(\frac{100x^2}{y}\right)$ $(xy)^3 = \frac{100x^2}{y}$ $x^3y^3 = \frac{100x^2}{y}$ $x = \frac{100}{y^4}$ OR $3\lg(xy) = 2 + 2\lg x - \lg y$ $\lg(xy)^3 = 2 + \lg x^2 - \lg y$ $\lg(x^3y^3) = 2 + \lg x^2 - \lg y$ $\lg x^3 + \lg y^3 = 2 + \lg x^2 - \lg y$ $\lg x^3 - \lg x^2 = 2 - \lg y - \lg y^3$ $\lg x = 2 - (\lg y + \lg y^3)$ $\lg x = 2 - \lg y^4$ $x = 10^{2-\lg y^4}$ or $x = 10^{2-4\lg y}$
	(ii) $\log_4(x+2) - 4\log_{16}(x-1) = 1$ $\log_4(x+2) - \log_{16}(x-1)^4 = 1$

	$\log_4(x+2) - \dfrac{\log_4(x-1)^4}{\log_4 16} = 1$ $\log_4(x+2) - \dfrac{4\log_4(x-1)}{2} = 1$ $\log_4(x+2) - 2\log_4(x-1) = 1$ $\log_4 \dfrac{x+2}{(x-1)^2} = 1$ $\dfrac{x+2}{(x-1)^2} = 4$ $x+2 = 4(x-1)^2$ $x+2 = 4x^2 - 8x + 4$ $4x^2 - 9x + 2 = 0$ $(4x-1)(x-2) = 0$ $4x-1=0$ or $x-2=0$ $x = \dfrac{1}{4}$ (N.A.) $x = 2$

7 $f(x)$ is a cubic polynomial such that $f(x)=(x+1)(x-m)(x-3m)$, where m is an integer. It is given that $f(x)$ has a remainder of 10 when divided by $(x-1)$.

(i) Find the value of m. [3]

(ii) With the value of m found in **(i)**, write down the expression for $f(x)$ in descending powers of x. [2]

(iii) Hence, solve the equation $(y+1)^3-7(y+1)^2+4y+16=0$. [2]

Solutions:	
7	**(i)** $f(x)=(x+1)(x-m)(x-3m)$ By remainder theorem, $f(1)=(1+1)(1-m)(1-3m)=10$ $2(1-m)(1-3m)=10$ $3m^2-4m-4=0$ $(m-2)(3m+2)=0$ $m-2=0$ or $3m+2=0$ $m=2$ $\quad m=-\frac{2}{3}$ (N.A.)
	(ii) $f(x)=(x+1)(x-2)(x-6)$ $=(x^2-x-2)(x-6)$ $=x^3-7x^2+4x+12$
	(iii) Sub $x=y+1$ into $x^3-7x^2+4x+12=0$ $(y+1)^3-7(y+1)^2+4(y+1)+12=0$ $(y+1)^3-7(y+1)^2+4y+4+12=0$ $(y+1)^3-7(y+1)^2+4y+16=0$ $y+1=-1$ or $y+1=2$ or $y+1=6$ $y=-2$ $\quad y=1$ $\quad y=5$

8 **(a)** A circle C_1 has an equation given by $x^2 + y^2 - 2x - 6y - 8 = 0$.

(i) Find the centre and radius of circle C_1. [2]

(ii) Given that the equation of the tangent to C_1 at point P is $y = x - 4$, find the coordinates of P. [4]

(b) A circle C_2 has a radius of 6 units and a centre $(2,4)$. The lowest point on the circle C_2 is T. Another circle C_3 has its highest point and lowest point at the centre of C_2 and T respectively. Find the equation of C_3. [3]

Solutions:

8

(i) $x^2 + y^2 - 2x - 6y - 8 = 0$

$2f = -2 \qquad 2g = -6$

$f = -1 \qquad g = -3$

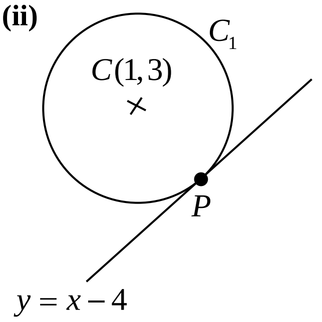
$\therefore C(1,3)$

Radius $= \sqrt{1^2 + 3^2 - (-8)}$

$= \sqrt{18}$

$= 3\sqrt{2}$ units

(ii)



$y = x - 4$

Sub $y = x - 4$ into C_1:

$x^2 + (x-4)^2 - 2x - 6(x-4) - 8 = 0$

$2x^2 - 16x + 32 = 0$

$x^2 - 8x + 16 = 0$

$(x-4)^2 = 0$

$x = 4$

$y = 4 - 4 = 0$

$\therefore P(4,0)$

OR

$y = x - 4$

Gradient of tangent $= 1$

Gradient of CP $= -1$

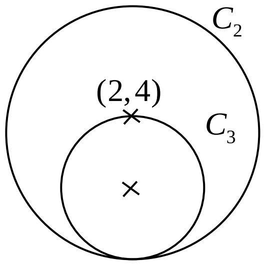
$y - y_1 = m(x - x_1)$

$y - 3 = -1(x - 1)$

$y = -x + 4$

Sub $y = -x + 4$ into $y = x - 4$

$-x + 4 = x - 4$

	$\therefore x=4$ Sub $x=4$ into $y=x-4$ $y=0$ $\therefore P(4,0)$
	(b)  Radius of $C_2=6$ units Radius of $C_3=3$ units Centre of $C_3=(2,1)$ $(x-2)^2+(y-1)^2=3^2$