Q1	**Numbers and Proofs**
(a)(i)	Let $\gcd(a,b)=x$; $\gcd(a,b-a)=y$ Since $x\mid a$ and $x\mid b$, $x\mid(b-a)$ $\therefore$ as $x\mid a$ and $x\mid(b-a)\Rightarrow x\mid y$ Similiarly, since $y\mid a$ and $y\mid(b-a)$, $y\mid\left(a+(b-a)\right)=b$ $\therefore$ as $y\mid a$ and $y\mid b\Rightarrow y\mid x$ Hence $x=y$, i.e. $\gcd(a,b)=\gcd(a,b-a)$
(a)(ii)	$\gcd(72,120)=\gcd(72,120-72)=\gcd(72,48)$ Similarly, $\gcd(48,72)=\gcd(48,24)=24$
(b)	Let $\gcd\left(a,\gcd(b,c)\right)=d_1$ and $\gcd\left(\gcd(a,b),c\right)=d_2$ Since $\gcd\left(a,\gcd(b,c)\right)=d_1$, $\Rightarrow d_1$ | a and $d_1\mid\gcd(b,c)$ $\Rightarrow d_1$ | a and $d_1\mid b$ and $d_1\mid c$ $\Rightarrow d_1\mid\gcd(a,b)$ and $d_1\mid c$ $\Rightarrow d_1\mid\gcd\left(\gcd(a,b),c\right)$ $\Rightarrow d_1\mid d_2$ Similiarly, since $\gcd\left(\gcd(a,b),c\right)=d_2$, $\Rightarrow d_2\mid\gcd(a,b)$ and $d_2\mid c$ $\Rightarrow d_2\mid a$ and $d_2\mid b$ and $d_2\mid c$ $\Rightarrow d_2\mid a$ and $d_2\mid\gcd(b,c)$ $\Rightarrow d_2\mid\gcd\left(a,\gcd(b,c)\right)$ $\Rightarrow d_2\mid d_1$ Since $d_2\mid d_1$ and $d_1\mid d_2$, $\Rightarrow$ $d_1=d_2$

Q2	Counting
(a)(i)	Equivalent to $x_1+x_2+x_3+x_4+x_5=13$, $x_i \in \mathbb{Z}^+ \cup \{0\}$ Number of ways $=\binom{13+4}{4}$ $=2380$
(a)(ii)	Equivalent to $x_1+x_2+x_3+x_4+x_5=13$, $x_1 \in \{1,2,3\}$, $x_i \in \mathbb{Z}^+$ Equivalent to $y_1+y_2+y_3+y_4+y_5=8$, $y_i \in \{0,1,2\}$, $y_i \in \mathbb{Z}^+ \cup \{0\}$ Number of ways $=\binom{8+4}{4}$ Complement is equivalent to $z_1+z_2+z_3+z_4+z_5=8, z_i \ge 3, z_i \in \mathbb{Z}^+ \cup \{0\}$ [at least 3 5 cent coins] equivalent to $w_1+w_2+w_3+w_4+w_5=5$, $w_i \in \mathbb{Z}^+ \cup \{0\}$ Required number of ways $=\binom{5+4}{4}$ Therefore, required number of ways $=\binom{8+4}{4}-\binom{5+4}{4}$ $=369$
(b)(i)	Number of ways $=5\times 4^{12}=83886080$
(b)(ii)	Number of ways $=5^{13}-\binom{5}{1}\times 4^{13}+\binom{5}{2}\times 3^{13}-\binom{5}{3}\times 2^{13}+\binom{5}{4}\times 1^{13}$ $=901020120$

