RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 13A : Matrix Theory

SYLLABUS INCLUDES

- Use of matrices to represent a set of linear equations
- Operations on 3×3 matrices
- Determinant of a square matrix and inverse of a non-singular matrix (2×2 and 3×3 matrices only)
- Use of matrices to solve a set of linear equations (including row reduction and echelon forms, and geometrical interpretation of the solution)

PRE-REQUISITES

- Vectors

CONTENT

Task: Secret Code Message

Suppose Adam wants to send the following secret message to Bob,

MEET TOMORROW

He uses the following encoding system with each letter assigned its position in the alphabet as follows:

A	B	…	X	Y	Z
1	2	…	24	25	26

Thus, the original message is encoded as:

13 5 5 20 20 15 13 15 18 18 15 23

To encrypt the original message, he first breaks up the message into blocks of three and writes them as vectors as follows:

$$\begin{pmatrix}13\\5\\5\end{pmatrix},\begin{pmatrix}20\\20\\15\end{pmatrix},\begin{pmatrix}13\\15\\18\end{pmatrix},\begin{pmatrix}18\\15\\23\end{pmatrix}.$$

Then he performs the following matrix multiplication:

$$\mathbf{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&2&3\\1&1&2\\0&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

He gets

$$\begin{pmatrix}38\\28\\15\end{pmatrix},\begin{pmatrix}105\\70\\50\end{pmatrix},\begin{pmatrix}97\\64\\51\end{pmatrix}\text{ and }\begin{pmatrix}117\\79\\61\end{pmatrix}\text{ respectively.}$$

Thus the encrypted message code is

38 28 15 105 70 50 97 64 51 117 79 61

When Bob received the encrypted message, he needs to decode it.

Questions:

- What must Bob know in order to decode the message?
- Could Adam use any matrix other than A? What properties must A have?

Eve intercepts the encrypted message. What must she know in order to break the code?

1 Matrices

An $m \times n$ (read as m by n) **matrix** is a rectangular array of numbers, consisting of m rows (horizontal) and n columns (vertical).

Let **A** be an $m \times n$ matrix. Then, an **entry** is a general element found in the ith row and jth column, denoted by a_{ij}, which could be a real or complex number:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

A shorthand way of writing the above matrix is $\left(a_{ij}\right)_{m \times n}$.

For the current syllabus, we deal with real values of a_{ij}'s only.

For example,

(a) $\begin{pmatrix} 3 & 0 & -1 & 6 \\ -2 & 4 & 0 & 1 \\ 2 & 1 & 7 & 5 \end{pmatrix}$ is a 3×4 matrix.

The (1,4)-entry is 6 and the (3,2)-entry is 1

(b) $(-1 \quad 0 \quad 3 \quad 10 \quad -5)$ is a 1×5 matrix.

(c) $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ is a 4×1 matrix.

$-1(1+2)$

12.

(d) $\left(a_{ij}\right)_{3\times3}$, where $a_{ij} = (-1)^{i+j}(i-j)$, is a 3×3 matrix. 13 21 $-1($ 31

This matrix can also be written as $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

A matrix with only one row is called a **row matrix** (**(b)** above).

A matrix with only one column is called a **column matrix** (**(c)** above).

1.1 Special Matrices

1) An $n\times n$ matrix is called a **square matrix of order n**. In a square matrix $(a_{ij})_{n\times n}$, the entries $a_{11}, a_{22}, \ldots, a_{nn}$, are called **diagonal entries.**

For example, $\begin{pmatrix} 2 & 0 \\ -1 & 5 \end{pmatrix}$ is a square matrix of order 2, with the elements 2 and 5 being the diagonal entries.

2) A **diagonal matrix** of order n is a square matrix such that all its non-diagonal entries are zero. That is, $a_{ij} = 0$ whenever $i \neq j$.

For example, $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are a diagonal matrices.

3) The **identity matrix** or **unit matrix** of order n, denoted by $\mathbf{I}_n$, is an $n\times n$ **square matrix** $(a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

That is,

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

For example, $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity matrix of order 3.

When the order of the identity matrix is understood, we denote the identity matrix by **I** instead.

4) A matrix with all entries zero is called a **zero matrix** or **null matrix,** and is denoted by **0**.

For example, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the zero matrix of order 3.

1.2 Basic Matrix Operations

1) The **matrix addition** of two $m \times n$ matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ is given by $(a_{ij} + b_{ij})_{m\times n}$:

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

For example, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{pmatrix}$,

$$\begin{pmatrix} 2 & 4 \\ 6 & -8 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -3 & 4 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -4 \\ 7 & 13 \end{pmatrix}.$$

NOTE:

(a) Two matrices may be added if they share the same number of rows and columns. So, the following matrix addition is not valid:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & 5 \end{pmatrix}$$

(b) The addition of matrices is **commutative**. That is,
$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}.$$

(c) The addition of matrices is **associative**. That is,
$$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}).$$

(d) $\mathbf{A} + \mathbf{0} = \mathbf{A} = \mathbf{0} + \mathbf{A}$.

2) Let $\alpha \in \mathbb{R}$. The **scalar multiplication** of α to an $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m\times n}$ is given by $(\alpha a_{ij})_{m\times n}$:

$$\alpha\mathbf{A} = \alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}.$$

For example, $3\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 15 & 21 \end{pmatrix}$

We shall denote $(-1)\mathbf{A}$ by $-\mathbf{A}$. So, the matrix subtraction can be defined using the matrix addition: Given two matrices **A** and **B** of the same size, **A**−**B** is defined by the matrix **A**+ (−**B**).

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}-\begin{pmatrix}2&1&2\\2&1&3\\4&0&5\end{pmatrix}=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}+\begin{pmatrix}-2&-1&-2\\-2&-1&-3\\-4&0&-5\end{pmatrix}=\begin{pmatrix}-1&1&1\\2&4&3\\3&8&4\end{pmatrix}$$

NOTE

Scalar multiplication of matrix is **distributive**:

(i) $\lambda(\mathbf{A}+\mathbf{B})=\lambda\mathbf{A}+\lambda\mathbf{B}$, where $\lambda\in\mathbb{R}$.

