RAFFLES INSTITUTION
RAFFLES PROGRAMME 2023
YEAR 4 MATHEMATICS
TOPIC 2: REMAINDER & FACTOR THEOREMS, PARTIAL FRACTIONS (MATH 1)

WORKSHEET 3

Name: ( ) Class: 4 ( ) Date:

WORKSHEET 3: CUBIC EQUATIONS

think! Add Math Textbook A Chapter 4 p.65

KEY UNDERSTANDING(S)
Students will understand that

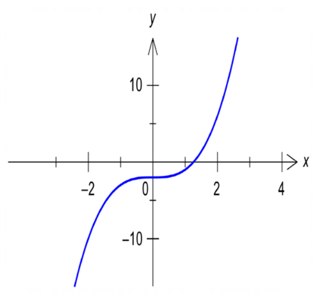
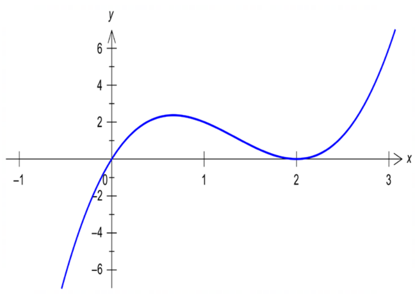
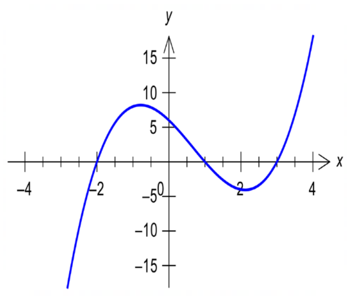
- Remainder and Factor Theorems can be used to factorise cubic expressions and hence solve cubic equations.

LEARNER OUTCOMES
At the end of the worksheet, students will be able to

- Factorize cubic expressions or solve cubic equations using the Remainder and Factor Theorems

(1) INTRODUCTION

In general, a cubic equation $ax^3 + bx^2 + cx + d = 0$ $(a \neq 0)$ has at least one real root and at most 3 real roots as shown in the diagram below.

No of real roots = 1	No of real roots = 2	No of real roots = 3
Eg: $x^3 - 2 = 0$ has 1 real root as the graph of $y = x^3 - 2$ intersects the x-axis at only 1 point	Eg: $2x^3 - 8x^2 + 8x = 0$ has 1 repeated real root and 1 real root as the graph of $y = 2x^3 - 8x^2 + 8x$ touches the x-axis at one point and intersects at another point	Eg: $x^3 - 2x^2 - 5x + 6 = 0$ has 3 real roots as the graph of $y = x^3 - 2x^2 - 5x + 6$ intersects the x-axis at 3 points
		

Question: Is it possible for a cubic equation to have no real roots?

(2) CUBIC EQUATIONS

Factor Theorem can be used to factorise quadratic expressions with ***integer*** coefficients so as to solve the corresponding quadratic equations as follow:

Eg To solve quadratic equation $x^2 - 2x - 3 = 0$,
we factorise the quadratic expression and obtain two linear factors which has ***integer*** coefficients: $x^2 - 2x - 3 = (x-3)(x+1)$
Hence,

$$x^2 - 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x = -3 \text{ or } 1$$

Similarly, we can apply Factor Theorem to solve cubic equations.

Eg To solve cubic equation $x^3 - 2x^2 - 5x + 6 = 0$,
we first find the 1st factor: 1st factor is $(x-1)$
We factorise the cubic expression into a linear and a quadratic factors,

$$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$$

We then further factorise the quadratic expression and obtain two linear factors which has ***integer*** coefficients if it is factorisable, i.e., $(x^2 - x - 6) = (x+2)(x-3)$
Hence,

$$x^3 - 2x^2 - 5x + 6 = 0$$
$$(x-1)(x+2)(x-3) = 0$$
$$x = 1, \quad -2 \text{ or } 3$$

Question: How to find the first factor?

EG 1 Solve the cubic equation $x^3 - 4x^2 + x + 6 = 0$

Step 1: Use **trial and error** and the factor theorem to determine a linear factor

$$f(x) = x^3 - 4x^2 + x + 6.$$

Interesting point to note:
For the trial factor $x-k$ where *k* is integer, *k* is always a factor of the constant term of the cubic expression.
For example, if $(x-k)$ is a factor of $f(x)=x^3-4x^2+x+6$,
then $f(x)=(x-k)(x^2+bx+c)$ and $-kc=6$ (constant term).
Thus, the possible values of *k* are ± 1, ± 2, ± 3 and ± 6
i.e. *k* is a factor of 6. (*Integral Root Theorem*)

<u>Step 2</u>: Determine the other factor(s) of $f(x)$ by division.

