Name:	Index No.:	Class:

PRESBYTERIAN HIGH SCHOOL

ADDITIONAL MATHEMATICS **4049/01**
Paper 1

18 August 2023 Friday 2 hours 15 min

2023 SECONDARY FOUR EXPRESS / FIVE NORMAL (ACADEMIC)
PRELIMINARY EXAMINATIONS

MARK SCHEME

Cynthia – Q1 to 10
Sabrina – Q11 to 13

1 The line $y = 2x + 15$ intersects the curve $y = x^2 + 6x + 3$ at points A and B.
Find the value of p for which the distance AB can be expressed as $p\sqrt{5}$. [5]

$x^2 + 6x + 3 = 2x + 15$	M1 (equate curve to line)
$x^2 + 4x - 12 = 0$	
$(x-2)(x+6) = 0$	M1 (factorise)
$x = 2$ *or* $x = -6$	
$y = 19$ *or* $y = 3$	M1 (find y)
$AB = \sqrt{(2-(-6))^2 + (19-3)^2}$	M1 (apply distance formula)
$AB = \sqrt{320}$	
$AB = 8\sqrt{5}$	
$p = 8$	A1

2 A curve is such that $\dfrac{d^2y}{dx^2} = 12e^{2x} + e^{-x}$. The curve intersects the y-axis at $P(0, 5)$ and the tangent to the curve at P is parallel to $y = 4x + 3$. Find the equation of the curve. [6]

$\dfrac{dy}{dx} = \int \left(12e^{2x} + e^{-x}\right) dx = 6e^{2x} - e^{-x} + c_1$	M1 (any 2 correct terms)
At $(0,5)$, $\dfrac{dy}{dx} = 4$	M1 (seen gradient at P = 4)
$6e^{2(0)} - e^{-(0)} + c_1 = 4$	M1 (sub. gradient at x = 0, attempt to find c_1)
$\Rightarrow c_1 = -1$	
$y = \int \left(6e^{2x} - e^{-x} - 1\right) dx = 3e^{2x} + e^{-x} - x + c$	M1 (any 2 correct terms)
At $(0,5)$, $3e^{2(0)} + e^{-(0)} - 0 + c = 5$	M1 (sub. x = 0 & y = 5, attempt to find c)
$\Rightarrow c = 1$	
$\therefore y = 3e^{2x} + e^{-x} - x + 1$	A1

3 A function is defined by $\mathrm{f}(x) = x^2 + 2kx + 2k + 3$ for all real values of x, where k is a constant.

(a) Find the discriminant of $\mathrm{f}(x)$ in terms of k. [2]

For $\mathrm{f}(x) = x^2 + 2kx + 2k + 3$,

$b^2 - 4ac = (2k)^2 - 4(1)(2k+3)$ M1 (apply discriminant)

$= 4k^2 - 8k - 12$ A1

(b) Show that the discriminant of $\mathrm{f}(x)$ in **part (a)** can be expressed in the form $4(k-a)^2 - b$, where a and b are integers. [2]

$4k^2 - 8k - 12 = 4\left[k^2 - 2k + 1^2 - 1^2\right] - 12$

$= 4\left[(k-1)^2 - 1\right] - 12$ M1 (completing the square)

$= 4(k-1)^2 - 16$ A1

(c) Find the range of values of k for which $\mathrm{f}(x) = 0$ has no real roots. [3]

$b^2 - 4ac < 0$

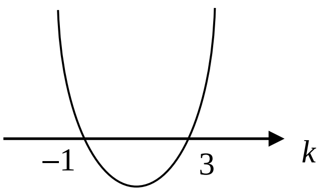
$4(k-1)^2 - 16 < 0$ M1 (apply discriminant < 0)

$(k-1)^2 - 2^2 < 0$

$(k-1+2)(k-1-2) < 0$

$(k+1)(k-3) < 0$ M1 (factorise)

$-1 < k < 3$ A1



4 It is given that $\mathrm{f}(x) = 2x^3 - 5x^2 - 4x + 12$.

(a) Show that $2x+3$ is a factor of $\mathrm{f}(x)$. [2]

$$\mathrm{f}\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 - 5\left(-\frac{3}{2}\right)^2 - 4\left(-\frac{3}{2}\right) + 12 \qquad \text{M1 (apply factor theorem)}$$

$$= -\frac{27}{4} - \frac{45}{4} + 6 + 12$$

$$= 0$$

By the Factor Theorem, $(2x+3)$ is a factor of $\mathrm{f}(x)$. (shown) AG1

(b) Factorise $\mathrm{f}(x)$ completely. [2]

$$\mathrm{f}(x) = 2x^3 - 5x^2 - 4x + 12$$

$$= (2x+3)\left(x^2 - 4x + 4\right) \qquad \text{M1 (long division or comparing coefficients)}$$

$$= (2x+3)(x-2)^2 \qquad \text{A1}$$

(c) Hence find the roots of the equation $2\left(2^{3y}\right) - 5\left(2^{2y}\right) - 4\left(2^{y}\right) + 12 = 0$. [3]

$$2\left(2^y\right)^3 - 5\left(2^y\right)^2 - 4\left(2^y\right) + 12 = 0$$

Let $x = 2^y$,

$$2\left(2^y\right)^3 - 5\left(2^y\right)^2 - 4\left(2^y\right) + 12 = 0$$

$$\left[2\left(2^y\right) + 3\right]\left[\left(2^y\right) - 2\right]^2 = 0$$

$$2\left(2^y\right) + 3 = 0 \quad \text{or} \quad \left(2^y\right) - 2 = 0$$

M1 (attempt to let $x = 2^y$ and solve)