(ii) $(\lambda+\mu)\mathbf{A}=\lambda\mathbf{A}+\mu\mathbf{A}$, where $\lambda,\mu\in\mathbb{R}$.

3) Let $\mathbf{A}=\left(a_{ij}\right)_{m\times p}$ and $\mathbf{B}=\left(b_{ij}\right)_{p\times n}$.

The **matrix multiplication** of these two matrices, denoted by **AB** is defined by $\left(\sum_{k=1}^{p}a_{ik}b_{kj}\right)_{m\times n}$, that is, the (i,j)th-entry in the matrix **AB** is

$$\sum_{k=1}^{p}a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots+a_{ip}b_{pj}:$$

$$\mathbf{AB}=\begin{pmatrix}\boxed{a_{11}\quad a_{12}\quad\cdots\quad a_{1p}}\\a_{21}\quad a_{22}\quad\cdots\quad a_{2p}\\\vdots\qquad\vdots\qquad\qquad\vdots\\a_{m1}\quad a_{m2}\quad\cdots\quad a_{mp}\end{pmatrix}\begin{pmatrix}\boxed{\begin{matrix}b_{11}\\b_{21}\\\vdots\\b_{p1}\end{matrix}}&\begin{matrix}b_{12}\\b_{22}\\\vdots\\b_{p2}\end{matrix}&\begin{matrix}\cdots\\\cdots\\\\\cdots\end{matrix}&\begin{matrix}b_{1n}\\b_{2n}\\\vdots\\b_{pn}\end{matrix}\end{pmatrix}$$

$$=\begin{pmatrix}\boxed{a_{11}b_{11}+a_{12}b_{21}+\ldots+a_{1p}b_{p1}}&a_{11}b_{12}+a_{12}b_{22}+\ldots+a_{1p}b_{p2}&\cdots&a_{11}b_{1n}+a_{12}b_{2n}+\ldots+a_{1p}b_{pn}\\a_{21}b_{11}+a_{22}b_{21}+\ldots+a_{2p}b_{p1}&a_{21}b_{12}+a_{22}b_{22}+\ldots+a_{2p}b_{p2}&\cdots&a_{21}b_{1n}+a_{22}b_{2n}+\ldots+a_{2p}b_{pn}\\\vdots&\vdots&&\vdots\\a_{m1}b_{11}+a_{m2}b_{21}+\ldots+a_{mp}b_{p1}&a_{m1}b_{12}+a_{m2}b_{22}+\ldots+a_{mp}b_{p2}&\cdots&a_{m1}b_{1n}+a_{m2}b_{2n}+\ldots+a_{mp}b_{pn}\end{pmatrix}$$

Note: An $m\times p$ matrix multiplied with a $p\times n$ matrix gives an $m\times n$ matrix. That is, we can only multiply two matrices **A** and **B** to give **AB** when the number of columns of **A** is equal to the number of rows of **B**, For example,

(i) $$\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}0-2 & 1+4\\ 0-1 & 3+2\end{pmatrix}=\begin{pmatrix}-2 & 5\\ -1 & 5\end{pmatrix}$$

(ii) $$\begin{pmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 2 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\\ 3 & 0\end{pmatrix}$$
$$=\begin{pmatrix}1\times1+2\times0+(-1)\times3 & 1\times0+2\times2+(-1)\times0\\ 0\times1+1\times0+0\times3 & 0\times0+1\times2+0\times0\\ 2\times1+(-1)\times0+1\times3 & 2\times0+(-1)\times2+1\times0\end{pmatrix}$$
$$=\begin{pmatrix}-2 & 4\\ 0 & 2\\ 5 & -2\end{pmatrix}$$

(iii) $$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 0\\ 0 & 1 & 1\end{pmatrix}=\begin{pmatrix}3 & 8 & 3\\ 9 & 20 & 6\\ 15 & 32 & 9\end{pmatrix}$$ ✓

(iv) $$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}= (4+18) = (32)$$ ✓

(v) $$\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}= \begin{pmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ 6 & 12 & 18\end{pmatrix}$$

(vi) $$\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}\begin{pmatrix}-1\\ 3\\ 5\end{pmatrix}=$$ impossible

NOTE

(a) Multiplication of matrices is <u>NOT</u> commutative. That is, in general, $\mathbf{AB} \neq \mathbf{BA}$.

(b) Multiplication of matrices is associative. That is, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

(c) The operation of multiplication of matrices is distributive over addition. That is,
$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC};$$
$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$

(d) If α is a scalar, then $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$.

(e) $\mathbf{A0} = \mathbf{0}$; $\quad \mathbf{0A} = \mathbf{0}$

(f) $\mathbf{AI} = \mathbf{A}$; $\quad \mathbf{IA} = \mathbf{A}$

4) The **transpose** of a matrix $\mathbf{A}=\left(a_{ij}\right)_{m\times n}$ denoted by $\mathbf{A}^{\mathrm{T}}$, is obtained by interchanging the rows and columns of $\mathbf{A}$. That is, $\mathbf{A}^{\mathrm{T}}=\left(a_{ji}\right)_{n\times m}$.

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \Rightarrow \quad \mathbf{A}^{\mathrm{T}}=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

Note that if $\mathbf{A}$ is an $m\times n$ matrix, then $\mathbf{A}^{\mathrm{T}}$ is an $n\times m$ matrix.

For example, if $\mathbf{A}=\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$.

NOTE

If $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that the below operations are well-defined, and k is a real constant, then

(a) $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}=\mathbf{A}$

(b) $\left(\mathbf{A}+\mathbf{B}\right)^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}}+\mathbf{B}^{\mathrm{T}}$

(c) $\left(k\mathbf{A}\right)^{\mathrm{T}}=k\mathbf{A}^{\mathrm{T}}$

(d) $\left(\mathbf{AB}\right)^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

2 Determinant of a square matrix

If A is a 2×2 matrix, say $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then determinant of A is det (A) = $a_{11}a_{22}-a_{21}a_{12}$.

If A is a 3×3 matrix, say $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then determinant of A is

$$\det(\mathbf{A}) = a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

Note that determinants of 4×4 matrices and above are **NOT** in the current syllabus.