Note: A cubic polynomial can be expressed as a product of 1 linear and 1 quadratic factor or as a product of 3 linear factors. In other words, all cubic polynomials have at least one linear factor. Use Desmos or other graph plotting apps to visualise this in a graphical form and convince yourself of its validity.

<u>Step 3</u>: Factorise $f(x)$ completely and solve $f(x)=0$.

Note: The above method works only when you are given the first root / factor or you can use trial and error to find the first integral root.

EG 2 Factorise the expression $2x^3 + 5x^2 - 4x - 12$ completely.

EG 3 The expression $px^3 - 5x^2 + qx + 10$ has factor $2x - 1$ but leaves a remainder of -20 when divided by $x + 2$. Find the values of p and q and factorise the expression completely.

EG 4 Solve the equation $2x^3 + 5x^2 - 7x - 12 = 0$.

Hence sketch the graph of $y = 2x^3 + 5x^2 - 7x - 12$, showing the intercepts with the axes and solve the inequality $2x^3 + 5x^2 - 7x - 12 > 0$.

Blended Learning Online Activity (OPTIONAL): *Students to access HeyMath! online lesson: "Year 4 Maths – Remainder and Factor Theorem: "Solving Cubic Equations – Summary and EG 1" [6:46] to consolidate learning before doing Homework 1.*

HOMEWORK 1

LEVEL 1

1 Solve the equation

(a) $2x^3 - 7x^2 - 7x + 30 = 0$

(b) $2x^3 + 9x^2 = 20x + 12$

(c) $2x^3 - 5x^2 + 4 = 0$

2 The expression $2x^3 + ax^2 + bx + 4$ is divisible by $x - 1$ but leaves a remainder of 3 when divided by $x + 2$. Find the values of a and b and factorise the expression.

3 Find the x-coordinate of each of the three points of intersection of the curves $y = 6x^2 - 5$ and $y = 17x - \frac{6}{x}$.

[Ans: (1a) $-2,\ 2\frac{1}{2}$ or 3 *(b)* $-6, -\frac{1}{2}$ or 2 *(c)* 2 or $\frac{1}{4} \pm \frac{1}{4}\sqrt{17}$ *(2)* $a = \frac{1}{2},\ b = -6\frac{1}{2}$ *(3)* $-\frac{2}{3},\ \frac{1}{2}$ or 3*]*

LEVEL 2

1 Given that $4x^4 - 12x^3 - a^2x^2 - 5ax - 1$ is exactly divisible by $2x + a$.

(a) show that $3a^3 + 5a^2 - 2 = 0$,

(b) find the possible values of *a*.

2 The functions $f(x)$ and $g(x)$ are defined as

$$f(x) = x^4 + 3x^3 - 12x^2 + 2x + 4,$$
$$g(x) = x^4 + 2x^3 - 8x^2 + x - 2.$$

(i) Solve completely the equation $f(x) - g(x) = 0$.

(ii) $f(x)$ and $g(x)$ have a common factor $x - a$. Find the value of a.

[Ans: (1b) $a = -1$ or $\dfrac{-1 \pm \sqrt{7}}{3}$ *(2)* $x = -1, 2$ or 3, $a = 2$*]*

(3) SUM AND DIFFERENCE OF CUBES

Recall: Difference of two Squares

$$a^2 - b^2 = (a+b)(a-b)$$

How about difference of two cubes and sum of two cubes?

Are you able to visualise how a^3 - b^3 and a^3+b^3 look like geometrically?

<u>Blended Learning Online Activity (SDL)</u>: Students to access HeyMath! online lesson: "Year 4 Maths – Remainder and Factor Theorem: Sum and Difference of Cubes – Geometric Method" [5:24] and then <u>complete EG 5.</u>

<u>EG 5</u> Use Remainder and Factor Theorem to prove

(a) $a^3 + b^3 = \left(a+b\right)\left(a^2 - ab + b^2\right)$,

Hence, prove

(b) $a^3 - b^3 = \left(a-b\right)\left(a^2 + ab + b^2\right)$.