Q3	Numbers and Proofs
3(i)	$(x+y)^p = x^p + y^p + \sum_{i=1}^{p-1} \binom{p}{i} x^i y^{p-i}$ Note that $\binom{p}{i} = \frac{p(p-1)\cdots(p-i+1)}{i!}$ For $1 \le i \le p-1$, since $i < p$ and p is prime, thus $i! \mid (p-1)\cdots(p-i+1)$ and p is a factor of $\binom{p}{i}$. Accordingly, $(x+y)^p \equiv x^p + y^p \pmod p$
3(ii)	Let P_a be the proposition that $a^p \equiv a \pmod p$ for all positive integers a. Clearly, $1^p = 1$. Thus P_1 is true. Suppose P_k is true for some $k \in \mathbb{Z}^+$. Consider P_{k+1}. $(k+1)^p \equiv k^p + 1 \pmod p$ $\equiv k+1 \pmod p$ (by induction hypothesis) Thus P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, $a^p \equiv a \pmod p$ for all positive integers a.
3(iii)	Since n is not a multiple of $p-1$, we must have $n = k(p-1)+r$ for some $k \in \mathbb{Z}^+$ and $r = 1, 2, 3$. Using (ii), for $a < p$, we must have $a \cdot a^{p-1} \equiv a \pmod p \Rightarrow a^{p-1} \equiv 1 \pmod p$ Now for $i = 1, 2, 3, 4$, $i^n = i^{k(p-1)+r} = i^{k(p-1)} i^r \equiv 1^k i^r = i^r \pmod p$ Thus $\sum_{i=1}^{4} i^n \equiv \sum_{i=1}^{4} i^r \pmod 5$ For $r = 1, 2, 3,$, $\sum_{i=1}^{4} i^r = 10, 30, 100$ respectively. Thus $\sum_{i=1}^{4} i^n \equiv 0 \pmod 5$

Q4	Inequalities
(a)	$\left(\sqrt{a}\cdot\sqrt{p}+\sqrt{b}\cdot\sqrt{q}+\sqrt{c}\cdot\sqrt{r}\right)^2 \le (a+b+c)(p+q+r)$ $\left(\sqrt{a}\cdot\sqrt{p}+\sqrt{b}\cdot\sqrt{q}+\sqrt{c}\cdot\sqrt{r}\right) \le \sqrt{(a+b+c)(p+q+r)}$ $\sqrt{ap}+\sqrt{bq}+\sqrt{cr} \le \sqrt{(a+b+c)(p+q+r)}$
(b)	$x+y \ge 2\sqrt{xy}$ $\frac{1}{x+y} \le \frac{1}{2\sqrt{xy}}$ $\frac{x}{x+y} \le \frac{x}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\frac{x}{y}} - (1)$ $\frac{y}{y+z} \le \frac{1}{2}\sqrt{\frac{y}{z}} - (2)$ $\frac{z}{z+x} \le \frac{1}{2}\sqrt{\frac{z}{x}} - (3)$ $\frac{x}{(x+y)}\cdot\frac{y}{(y+z)}\cdot\frac{z}{(x+z)} \le \frac{1}{2}\sqrt{\frac{x}{y}}\cdot\frac{1}{2}\sqrt{\frac{y}{z}}\cdot\frac{1}{2}\sqrt{\frac{z}{x}}$ $\frac{xyz}{(x+y)(y+z)(x+z)} \le \frac{1}{8}\sqrt{\frac{x}{y}}\cdot\sqrt{\frac{y}{z}}\cdot\sqrt{\frac{z}{x}}$ $\frac{xyz}{(x+y)(y+z)(x+z)} \le \frac{1}{8}$