Remarks:

1) The second formula for computing the determinant of a 3×3 matrix can be obtained by recopying the first and second columns of A as shown below:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

The determinant is then computed by summing the products on the downward arrows and subtracting the products on the upward arrows.
Note that this method DOES NOT WORK for 4×4 matrices and above.

2) The determinant of a square matrix is also denoted as |A|,
i.e. if $\mathbf{A}=\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then det (A)= $|\mathbf{A}|=\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$. $=2$

Example 1

(a) Given $\mathbf{A}=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, find det (A). $= 18-20=-2$ ✓

(b) Given $\mathbf{B}=\begin{pmatrix} 1 & 4 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$, find det (B). $\begin{pmatrix} 1 & 4 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}\begin{matrix} 1 & 4 \\ -2 & 0 \\ 2 & 1 \end{matrix} = -8-8-(-1-24)=-16+25=9$

×
13

Solution:

2.1 Minor and Cofactor of a Matrix

If $\mathbf{A} = (a_{ij})$ is an $n \times n$ matrix, i.e. a square matrix, then the **minor** of the entry a_{ij}, denoted by M_{ij} is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from $\mathbf{A}$.

The **cofactor** of the entry a_{ij}, denoted by C_{ij}, is defined to be $(-1)^{i+j}M_{ij}$.

For example, if $\mathbf{A}=\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$, then

the minor of entry a_{11} is $M_{11}=\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}=5\times8-6\times4=16$, and
the cofactor of a_{11} is $C_{11}=(-1)^{1+1}M_{11}=16$.

Similarly, the minor of entry a_{32} is $M_{32}=\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}=26$, and
the cofactor of a_{32} is $C_{32}=(-1)^{3+2}M_{32}=-26$.

Notice that C_{ij} and M_{ij} differs only in sign, that is $C_{ij}= =\begin{cases} M_{ij}, & \text{if } i+j \text{ is even;} \\ -M_{ij}, & \text{if } i+j \text{ is odd.} \end{cases}$

For 2×2 and 3×3 matrices, a quick way to check whether to use the '+' or '–' sign is to use the fact that the sign relating C_{ij} and M_{ij} on the ith row and jth column of the matrix is $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ for 2×2 matrices and $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ for 3×3 matrices.

For example, $C_{11}= M_{11}$, $C_{21}= -M_{21}$, etc

Similarly, for 4×4 matrices, the sign relation between C_{ij} and M_{ij} is $\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$.

Consider a general 3×3 matrix $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$\begin{aligned}\det(\mathbf{A}) &= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{22}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32} \\ &= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{21}(-a_{13}a_{32}+a_{12}a_{33})+a_{31}(a_{12}a_{23}-a_{13}a_{22}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}+a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}\end{aligned}$$

This method of evaluating the determinant of **A** is called the **cofactor expansion along the first column of A.**

By rearranging the terms in the above equation and taking out appropriate factors, it is possible to obtain other formulas:

$$\begin{aligned}\det(\mathbf{A}) &= a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31} = a_{11}M_{11}-a_{21}M_{21}+a_{31}M_{31} \leftarrow \text{cofactor expansion along the 1}^{\text{st}}\text{ col}\\ &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13} = a_{11}M_{11}-a_{12}M_{12}+a_{13}M_{13} \leftarrow \text{cofactor expansion along the 1}^{\text{st}}\text{ row}\\ &= a_{12}C_{12}+a_{22}C_{22}+a_{32}C_{32} = -a_{12}M_{12}+a_{22}M_{22}-a_{32}M_{32} \leftarrow \text{cofactor expansion along the 2}^{\text{nd}}\text{ col}\\ &= a_{21}C_{21}+a_{22}C_{22}+a_{23}C_{23} = -a_{21}M_{21}+a_{22}M_{22}-a_{23}M_{23} \leftarrow \text{cofactor expansion along the 2}^{\text{nd}}\text{ row}\\ &= a_{13}C_{13}+a_{23}C_{23}+a_{33}C_{33} = a_{13}M_{13}-a_{23}M_{23}+a_{33}M_{33} \leftarrow \text{cofactor expansion along the 3}^{\text{rd}}\text{ col}\\ &= a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33} = a_{31}M_{31}-a_{32}M_{32}+a_{33}M_{33} \leftarrow \text{cofactor expansion along the 3}^{\text{rd}}\text{ row}\end{aligned}$$

Notice that in each of the expansions, the entries and cofactors all come from the same row or column. These equations are called the **cofactor expansions of A.**

Note: This is the way of finding determinants for matrices of higher order.

Example 2

Find the determinant of the matrix $\mathbf{A}=\begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}$, using the cofactor expansion along the

(i) first column, 3 −16

(ii) first row.

Solution i) $\det(A) = 3(8-12) - (-2)(-2) + 5(3) = -12-4+15 = -1$ ✓

ii) $\det(A) = 3(-4) - (4-15) + 0 = -12+11 = -1$ ✓

$$\begin{pmatrix}3 & 1 & 0\\ -2 & -4 & 3\\ 5 & 4 & -2\end{pmatrix}\begin{matrix}3 & 1\\ -2 & -4\\ 5 & 4\end{matrix}$$

Remark

In Example **2(ii)**, it is not necessary to compute the third cofactor since it was multiplied by zero.

In general, the best strategy to evaluate the determinant by cofactor expansion is to expand along a row or column with the most number of zeroes.

Some Useful Results

(a) $\det(\mathbf{I}) = 1$

(b) $\det(\mathbf{0}) = 0$

(c) If **A** is a $n \times n$ matrix, then $\det(\mathbf{A}) = \det(\mathbf{A}^{\mathbf{T}})$, where $\mathbf{A}^{\mathbf{T}}$ denotes the transpose of the matrix **A**.

(d) If **A** and **B** are $n \times n$ matrices, then

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{B})\det(\mathbf{A}) = \det(\mathbf{BA}),$$

although **AB** is **not equal** to **BA** in general.

3 Inverse of a Non-Singular Square Matrix

A square matrix **A** of size $n \times n$ is said to be **invertible** or **non-singular** if there is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

$\mathbf{A}^{-1}$ is called the **inverse of A** and is unique. If no such matrix exists, then **A** is said to be **non-invertible** or **singular**.