(4) FURTHER EQUATIONS

Blended Learning Online Activity (SDL)*: Students to access HeyMath! online lesson: "Year 4 Maths – Remainder and Factor Theorem: "Solving Cubic Equations – EG 2" [3:18] and then *complete EG 6.

EG 6 Given that $x^2 - 3x + 2$ is a factor of $2x^4 + px^3 + x^2 + qx - 12$, find the values of p and q. Hence solve the equation $2x^4 + px^3 + x^2 + qx - 12 = 0$.

EG 7 Solve the equation $2x^3 - 3x^2 - 8x - 3 = 0$. Hence, solve

(a) $3y^3 + 8y^2 + 3y - 2 = 0$,

(b) $2z^3 + 3z^2 - 8z + 3 = 0$,

(c) $2\cos^3\theta - 8\cos\theta = 6 - 3\sin^2\theta$ for $0^\circ \le \theta \le 180^\circ$.

EG 8 The term containing the highest power of x in the polynomial $f(x)$ is $2x^4$. Two of the roots of the equation $f(x)=0$ are -1 and 2. Given that x^2-3x+1 is a quadratic factor of $f(x)$, find

(i) an expression for $f(x)$ in descending powers of x,

(ii) the number of real roots of the equation $f(x)=0$, justifying your answer,

(iii) the remainder when $f(x)$ is divided by $2x-1$.

(GCE 'O' levels, NOV 2008)

HOMEWORK 2

LEVEL 1

1 Factorise $2x^3+3x^2-3x-2$ completely.

Hence, solve the equation $2(y-1)^3+3(y-1)^2-3y+1=0$.

2 Solve the equation $2x^3=3x^2+8x-12$. Hence solve the following equations:

(a) $12y^3-8y^2-3y+2=0$,

(b) $2z^3+3z^2-8z-12=0$.

[Ans: (1) $(x-1)(2x+1)(x+2)$; $y=-1$, $\frac{1}{2}$ or 2 *(2)* ±2 or $1\frac{1}{2}$ *(a)* $\pm\frac{1}{2}$ or $\frac{2}{3}$ *(b)* ±2 or $-1\frac{1}{2}$*]*

LEVEL 2

1 The expression $x^{2n}+x^3-6x^2-4x+p$ has a factor $(x+2)^2$ and leaves a remainder of 6 when divided by $x+1$. Calculate the value of p and of n.

Hence or otherwise, factorise the expression completely.

2 Solve the equation $2x^3 + 7x^2 - 10x - 24 = 0$.
Hence, solve the following equations

(a) $2(y+2)^3 + 7(y+2)^2 - 10y - 44 = 0$

(b) $2z^3 - 7z^2 - 10z + 24 = 0$

(c) $\dfrac{2e^{3w} + 7e^{2w}}{10e^{w} + 24} = 1$

[Ans: (1) p = 8, n = 2; $\mathbf{(x+2)^2(x-1)(x-2)}$

(2) $\mathbf{x = 2, -1\frac{1}{2}}$ **or** $\mathbf{-4}$ *(a)* $\mathbf{y = 0, \ -3\frac{1}{2}}$ **or** $\mathbf{-6}$ *(b)* $\mathbf{z = -2, 1\frac{1}{2}}$ **or 4** *(c) w = ln2]*

LEVEL 3

1 The equation $x^3 + ax^2 + bx + c = 0$ has three distinct integral solutions. It is also given that c is a prime number. Find the largest possible value of a. *[Ans: –2]*

(5) FOR YOUR INTEREST

(5.1) THE "CUBIC FORMULA"

Knowledge of the quadratic formula is older than the Pythagoras' Theorem. Solving a cubic equation, on the other hand, was the first major success story of Renaissance mathematics in Italy. The solution was first published by Girolamo Cardano (1501-1576) in his Algebra book *Ars Magna*.
By Cardano's Method, for any cubic equation $ax^3 + bx^2 + cx + d = 0 \ (a \neq 0)$, one root is given by

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \frac{b}{3a}$$

where $m = \dfrac{3ac - b^2}{3a^2}$ and $n = \dfrac{-2b^3 + 9abc - 27a^2 d}{27a^3}$

You may click on the following link or the QR code to find out more on Cardano's Method of solving cubic equations.

https://en.wikipedia.org/wiki/Cubic_equation

(5.2) APPLICATIONS OF CUBIC EQUATIONS: VAN DER WAALS EQUATION

(Applied Learning (AL): Real life applications of cubic equation in Physics)

The **Van der Waals equation** is an equation that related several properties of real gases, which may be written as a **cubic equation in *v*** as shown below:

$$v^3 - \frac{1}{3}\left(1 + \frac{8T}{p}\right)v^2 + \frac{3}{p}v - \frac{1}{p} = 0,$$

where *v* *(dm³)*, *T (Kelvin)* and *p (atm)* are the volume, temperature and pressure of the gas respectively. At the critical temperature, i.e. $T = 1$ K, $p = 1$ atm, the equation becomes $v^3 - 3v^2 + 3v - 1 = 0$, which can be solve using Remainder and Factor Theorem.