(c)	$\sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}}$
	$=\sqrt{\frac{2x(y+z)(z+x)}{(x+y)(y+z)(z+x)}}+\sqrt{\frac{2y(x+z)(x+y)}{(y+z)(x+y)(z+x)}}+\sqrt{\frac{2z(x+y)(y+z)}{(z+x)(x+y)(y+z)}}$
	$=\frac{\sqrt{2x(y+z)(z+x)}+\sqrt{2y(x+z)(x+y)}+\sqrt{2z(x+y)(y+z)}}{\sqrt{(x+y)(y+z)(z+x)}}$
	$\le\frac{\sqrt{\{2x(y+z)+2y(x+z)+2z(x+y)\}\{(z+x)+(x+y)+(y+z)\}}}{\sqrt{(x+y)(y+z)(z+x)}}$
	$=\frac{\sqrt{2}\sqrt{2}\sqrt{2(xy+yz+xz)(x+y+z)}}{\sqrt{(x+y)(y+z)(z+x)}}$
	$=\frac{2\sqrt{2}\sqrt{(xy+yz+xz)(x+y+z)}}{\sqrt{(x+y)(y+z)(z+x)}}$
	$=\frac{2\sqrt{2}\sqrt{(x+y)(y+z)(z+x)+xyz}}{\sqrt{(x+y)(y+z)(z+x)}}$
	$=2\sqrt{2}\sqrt{\frac{(x+y)(y+z)(z+x)+xyz}{(x+y)(y+z)(z+x)}}$
	$=2\sqrt{2}\sqrt{1+\frac{xyz}{(x+y)(y+z)(z+x)}}$
	$\le 2\sqrt{2}\sqrt{1+\frac{1}{8}}$
	$=2\sqrt{2}\frac{3}{2\sqrt{2}}$
	$=3$

Q5	**Counting**
(i)	Eugene does not do a threshold run on two consecutive days and he does not do a recovery run for more than two consecutive days. Call this condition (*). For a_{n+1}, Day 1 is a threshold run. Day 2 cannot be a threshold run. Days 2 to $n+1$ is a sequence of *n* runs satisfying (*) where Day 2 is a tempo run or recovery run. By Addition Principle, $a_{n+1} = b_n + c_n$ For b_{n+1}, Day 1 is a tempo run. Days 2 to $n+1$ is a sequence of *n* runs satisfying (*) where Day 2 can be any run. By Addition Principle, $b_{n+1} = a_n + b_n + c_n$ For c_{n+2}, Day 1 is a recovery run. Case 1: Day 2 is a threshold run Days 2 to $n+2$ is a sequence of $n+1$ runs satisfying (*) where Day 2 is a threshold run. Case 2: Day 2 is a tempo run Days 2 to $n+2$ is a sequence of $n+1$ runs satisfying (*) where Day 2 is a tempo run. Case 3: Day 2 is a recovery run Day 3 cannot be a recovery run. Case 3A: Day 3 is a threshold run Days 3 to $n+2$ is a sequence of *n* runs satisfying (*) where Day 3 is a threshold run. Case 3B: Day 3 is a tempo run Days 3 to $n+2$ is a sequence of *n* runs satisfying (*) where Day 3 is a tempo run. By Addition Principle, $c_{n+2} = a_{n+1} + b_{n+1} + \left(a_n + b_n\right)$

(ii)	Doing a replacement yields $a_{n+4} = b_{n+3} + c_{n+3} \ldots (1)$ $b_{n+4} = a_{n+3} + b_{n+3} + c_{n+3} \ldots (2)$ $c_{n+3} = a_{n+2} + b_{n+2} + a_{n+1} + b_{n+1} \ldots (3)$ Sub (1) into (2), $b_{n+4} = a_{n+3} + a_{n+4} \ldots (4)$ Doing a replacement yields $b_{n+1} = a_n + a_{n+1} \ldots (5)$ $b_{n+2} = a_{n+1} + a_{n+2} \ldots (6)$ $b_{n+3} = a_{n+2} + a_{n+3} \ldots (7)$ Sub (5) and (6) into (3), $c_{n+3} = a_{n+2} + \left(a_{n+1} + a_{n+2}\right) + a_{n+1} + \left(a_n + a_{n+1}\right)$ $= 2a_{n+2} + 3a_{n+1} + a_n \ldots (8)$ Sub (7) and (8) back into (1), a_{n+4} $= \left(a_{n+2} + a_{n+3}\right) + \left(2a_{n+2} + 3a_{n+1} + a_n\right)$ $= a_{n+3} + 3a_{n+2} + 3a_{n+1} + a_n$
(iii)	<u>**Method 1: Recurrence**</u> $a_1 = 1$ $a_2 = 1 \times 2 = 2$ $a_3 = 1 \times 2 \times 3 = 6$ $a_4 = \underbrace{2+3+3}_{\substack{\text{1st day THR} \\ \text{2nd day TEM}}} + \underbrace{2+3+2}_{\substack{\text{1st day THR} \\ \text{2nd day REC}}} = 15$ $a_5 = 15 + 3(6+2) + 1$ $= 40$ (shown) <u>**Method 2:**</u> For a_5, Day 1 is a threshold run Day 2 can be only be a tempo run or recovery run. Case 1: Day 2 is tempo run