For example, $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ since $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Important Result

If **A** is an $n \times n$ matrix, then **A** is invertible if and only if $\det(\mathbf{A}) \neq 0$.

If an $n \times n$ matrix **A** is invertible, then $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$.

3.1 Finding inverse of a matrix using cofactor and adjoint.

If **A** is a $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & \cdots & C_{nn} \end{pmatrix}$$ is called the **matrix of cofactors of A**.

The transpose of this matrix is called the **adjoint of A** and is denoted by **adj(A)**, that is,

$$\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} C_{11} & C_{21} & \cdots & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & \cdots & C_{nn} \end{pmatrix}$$

Example 3

Find the adjoint of $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$.

Solution:
The cofactors of **A** are

$C_{11} = 12$ $\quad C_{12} = 6$ $\quad C_{13} = -16$

$C_{21} = -4$ $\quad C_{22} = 2$ $\quad C_{23} = 16$

$C_{31} = 12$ $\quad C_{32} = -10$ $\quad C_{33} = 16$

so that the matrix of cofactors is $\begin{pmatrix} 12 & 6 & -16 \\ -4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$ and the adjoint of **A** is $\begin{pmatrix} 12 & -4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$

Theorem 3.1

If **A** is an invertible square matrix, then $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A})$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$, then $\det(\mathbf{A}) = 64$.

Thus, $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}) = \dfrac{1}{64}\begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$

Remark

To check whether the inverse matrix that you have found is correct, you can verify by checking that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n$ or $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$.

Using $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$ as an example, then

$$\mathbf{A}\mathbf{A}^{-1} = \frac{1}{64}\begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$$

$$= \frac{1}{64}\begin{pmatrix} 3\times12+2\times6+(-1)\times(-16) & 3\times4+2\times2+(-1)\times16 & 3\times12+2\times(-10)+(-1)\times16 \\ 1\times12+6\times6+3\times(-16) & 4\times1+6\times2+3\times16 & 1\times12+6\times(-10)+3\times16 \\ 2\times12+(-4)\times6+0 & 2\times4+(-4)\times2+0 & 2\times12+(-4)\times(-10)+0 \end{pmatrix}$$

$$= \frac{1}{64}\begin{pmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}_3$$

3.2 Properties of Inverse

If **A** and **B** are two square matrices such that the below operations are well-defined, and k is a non-zero constant, then

(a) $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$

(b) $\left(k\mathbf{A}\right)^{-1} = \frac{1}{k}\mathbf{A}^{-1}$

(c) $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(d) $\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$

a)

$BB^{-1} = I$

$(A^{-1})(A^{-1})^{-1} = I$

same

$A^{-1}A = I$

b)

$AA^{-1} = I$

$k \times \frac{1}{k} AA^{-1} = I$

$(kA)(\frac{1}{k}A^{-1}) = I$

c)

$A^{-1}A = AA^{-1} = I$

let $A = B^{-1} \Rightarrow A^{-1} = B$

$A^{-1}B^{-1} =$

$AB = B^{-1}A^{-1}$

$(AB)^{-1} = BA$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

d)

$(AB)^{T} = B^{T}A^{T}$

$(AA^{-1})^{T} = (A^{-1})^{T}A^{T}$

$A^{T}(A^{-1})^{T} = (A^{-1})^{T}A^{T}$

$(A^{T})(A^{-1})^{T}$

$= (A^{-1}A)^{T} = I$

4 Elementary Row Operations and Row-Echelon Form

Let **A** be an $m \times n$ matrix.
There are three types of **elementary row operations**:

(a) $R_i \leftrightarrow R_j$ Swapping Row (i) with Row (j).
(b) $R_i \to \alpha R_i$ Multiplying Row(i) with a real scalar $\alpha \neq 0$.
(c) $R_i \to R_i + \alpha R_j$ Multiplying Row(j) by a real scalar α and add it to Row(i).

For example, $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow[R_3 \to 3R_3]{R_1 \leftrightarrow R_2} \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 21 & 24 & 27 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 21R_2]{R_1 \to R_1 - 6R_2} \begin{pmatrix} 0 & -7 & -14 \\ 1 & 2 & 3 \\ 0 & -18 & -36 \end{pmatrix}$

Two matrices are said to be **row equivalent** if one can be obtained from the other by a series of elementary row operations. Thus, $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{pmatrix}$, $\begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 21 & 24 & 27 \end{pmatrix}$ and $\begin{pmatrix} 0 & -7 & -14 \\ 1 & 2 & 3 \\ 0 & -18 & -36 \end{pmatrix}$ are row equivalent.

4.1 Elementary Matrices

Consider the following matrix multiplication:

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{12} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{31} \end{pmatrix}$ $R_1 \leftrightarrow R_2$

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$ $R_3 \to 3R_3$

(iii) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$ $R_1 \to R_1 + 2R_2$

An $n \times n$ square matrix is called an **elementary matrix** if it can be obtained from the $n \times n$ identity matrix by performing a <u>single</u> elementary row operation.

From the above example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$12+4-10=6$ $\quad -\frac{1}{6}\begin{pmatrix} 2 & 8 & 2 \\ -2 & 4 & 1 \\ -6 & 6 & 3 \end{pmatrix}^T = \frac{1}{6}\begin{pmatrix} 2 & -2 & -6 \\ 8 & 4 & 6 \\ 2 & 1 & 3 \end{pmatrix}$

NOTE

All elementary matrices are invertible and their inverses are also elementary matrices.

Example 4

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 10 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

Write down the elementary matrices that represent each of these elementary row operations, and write down their corresponding inverse.

Solution

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$

$\cancel{E_1^T = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$ $\quad \cancel{E_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$ $\quad \cancel{E_3^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}$

$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Properties of Determinant of a Matrix Undergoing Row Operations

It can be shown that

- If two rows (columns) of a matrix are equal, then $\det(\mathbf{A}) = 0$.
- If one row (column) of a matrix is zero, then $\det(\mathbf{A}) = 0$.
- If two rows (columns) of a matrix **A** are interchanged, the determinant of the resulting matrix is $-\det(\mathbf{A})$.
- If a single row (column) of a matrix **A** is multiplied by a scalar k, then the determinant of the resulting matrix is $k\det(\mathbf{A})$.
- If a scalar multiple of a row (column) of a matrix **A** is added to another row (column) of **A**, then the determinant of the resulting matrix is the same as $\det(\mathbf{A})$.