Solution

Let $f(v) = v^3 - 3v^2 + 3v - 1$.

The positive and negative factors of -1 are ± 1.

By trial and error, $f(1) = (1)^3 - 3(1)^2 + 3(1) - 1 = 0$.

By the Factor Theorem, $v - 1$ is a factor of $f(v)$.

$\therefore\ f(v) = v^3 - 3v^2 + 3v - 1 = (v - 1)(av^2 + bv + c)$

By observation, $a = 1$ and $c = 1$.

$\therefore\ f(v) = (v - 1)(v^2 + bv + 1)$

Equating coefficients of v: $3 = -b + 1$

$b = -2$

$\therefore\ f(v) = (v - 1)(v^2 - 2v + 1) = (v - 1)^3$

Hence, $v^3 - 3v^2 + 3v - 1 = 0$

$(v - 1)^3 = 0$

$v = 1$

$\therefore$ At the critical temperature, the volume is 1 dm^3.

(5.3) SOLVING CUBIC EQUATION BY BISECTION METHOD

BISECTION METHOD is a straightforward technique to find numerical solutions of an equation with one unknown. Among all the numerical methods, the bisection method is the simplest one to solve the equation.

To solve an equation $f(x) = 0$ where $f(x)$ is a continuous function. On an interval $[a, b]$, if $f(a) < 0$ and $f(b) > 0$, then there is a value c such that $c \in [a, b]$ and $f(c) = 0$. That is to say, if $f(a)$ and $f(b)$ are of opposite signs, then there is a root c such that $c \in [a,b]$.

The **Bisection Method** is a successive *approximation* method that narrows down an interval that contains a root of the function $f(x)$. To initiate the bisection method, an interval $[a,b]$ that contains a root c must first be found: we often use the property that the signs of $f(a)$ and $f(b)$ are opposite for such initial interval.

ITERATION TASKS

Each iteration performs these steps:

1. Calculate c, the midpoint of the interval, $c = \dfrac{a+b}{2}$
2. Calculate the function value at the midpoint, $f(c)$.
3. Examine the sign of $f(c)$ and replace either $(a, f(a))$ or $(b, f(b))$ with $(c, f(c))$ so that there is a zero crossing within the new interval.
4. If convergence is satisfactory (that is, $c - a$ is sufficiently small, or $|f(c)|$ is sufficiently small), return c and stop iterating.

Example:

Show that one root of the equation $x^3 + x - 5 = 0$ lies between 1 and 2, and find this root correct to 2 decimal places.

Solution

$x^3 + x - 5 = 0$

Let $\mathrm{f}(x) = x^3 + x - 5$

$\mathrm{f}(1) = 1^3 + 1 - 5 = -3 < 0$

$\mathrm{f}(2) = 2^3 + 2 - 5 = 5 > 0$

Since $\mathrm{f}(1)$ and $\mathrm{f}(2)$ are of opposite signs, there is a root $c \in [1, 2]$. (Shown)

Iteration starting with $a = 1$ and $b = 2$:

a	b	$c = \frac{a+b}{2}$	$f(a)$	$f(c)$	$f(b)$	c lies in
1	2	1.5	–3	–0.125	5	[1.5, 2]
1.5	2	1.75	–0.125	2.109375	5	[1.5, 1.75]
1.5	1.75	1.625	–0.125	0.916015625	2.109375	[1.5, 1.625]
1.5	1.625	1.5625	–0.125	0.377197265	2.109375	[1.5, 1.5625]
1.5	1.5625	1.53125	–0.125	0.121612548	0.377197265	[1.5, 1.53125]
1.5	1.53125	1.515625	–0.125	–0.002803802	0.121612548	[1.515625, 1.53125]
1.515625	1.53125	1.5234375	–0.0028	0.05913	0.121612542	[1.515625, 1.5234375]
1.515625	1.5234375	1.5195313	–0.0028	0.02809	0.05912548	

From the last iteration, the root is between 1.515625 and 1.5195313.