	Case 1A: Day 3 is threshold run, Day 4 is tempo or recovery run, Day 5 is any run No. of ways $=2\times3=6$ Case 1B: Day 3 is tempo run, Day 4 is threshold run, Day 5 is tempo or recovery run No. of ways $=2$ Case 1C: Day 3 is tempo run, Day 4 is tempo or recovery run, Day 5 is any run No. of ways $=2\times3=6$ Case 1D: Day 3 is recovery run, Day 4 is threshold run, Day 5 is tempo or recovery run No. of ways $=2$ Case 1E: Day 3 is recovery run, Day 4 is tempo run, Day 5 is any run No. of ways $=3$ Case 1F: Day 3 is recovery run, Day 4 is recovery run, Day 5 is threshold or tempo run No. of ways $=2$ Case 2: Day 2 is recovery run Case 2A: Day 3 is threshold run, Day 4 is tempo or recovery run, Day 5 is any run No. of ways $=2\times3=6$ Case 2B: Day 3 is tempo run, Day 4 is threshold run, Day 5 is tempo or recovery run No. of ways $=2$ Case 2C: Day 3 is tempo run, Day 4 is tempo or recovery run, Day 5 is any run No. of ways $=2\times3=6$ Case 2D: Day 3 is recovery run, Day 4 is threshold run, Day 5 is tempo or recovery run No. of ways $=2$ Case 2E: Day 3 is recovery run, Day 4 is tempo run, Day 5 is any run No. of ways $=3$

	By Addition Principle, no. of ways $=40$ **Method 3:** For a_5, Day 1 is a threshold run Day 2 can be only be a tempo run or recovery run. Total no. of ways for Days 2 to 5 without restriction, without restrictions for Days 3 to 5 $=2\times3^3$ No. of ways where there are at least 3 consecutive days of recovery runs $= \underset{\substack{\text{D2 to D4: All REC} \\ \text{D5: Any}}}{3} + \underset{\substack{\text{D2: TEM} \\ \text{D3 to D5: All REC}}}{1}$ $=4$ No. of ways where there are at least 2 consecutive days of threshold runs $= \underset{\text{D2}}{2}\left(\underset{\text{D3 to D5: All THR}}{1} + \underset{\text{D3 to D5: 2 THR}}{2\times2}\right)$ $=10$ Required no. of ways $=2\times3^3-4-10=40$
(iv)	$b_5=a_4+a_5=55$ $c_5=2a_4+3a_3+a_2=50$ Required no. of ways $=55+50+40=145$