$\det(E A) = \det(E)\det(A)$

4.2 Row-Echelon and Reduced Row-Echelon Form

A matrix is said to be a **row-echelon matrix** (or in **row-echelon form**) if it satisfies the following conditions :

(a) If there are rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(b) If a row does not consist entirely of zeros, then the first non-zero number in the row is a "1" (called a **leading 1**).

(c) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs further to the right than the leading 1 in the higher row.

In addition, if a **row-echelon matrix** also satisfies the condition below:

(d) Each column that contains a leading 1 has zero everywhere else,

this matrix is said to be in **reduced row-echelon form** (or a **reduced row-echelon matrix**).

Note : A **reduced row-echelon matrix** is already in **row-echelon form.**
If a matrix is **not in row-echelon form,** then it is **not in reduced row-echelon form.**

A Quick Check :

(a) A matrix in **row-echelon form** must have zeros **below each leading 1.**

(b) A matrix in **reduced row-echelon form** must have zeros **below and above each leading 1.**

C D

Example 5

Determine which of the following matrices are in

(a) row-echelon form; **(b)** reduced row-echelon form. D

$$A=\begin{pmatrix}1 & 12 & 3 & 0\\ 0 & 0 & 6 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\ B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\ C=\begin{pmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix},\ D=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix},\ E=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix},$$

$$F=\begin{pmatrix}1 & 2 & 0 & 1 & 1\\ 0 & 0 & 2 & 2 & 3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},\ G=\begin{pmatrix}1 & 0 & 3 & 4\\ 0 & 1 & -2 & 5\\ 0 & 1 & 2 & 2\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

Solution :

Procedure for Reducing a Matrix to (Reduced) Row Echelon Form

Steps:

1) Locate the leftmost column that does not consist entirely of zeros.
2) Interchange the top row with another row, if necessary, so that the entry at the top of the column found in Step 1 is different from zero.
3) If the entry that is now at the top of the column found in Step 1 is $\alpha \neq 0$, multiply the first row by $1/\alpha$ in order to introduce a leading 1.
4) Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.
5) Cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue on this way until the entire matrix is in **row-echelon form.**
6) Beginning with the last non-zero row and working upwards, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Continue until the entire matrix is in **reduced row-echelon form.**

Example 6

Reduce $\begin{pmatrix} 1 & 1 & 2 & 5 \\ 3 & 9 & 8 & 17 \\ 1 & 7 & 4 & 7 \\ 3 & 6 & 7 & 16 \end{pmatrix}$ to reduced row echelon form.

Solution :

$$\begin{pmatrix} 1 & 1 & 2 & 5 \\ 3 & 9 & 8 & 17 \\ 1 & 7 & 4 & 7 \\ 3 & 6 & 7 & 16 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 6 & 2 & 2 \\ 0 & 6 & 2 & 2 \\ 0 & 3 & 1 & 1 \end{pmatrix} \xrightarrow[R3 \leftrightarrow R4]{R3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 6 & 2 & 2 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 6 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

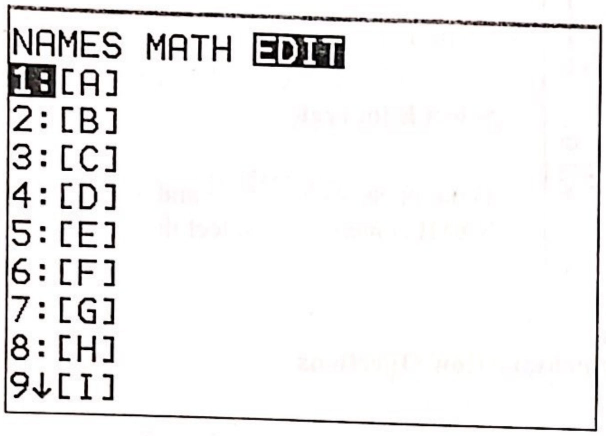
$$\downarrow$$

not unique $\to$ $\begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Row echelon

$$\downarrow$$

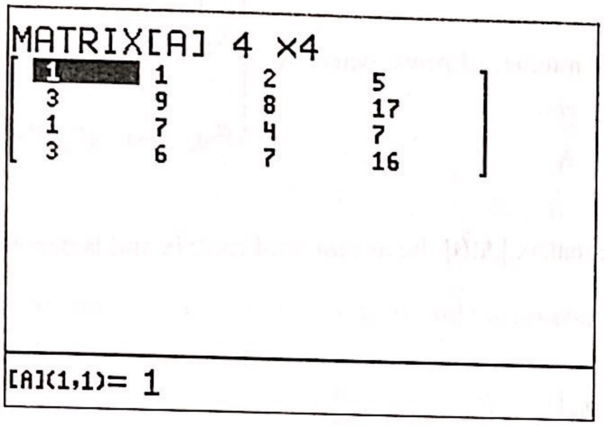
unique $\to$ $\begin{pmatrix} 1 & 0 & \frac{5}{3} & \frac{14}{3} \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Reduced Row Echelon

We can use GC to help use to do row operations on matrix, and also to find the row-echelon and reduced row-echelon matrix.



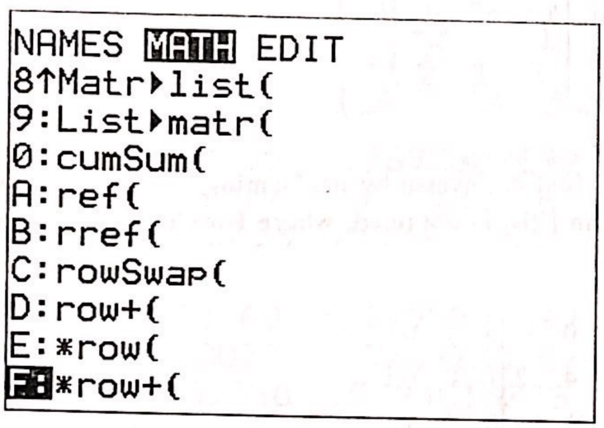
Step 1

Press [2nd] [MATRX] and then go to the **EDIT** menu. Press **1** to select entry to matrix **A**.