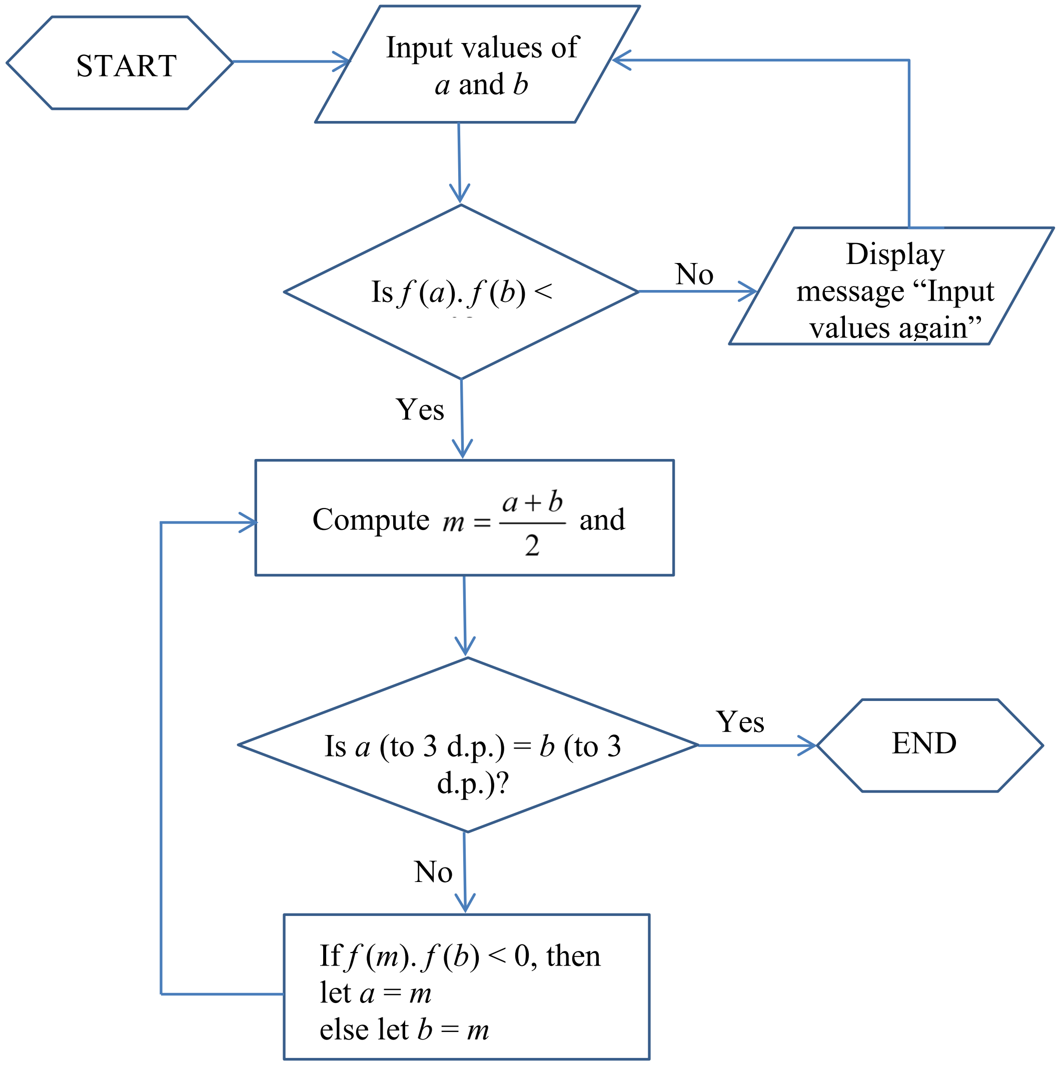
If we rounded up both values to 2 dp, both give the value 1.52

$\therefore\ x = 1.52$ (2 decimal places)

Note that Bisection Method can be used to solve other equations like $x^2 = e^x$ and $\sin x = \ln x$.

POGRAMMING

The algorithm of the **Bisection Method** of finding a root can be illustrated in the flow chart below.



You may click on link below or the QR code to find out more on ***Bisection Method.***

https://en.wikipedia.org/wiki/Bisection_method

Computational Thinking – Opportunities for Algorithmic Thinking by expressing the general solution or approach using a flowchart and then use programming to find the estimated root of cubic equation.