[Turn over

Q6	Functions and Graphs
(a)(i)	Substitute $x=\sin\theta$, then $\frac{dx}{d\theta}=\cos\theta$ and $\theta=\frac{\pi}{2}$, when $x=1$ and $\theta=0$, when $x=0$. $\int_0^1 (1-x^2)^n dx = \int_0^{\frac{\pi}{2}} (\cos^2\theta)^n (\cos\theta)\, d\theta$ $= \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n+1}\, d\theta$ $= I_{2n+1}$
(ii)	Substitute $x=\tan\theta$, then $\frac{dx}{d\theta}=\sec^2\theta$ and $\theta=\frac{\pi}{4}$, when $x=1$ and $\theta=0$, when $x=0$. $\int_0^1 (1+x^2)^{-n} dx = \int_0^{\frac{\pi}{4}} (\sec^2\theta)^{-n} (\sec^2\theta)\, d\theta$ $= \int_0^{\frac{\pi}{4}} (\cos\theta)^{2n} \left(\frac{1}{\cos^2\theta}\right) d\theta$ $= \int_0^{\frac{\pi}{4}} (\cos\theta)^{2n-2}\, d\theta$ $< \int_0^{\frac{\pi}{2}} (\cos\theta)^{2n-2}\, d\theta = I_{2n-2}$
(b)	From MF26, $e^{x^2} = 1+x^2+\frac{x^4}{2}+\frac{x^6}{6}+\dots$ and $(1-x^2)^{-1}=1+x^2+x^4+x^6+\dots$ hence, $1+x^2 \le e^{x^2} \le (1-x^2)^{-1}$ taking reciprocal, $1-x^2 \le e^{-x^2} \le (1+x^2)^{-1}$ (shown) Raising to n power and integrate, $\int_0^1 (1-x^2)^n dx \le \int_0^1 e^{-nx^2} dx \le \int_0^1 (1+x^2)^{-n} dx$ Substitute $y=\sqrt{n}x$, then $\frac{dy}{dx}=\sqrt{n}$ and $y=\sqrt{n}$, when $x=1$ and $y=0$, when $x=0$. $\int_0^1 e^{-nx^2} dx = \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} dy$

	Using (i), $$I_{2n+1}=\int_0^1\left(1-x^2\right)^n\mathrm{d}x\le\frac{1}{\sqrt{n}}\int_0^{\sqrt{n}}\mathrm{e}^{-y^2}\ \mathrm{d}y\le\int_0^1\left(1+x^2\right)^{-n}\mathrm{d}x<I_{2n-2}$$ $$\sqrt{n}I_{2n+1}\le\int_0^{\sqrt{n}}\mathrm{e}^{-y^2}\ \mathrm{d}y<\sqrt{n}I_{2n-2}\quad\text{(shown)}$$
(c)	As $n\to\infty$, $$\sqrt{n}I_{2n+1}=\frac{\sqrt{n}}{\sqrt{2n+1}}\sqrt{2n+1}\ I_{2n+1}=\sqrt{\frac{1}{2+1/n}}\sqrt{2n+1}\ I_{2n+1}\to\frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}}=\frac{\sqrt{\pi}}{2}$$ $$\sqrt{n}I_{2n-2}=\frac{\sqrt{n}}{\sqrt{2n-2}}\sqrt{2n-2}\ I_{2n-2}=\sqrt{\frac{1}{2-2/n}}\sqrt{2n-2}\ I_{2n-2}\to\frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}}=\frac{\sqrt{\pi}}{2}$$ hence, $\frac{\sqrt{\pi}}{2}\le\int_0^{\sqrt{n}}\mathrm{e}^{-y^2}\ \mathrm{d}y<\frac{\sqrt{\pi}}{2}$ as $n\to\infty$ therefore, $\int_0^{\infty}\mathrm{e}^{-y^2}\ \mathrm{d}y=\frac{\sqrt{\pi}}{2}$
(d)	Consider $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2n-1}\mathrm{e}^{-x^2}\right)=(2n-1)x^{2n-2}\mathrm{e}^{-x^2}-2x^{2n}\mathrm{e}^{-x^2}$ $$\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2n-1}\mathrm{e}^{-x^2}\right)\mathrm{d}x=(2n-1)\int_0^{\infty}x^{2n-2}\mathrm{e}^{-x^2}\mathrm{d}x-2\int_0^{\infty}x^{2n}\mathrm{e}^{-x^2}\mathrm{d}x$$ $$\left(x^{2n-1}\mathrm{e}^{-x^2}\right)_0^{\infty}=(2n-1)U_{n-1}-2U_n$$ $$0=(2n-1)U_{n-1}-2U_n\qquad\text{since } x^{2n-1}\mathrm{e}^{-x^2}\to0\text{ as }x\to\infty$$ $$U_n=\frac{2n-1}{2}U_{n-1}\ \text{(shown)}$$ $U_1=\frac{1}{2}U_0$ $U_2=\frac{3}{2}U_1=\frac{1\times3}{2^2}U_0$ $U_3=\frac{5}{2}U_2=\frac{1\times3\times5}{2^3}U_0$ $\vdots$ $$U_n=\frac{1\times3\times5\times\ldots\times(2n-1)}{2^n}U_0$$ $$=\frac{(2n)!}{2^n(2\times4\ldots\times2n)}U_0$$ $$=\frac{(2n)!}{2^{2n}(n!)}\left(\frac{\sqrt{\pi}}{2}\right)\qquad\text{by (i) }U_0=\int_0^{\infty}\mathrm{e}^{-x^2}\mathrm{d}x=\frac{\sqrt{\pi}}{2}$$ $$\int_0^{\infty}x^{2n}\mathrm{e}^{-x^2}\mathrm{d}x=\frac{(2n)!\sqrt{\pi}}{2^{2n+1}n!}$$