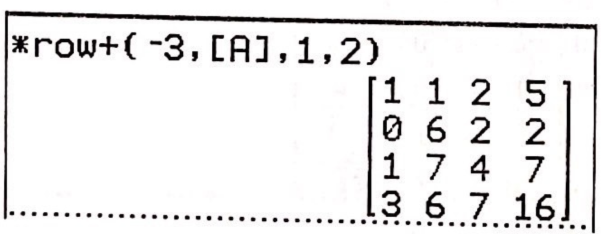
Step 2

Press 4 [ENTER] 4 [ENTER] to define a 4x4 matrix. Key in the entries.



Step 3

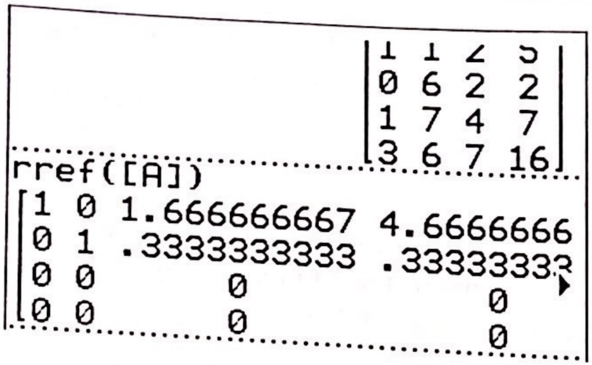
Press [2nd] [QUIT] to return to the home screen. Press [2nd] [MATRX] and then go to the **MATH** menu, and select **F** for ***row+(** to represent "multiplies the row, add to the second row".



Step 4

Key in **-3, [A], 1, 2**, where the entry **[A]** is obtained by pressing [2nd] [MATRX] and go to the **NAMES** menu, and select the matrix **[A]**.

The above entry means multiplies row **1** of matrix **[A]** by **-3** and then adds it to row **2**



Step 5

To find the rref of the matrix, press 2nd MATRX and go to the **MATH** menu. Select **B** for **rref(**.

Then, press 2nd MATRX and go to the **NAMES** menu, and select the matrix **[A]**.

4.2 Finding Inverse of Matrix using Elementary Row Opertions

Given two matrices **A** and **B** having the same number of rows, where **A**= $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

and **B**= $\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mk} \end{pmatrix}$, then we call the matrix **[A|B]** the **augmented matrix** and is denoted by

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & \cdots & b_{1k} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} & b_{m2} & \cdots & b_{mk} \end{array}\right)$$

Given a **non-singular** $n\times n$ matrix **A**, we can find its inverse by **performing elementary row operations on the matrix [A|I] until the form [I|B] is obtained, where I is the** $n\times n$ **identity matrix. Then B is the (unique) inverse of A.**
For example,

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} -2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to -\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 4 & \frac{3}{2} & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to\frac{1}{4}R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0\\ 0 & 1 & 2 & 0 & 1 & 1\end{array}\right)$$

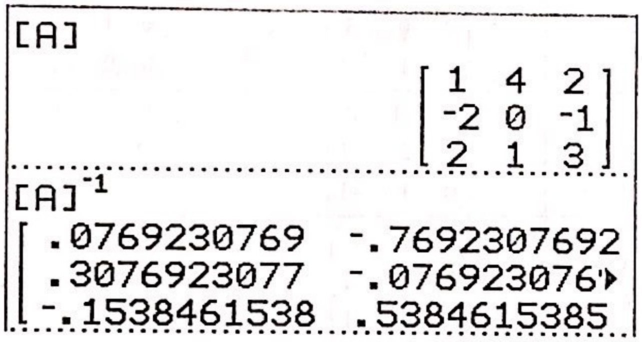
$$\xrightarrow{R_3\to R_3-R_2}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0\\ 0 & 0 & \frac{13}{8} & -\frac{1}{4} & \frac{7}{8} & 1\end{array}\right)$$

$$\xrightarrow{R_3\to\frac{8}{13}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 1 & \frac{3}{8} & \frac{1}{4} & \frac{1}{8} & 0\\ 0 & 0 & 1 & -\frac{2}{13} & \frac{7}{13} & \frac{8}{13}\end{array}\right)$$

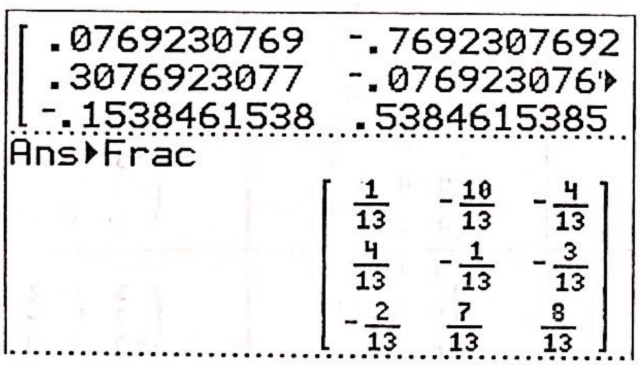
$$\xrightarrow[R_2\to R_2-\frac{3}{8}R_3]{R_1\to R_1-\frac{1}{2}R_3}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{13} & -\frac{10}{13} & -\frac{4}{13}\\ 0 & 1 & 0 & \frac{4}{13} & -\frac{1}{13} & -\frac{3}{13}\\ 0 & 0 & 1 & -\frac{2}{13} & \frac{7}{13} & \frac{8}{13}\end{array}\right)$$

Hence the inverse of $\begin{pmatrix}1 & 4 & 2\\ -2 & 0 & -1\\ 2 & 1 & 3\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{13} & -\frac{10}{13} & -\frac{4}{13}\\ \frac{4}{13} & -\frac{1}{13} & -\frac{3}{13}\\ -\frac{2}{13} & \frac{7}{13} & \frac{8}{13}\end{pmatrix}=\frac{1}{13}\begin{pmatrix}1 & -10 & -4\\ 4 & -1 & -3\\ -2 & 7 & 8\end{pmatrix}$.