Q7	**Functions and Graphs**
(i)	$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear points $\Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$ $(x_3 - x_2)y_2 - (x_3 - x_2)y_1 = (x_2 - x_1)y_3 - (x_2 - x_1)y_2$ $(x_3 - x_1)y_2 - (x_3 - x_2)y_1 - (x_2 - x_1)y_3 = 0$ $-(x_3 - x_2)y_1 - (x_1 - x_3)y_2 - (x_2 - x_1)y_3 = 0$ $(x_3 - x_2)y_1 + (x_1 - x_3)y_2 + (x_2 - x_1)y_3 = 0$
(ii)	Sub $y = -2x$, $\mathrm{g}(\mathrm{f}(x - 2x)) = \mathrm{f}(x) + (2x - 2x)\mathrm{g}(-2x)$ $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x)$ Sub $x = -x - y$, $\mathrm{g}(\mathrm{f}(-x - y + y)) = \mathrm{f}(-x - y) + (2(-x - y) + y)\mathrm{g}(y)$ $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(-x - y) - (2x + y)\mathrm{g}(y)$ but $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x)$, so $\mathrm{f}(x) = \mathrm{f}(-x - y) - (2x + y)\mathrm{g}(y)$ $\mathrm{f}(-x - y) = \mathrm{f}(x) + (2x + y)\mathrm{g}(y)$
(iii)	Let $x = -b$ and $y = a + b$ $\mathrm{f}(b - a - b) = \mathrm{f}(-b) + (-2b + a + b)\mathrm{g}(a + b)$ $\mathrm{f}(-a) = \mathrm{f}(-b) + (a - b)\mathrm{g}(a + b)$ WLOG, we have $\mathrm{f}(-b) = \mathrm{f}(-c) + (b - c)\mathrm{g}(b + c)$ $\mathrm{f}(-c) = \mathrm{f}(-a) + (c - a)\mathrm{g}(c + a)$
(iv)	Adding the three equations in (iii), $\mathrm{f}(-a) + \mathrm{f}(-b) + \mathrm{f}(-c) = \mathrm{f}(-a) + \mathrm{f}(-b) + \mathrm{f}(-c)$ $+ (a - b)\mathrm{g}(a + b) + (b - c)\mathrm{g}(b + c) + (c - a)\mathrm{g}(c + a)$ we have, $(a - b)\mathrm{g}(a + b) + (b - c)\mathrm{g}(b + c) + (c - a)\mathrm{g}(c + a) = 0$ Let $\mathrm{g}(a + b) = y_1$ then $x_1 = a + b$ $\mathrm{g}(b + c) = y_2$ then $x_2 = b + c$ $\mathrm{g}(c + a) = y_3$ then $x_3 = c + a$ and $[(a + c) - (b + c)]\mathrm{g}(a + b) + [(a + b) - (c + a)]\mathrm{g}(b + c) + [(b + c) - (a + b)]\mathrm{g}(c + a) = 0$ by (i), $(a + b, \mathrm{g}(a + b)), (b + c, \mathrm{g}(b + c))$ and $(c + a, \mathrm{g}(c + a))$ are collinear points on g(x), thus g(x) is linear. Let $\mathrm{g}(x) = Ax + B$ where A and B are constants. From (ii), sub $x = 0$ and $y = -y$