Alternatively, you can use a GC to find the inverse of a square matrix.



After entering the entries into matrix **[A]** in the GC, press 2nd MATRX and go to the **NAMES** menu, and select the matrix **[A]**. Then press x^{-1} or $\square^{-1}$ to invert the matrix



Press the **[MATH]** button and then press **[ENTER]** to select **1** for **Frac** which will give the matrix with rational entries, if possible.

How come this works?

If $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ representing elementary row operations, such that

$$\mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$$

This implies that $\mathbf{A}^{-1} = \mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1$. Thus, when we perform the multiplications on an augmented matrix $[\mathbf{A} \mid \mathbf{I}]$, we get

$$[\mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} \mid \mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{I}] = [\mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} \mid \mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1]$$
$$= [\mathbf{I} \mid \mathbf{A}^{-1}]$$

Thus, this provides us another way to find the inverse of a matrix without having to find the determinant.

Example 7

Reduce $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ to reduced row echelon form and write down the corresponding elementary matrices for each step.

Solution

A

Elementary Row Operations	Elementary Matrices	Resultant Matrix	
		$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$R_2 \to R_2 - 2R_1$	$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$R_1 \to R_1 + R_3$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$R_2 \to R_2 - 2R_3$		$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
$R_1 \to R_1 - R_2$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
$R_3 \to R_3 + R_2$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -2 & 1 & -1 \end{pmatrix}$
$R_2 \leftrightarrow R_3$		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 3 \\ -2 & 1 & -1 \\ -2 & 1 & -2 \end{pmatrix}$

I A^{-1}

Thus, for the above example, we can write that

$$\underbrace{\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\cdots\cdots\begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix}}_{\mathbf{A}^{-1}}\underbrace{\begin{pmatrix}1&-1&2\\2&0&3\\0&1&-1\end{pmatrix}}_{\mathbf{A}}=\underbrace{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}_{\mathbf{I}}$$

product of all the elementary matrices

Hence, the inverse of $\begin{pmatrix}1&-1&2\\2&0&3\\0&1&-1\end{pmatrix}$ is $\begin{pmatrix}3&-1&3\\-2&1&-1\\-2&1&-2\end{pmatrix}$

NOTE:

For a square matrix **A**, the following are equivalent:

- **A** is non-singular (invertible)
- $\mathbf{A}^{-1}$ exists
- $\det(\mathbf{A}) \neq 0$
- **A** is row equivalent to **I**

?

5 Applications of Matrices : System of Linear Equations

5.1 System of Linear Equations

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of m linear equations each in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &= \vdots \qquad \text{(I)} \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where b_i's and a_{ij}'s are constants.

The i and j in each a_{ij} are known as the subscripts which are used to identify the location of a_{ij}. For example, a_{12} = coefficient of the 2nd unknown in the first equation.

i.e. a_{ij} = coefficient of the jth unknown in the ith equation.

Note that we will only deal with the case where b_i's and a_{ij}'s are real.

We also call a system of linear equations as **a linear system.**

(a) A **solution of the system** (I) is a sequence of n numbers $\{s_1, s_2, \ldots, s_n\}$ which is a **solution of every equation in (I).**

(b) The set of all solutions of (I) is called the **solution set** of (I). To solve the system (I) means to find the solution set of (I).

(c) The system (I) is **consistent** if it has at least one solution. Otherwise if the system has no solution, then it is **inconsistent.**

Theorem 5.1

Every system of linear equations has either **no solution, exactly one solution** or **infinitely many solutions.**

Note: Theorem 5.1 is **not** true if the equations in the system are not all linear.

In two-dimensional space, the two equations in the system $\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$ represent two straight lines.

Then, the system has

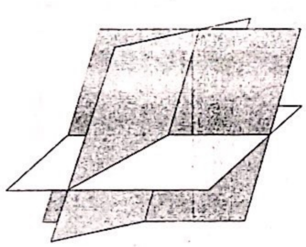
(i) no solution if and only if

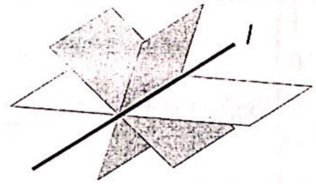
(ii) has exactly one solution if and only if

(iii) infinitely many solutions if and only if

$\underset{\sim}{r} \cdot \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = ??$

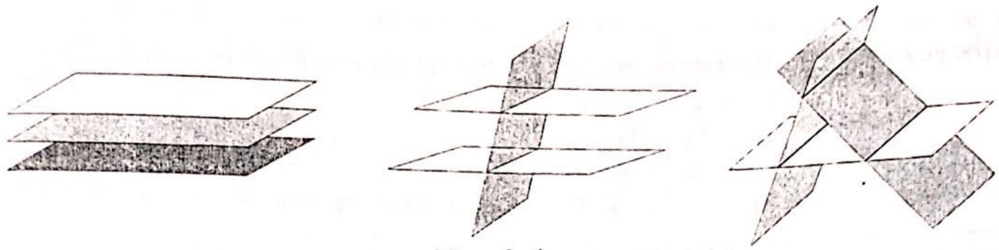
In three-dimensional space, the three equations in the system $\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$ represent three planes. In Chapter 5C: Vectors III, we have seen that there are 3 possible cases on the intersection of 3 planes:


Exactly one solution



Infinitely many solution (line)


Infinitely many solution (plane)



No solution

5.2 Matrix Representation of a Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (*) \\ \vdots \qquad\qquad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We may rewrite the linear system (*) as a matrix equation: $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

or $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

Solving for the unknowns $x_1, x_2, \ldots, x_n$ is the same as solving for the unknown vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$,

given that the coefficients a_{ij}'s and the values b_i's are known.

The matrix containing all the coefficients of the unknowns, namely $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$,

is called the **coefficient matrix.**

The matrix containing **A** as a sub-matrix and an additional last column vector **b**, which then represents the whole system of linear equations, denoted by $\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$

is called the **augmented matrix [A|b]**.