	$\mathrm{f}(y) = \mathrm{f}(0) + (-y)(A(-y) + B)$ $= Ay^2 - By + C \quad$ where $C = \mathrm{f}(0)$ $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x) \Rightarrow \quad \mathrm{g}(Ax^2 + Bx + C) = Ax^2 - Bx + C$ $A^2x^2 + ABx + AC + B = Ax^2 - Bx + C \quad ---(1)$ Comparing coefficient, $A^2 = A \Rightarrow A = 0 \text{ or } 1$ When $A = 0$, then (1) becomes $B = -Bx + C$ then $B = C = 0$ hence $\mathrm{f}(x) = \mathrm{g}(x) = 0$ When $A = 1$, then (1) becomes $x^2 + Bx + C + B = x^2 - Bx + C$ $2Bx + B = 0$ $B = 0$ then $\mathrm{f}(x) = x^2 + C$ and $\mathrm{g}(x) = x$

Q8	Sequences and Series
(a)(i)	For $k \geq 2,\ m^m \geq 2^m$ $$\frac{1}{m^m} \leq \frac{1}{2^m}$$ $x_n = \sum_{m=1}^{n} \frac{1}{m^m}$ $= 1 + \sum_{m=2}^{n} \frac{1}{m^m}$ $\leq 1 + \sum_{m=2}^{n} \frac{1}{2^m}$ for $m \geq 2$ $\leq 1 + \dfrac{\frac{1}{2^2}\left(1 - \frac{1}{2^{n-1}}\right)}{1 - \frac{1}{2}}$ $\leq 1 + \dfrac{\frac{1}{2^2}}{1 - \frac{1}{2}} = \dfrac{3}{2}$ $\therefore x_n \leq \dfrac{3}{2}$ for all $n \geq 1$ (shown)
(ii)	$x_n - x_{n-1} = \dfrac{1}{n^n} > 0,$ hence the sequence $\{x_n\}$ is strictly increasing By Monotone Convergence Theorem, since the sequence is bounded above by $\dfrac{3}{2}$ and is strictly increasing, the sequence $\{x_n\}$ is convergent.
(b)(i)	**Show base case *r* =1 is true** $LHS = F_1 = 1$ $RHS = 2^0 F_1 = 2 \geq LHS$ $\therefore F_r \leq 2^{r-1} F_1$ is true when r=1 **Assume *case r* = *k* is true** Assume $F_k \leq 2^{k-1} F_1$ is true for some $k \geq 1$ **To show *case r* = *k*+1 is true:** $F_{k+1} \leq 2^k F_1$ $F_{k+1} = F_k + F_{k-1}$ $\leq 2F_k$ since $F_{k-1} \leq F_k$ $\leq 2\left(2^{k-1} F_1\right)$ $\therefore F_{k+1} \leq 2^k F_1$ By Mathematical Induction, since $F_r \leq 2^{r-1} F_1$ is true when r=1, and by assuming $F_k \leq 2^{k-1} F_1$ is true for all $k \geq 1$, and $F_{k+1} \leq 2^k F_1$ is true when r=k+1. Therefore $F_r \leq 2^{r-1} F_1$ for all $r \geq 1$.