Example 8
Write down the augmented matrix representing the following system of linear equations :

$$\begin{aligned} x + 2y - z + w &= 0 \\ 3x - 5y + 5z - w &= 1 \\ y - z + 7w &= -7 \\ 11x + z &= 1 \end{aligned}$$

Solution :

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & -5 & 5 & -1 \\ 0 & 1 & -1 & 7 \\ 11 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -7 \\ 1 \end{pmatrix}$$

$$\downarrow$$

$$\left(\begin{array}{c|c} & 0 \\ & 1 \\ & -7 \\ & 1 \end{array}\right)$$

5.3 Solving a System of Linear Equations Using Matrices

Let's say we want to solve the following system of linear equations:

$$x+y+3z=0 - - - - - (1)$$
$$2x-2y+2z=4 - - - - - (2)$$
$$3x+9y=3 - - - - - (3)$$

Its augmented matrix is $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$.

If we want to solve "by hand", then:

Step	System of Linear Eqns	Row Operations	Augmented Matrix
	$x+y+3z=0-----(1)$ $2x-2y+2z=4-----(2)$ $3x+9y=3-----(3)$		$\left(\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$
$(2)\div 2$	$x+y+3z=0-----(1)$ $x-y+z=2-----(2)$ $3x+9y=3-----(3)$	$R_2 \to \frac{1}{2}R_2$	$\left(\begin{array}{ccc\|c} 1 & 1 & 3 & 0 \\ 1 & -1 & 1 & 2 \\ 3 & 9 & 0 & 3 \end{array}\right)$
Eliminate z in (1): $(1)-3\times(2)$	$-2x+4y=-6-----(1)$ $x-y+z=2-----(2)$ $3x+9y=3-----(3)$	$R_1 \to R_1 - 3R_2$	$\left(\begin{array}{ccc\|c} -2 & 4 & 0 & -6 \\ 1 & -1 & 1 & 2 \\ 3 & 9 & 0 & 3 \end{array}\right)$
Eliminate x in (3): $(3)+\frac{3}{2}\times(1)$	$-2x+4y=-6-----(1)$ $x-y+z=2-----(2)$ $15y=-6-----(3)$	$R_3 \to R_3 + \frac{3}{2}R_1$	$\left(\begin{array}{ccc\|c} -2 & 4 & 0 & -6 \\ 1 & -1 & 1 & 2 \\ 0 & 15 & 0 & -6 \end{array}\right)$
Find y in (3): $(3)\div 15$	$-2x+4y=-6-----(1)$ $x-y+z=2-----(2)$ $y=-\frac{2}{5}-----(3)$	$R_3 \to \frac{1}{15}R_3$	$\left(\begin{array}{ccc\|c} -2 & 4 & 0 & -6 \\ 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & -\frac{2}{5} \end{array}\right)$

Thus, the method of row operations is a "short-hand" notation way of solving system of linear equations by hand. The reduced row-echelon form of the augmented matrix will yield the solution easily.

With the GC app Plysmlt2, we can use it to help to solve a system of linear equations. Now, we shall explore how we can use matrices to help us in solving the system of linear equations.

First, represent the system of linear equations by its corresponding augmented matrix.
Then, reduce the augmented matrix to reduced row-echelon form.
In **reduced row-echelon form**, check

(i) any inconsistent row?
If so, then the linear system has **no solution.**

(ii) **[I|x]** emerges?
The linear system has a **unique solution given by x.**

(iii) others?
Then the linear system has **infinitely many solutions.**

Example 9
Solve the following system of equations

(a)
$$\begin{aligned} x_1+2x_2+2x_3&=-1 \\ x_1+3x_2+x_3&=4 \\ x_1+3x_2+2x_3&=3 \end{aligned}$$

(b)
$$\begin{aligned} x_1+2x_2-x_3+x_4&=0 \\ 3x_1-5x_2+5x_3-x_4&=1 \\ x_2-x_3+7x_4&=-7 \\ x_1+x_3&=1 \end{aligned}$$

(c)
$$\begin{aligned} x_1+2x_2-x_3+x_4&=0 \\ 3x_1-5x_2+2x_3+4x_4&=1 \\ x_2-x_3+7x_4&=-7 \\ x_1+x_3-13x_4&=14 \end{aligned}$$

Solution :

Example 10
Determine values of a such that

$$x+2y-z=1$$
$$2x+5y+az=3$$
$$3x+ay+6z=4$$

dont worry

has **(i)** no solution; **(ii)** more than one solution; **(iii)** a unique solution.
Solve the equations completely in cases **(ii)** & **(iii)**.

Solution :

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & a & 3 \\ 3 & a & 6 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & (a+2) & 1 \\ 0 & (a-6) & 9 & 1 \end{array}\right)$$

$$\downarrow R_3 \to R_3 - (a-6)R_2$$

a^2

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 21+4a-a^2 & 7-a \end{array}\right)$$

i) The equations have no solution if $21+4a-a^2=0$ and $a\neq 7$ } contradictory

$(7-a)(3+a)=0$
$a=-3$

ii) $21+4a-a^2=0$ and $7-a=0$ | both $=0$
$(7-a)(3+a)=0$, $a=7$
$a=7$

If $a=7$,

$$\left(\begin{array}{ccc|c} 1 & 0 & -19 & -1 \\ 0 & 1 & 9 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ ie } \rightsquigarrow \begin{array}{l} x-19z=-1 \\ y+9z=1 \end{array} \text{ ie } \begin{array}{l} x=-1+19z \\ y=1-9z \end{array}$$

Hence $x=-1+19t$, $y=1-9t$, $z=t$, where $t\in\mathbb{R}$

iii) $21+4a-a^2\neq 0$
$(7-a)(3+a)\neq 0$
$7\neq a$; $-3\neq a$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 2+a & 1 \\ 0 & 0 & 1 & \frac{7-a}{21+4a-a^2} \end{array}\right)$$

both $\neq 0$

So solutions are
$z=\dfrac{7-a}{21+4a-a^2}=\dfrac{7-a}{(7-a)(3+a)}=\dfrac{1}{3+a}$
$y=1-(2+a)z=\dfrac{1}{3+a}$
$x=1+z-2y=\dfrac{2+a}{3+a}$

Hence ~~solu~~ the eqns have unique sol if $a\neq 7$ or $a\neq -3$

Summary