(ii)	$LHS = 81\sum_{r=1}^{n}\frac{F_{r+1}}{9^{r+1}} - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \sum_{r=1}^{n}\frac{F_{r-1}}{9^{r-1}}$ replace r with r -1 replace r with $r+1$ $= 81\sum_{r=2}^{n+1}\frac{F_r}{9^r} - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \sum_{r=0}^{n-1}\frac{F_r}{9^r}$ $= 81\left(\sum_{r=1}^{n+1}\frac{F_r}{9^r} - \frac{F_1}{9}\right) - 9\sum_{r=1}^{n}\frac{F_r}{9^r} - \left(\sum_{r=1}^{n-1}\frac{F_r}{9^r} + \frac{F_0}{9^0}\right)$ $= 81S_{n+1} - 9F_1 - 9S_n - S_{n-1} - F_0$ $= 81(S_n + u_{n+1}) - 9F_1 - 9S_n - (S_n - u_n) - F_0$ where u_n is the n^{th} term $= (81-9-1)S_n - 9F_1 - F_0 + 81\frac{F_{n+1}}{9^{n+1}} + \frac{F_n}{9^n}$ $= 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ =RHS (shown)
(iii)	$\sum_{r=1}^{\infty}\frac{F_r}{9^r} = S_\infty = \lim_{n\to\infty} S_n$ From b(ii), $\sum_{r=1}^{n}\frac{1}{9^{r-1}}(F_{r+1} - F_r - F_{r-1}) = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ $0 = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$ since $F_{r+1} = F_r + F_{r-1}$ for $r \geq 1$ $S_n = \frac{1}{71}\left(9F_1 + F_0 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $= \frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ From b(i),

	$F_r \le 2^{r-1}F_1$ $F_n \le 2^{n-1}F_1$ $\le 2^{n-1}$ $\le 2^n\left(\frac{1}{2}\right)$ $\therefore \frac{F_n}{9^n} \le \frac{2^n}{9^n}\left(\frac{1}{2}\right)$ and $\frac{F_{n+1}}{9^{n-1}} \le \frac{2^{n+1}}{9^{n-1}}\left(\frac{1}{2}\right)$ As $n \to \infty, \left(\frac{2}{9}\right)^n\left(\frac{1}{2}\right) \to 0$ and $\left(\frac{2}{9}\right)^{n+1}(81)\left(\frac{1}{2}\right) \to 0,$ $\therefore \frac{F_n}{9^n} \to 0$ and $\frac{F_{n+1}}{9^{n-1}} \to 0$ $\sum_{r=1}^{\infty}\frac{F_r}{9^r} = S_\infty$ $= \lim_{n\to\infty}\frac{1}{71}\left(9 - \frac{F_n}{9^n} - \frac{F_{n+1}}{9^{n-1}}\right)$ $= \frac{9}{71}$ (shown)
(iv)	$\frac{9}{71} = \sum_{r=1}^{\infty}\frac{F_r}{9^r}$ $= \frac{F_1}{9^1}+\frac{F_2}{9^2}+\frac{F_3}{9^3}+\frac{F_4}{9^4}+\frac{F_5}{9^5}+\frac{F_6}{9^6}+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $= \frac{1}{9^1}+\frac{1}{9^2}+\frac{2}{9^3}+\frac{3}{9^4}+\frac{5}{9^5}+\frac{8}{9^6}+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $= 0.1267572506+\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ $\frac{1}{71} = 0.014084139+\frac{1}{9}\sum_{r=7}^{\infty}\frac{F_r}{9^r}$ where $\frac{1}{9}\sum_{r=7}^{\infty}\frac{F_r}{9^r} < \frac{1}{9}(2\times10^{-6}) = 2.\dot{2}\times10^{-7} = 0.000000\dot{2}$ Hence the first 6 digits of the decimal of $\frac{1}{71}$ are 0.014084.