$2^y = -\frac{3}{2}$ *or* $2^y = 2$ M1 (seen either one)

(rejected)

$\therefore y = 1$ A1

5 **(a)** Using long division, show that $\dfrac{x^3-2x^2+5x-10}{x^2+5}=x-2$. [2]

$$
\begin{array}{r l}
 & \quad x-2 \\
x^2+5 \,\big) & \overline{\;x^3-2x^2+5x-10} \\
 & \underline{-(x^3 \qquad\quad +5x)} \\
 & \qquad -2x^2 -10 \\
 & \quad \underline{-(-2x^2-10)} \\
 & \qquad\qquad 0
\end{array}
$$

M1 (attempt to use long division)

$\therefore \dfrac{x^3-2x^2+5x-10}{x^2+5}=x-2$ (shown) A1

(b) Hence, by first expressing the denominator as a product of two factors, express $\dfrac{2x^2+1}{x^3-2x^2+5x-10}$ in partial fractions. [5]

$$\frac{2x^2+1}{x^3-2x^2+5x-10}=\frac{2x^2+1}{(x-2)(x^2+5)}$$

$$\frac{2x^2+1}{(x-2)(x^2+5)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+5}$$ M1 (seen both partial fractions)

$$2x^2+1=A(x^2+5)+(Bx+C)(x-2)$$

Sub. $x=2$,

$9=9A$

$A=1$ M1 (seen substitution or comparing coefficients)

Comparing constant term,

$1=5A-2C$

$1=5-2C$

$C=2$

Comparing x^2 term,

$2=A+B$

$2=1+B$ A2 (any 2 correct)

$B=1$

$\therefore \dfrac{2x^2+1}{x^3-2x^2+5x-10}=\dfrac{1}{x-2}+\dfrac{x+2}{x^2+5}$ A1

6 **(a)** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(2+\frac{ax}{4}\right)^8$,

where a is a non-zero constant. Give each term in its simplest form. [2]

$$\left(2+\frac{ax}{4}\right)^8 = 2^8 + \binom{8}{1}(2)^7\left(\frac{ax}{4}\right) + \binom{8}{2}(2)^6\left(\frac{ax}{4}\right)^2 + \ldots$$ M1 (apply Binomial theorem)

$$\left(2+\frac{ax}{4}\right)^8 = 256 + 256ax + 112a^2x^2 + \ldots$$ A1

(b) Given that the coefficient of x^2 is –320 in the expansion of $(3-x)^2\left(2+\frac{ax}{4}\right)^8$,

find the possible value(s) of a. [4]

$$(3-x)^2\left(2+\frac{ax}{4}\right)^8 = \left(9-6x+x^2\right)\left[256+256ax+112a^2x^2+\ldots\right]$$ M1 (expansion)

$$(9)\left(112a^2\right)+(-6)\left(256a\right)+(1)(256) = -320$$ M1 (comparing)

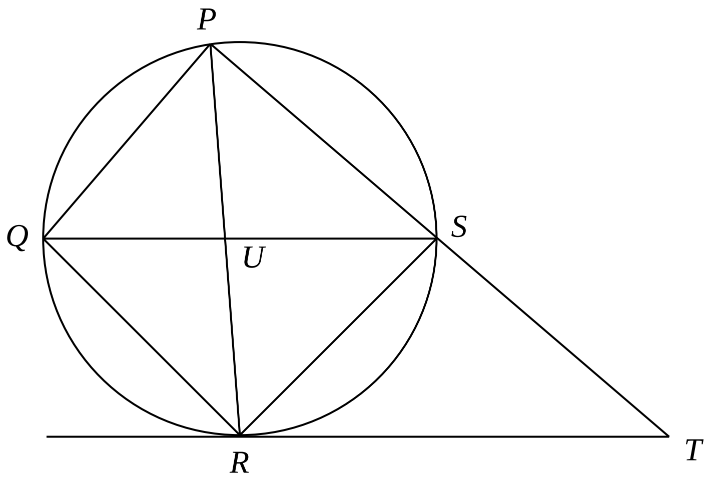
$$1008a^2 - 1536a + 576 = 0$$

$$21a^2 - 32a + 12 = 0$$

$$(3a-2)(7a-6) = 0$$

$$a = \frac{2}{3} \quad or \quad a = \frac{6}{7}$$ A1, A1

7



The diagram shows a quadrilateral $PQRS$ whose vertices lie on the circumference of a circle.

The diagonals PR and QS intersect at U. The tangent at R meets PS produced at T.

If $QR = RS$, prove that

(a) QS // RT, [3]

$\angle RQS = \angle RSQ$ (base $\angle$s of isos. Δ)	B1
$\angle RQS = \angle TRS$ (alt. segment theorem)	B1
Since $\angle RSQ = \angle TRS$, $\therefore QR // RT$ (alt. $\angle$s are equal)	AG1

(b) triangle PQR is similar to triangle QUR. [3]

$\angle RPQ = \angle RSQ$ ($\angle$s in the same segment)	B1
$\angle RPQ = \angle RSQ = \angle RQS$ (from part (a))	
$\angle PRQ = \angle QRU$ (common $\angle$)	B1
Triangle PQR is similar to triangle QUR. (AA similarity)	AG1

8 **(a)** The equation of a curve is $y = \ln\left(xe^{-3x}\right)$. The normal to the curve at the point P has a gradient of $\frac{1}{2}$. Find the coordinates of P. [4]

$y = \ln\left(xe^{-3x}\right) = \ln x - 3x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x} - 3$ M1

Gradient at point $P = -1 \div \frac{1}{2} = -2$ M1

$$\left.\begin{aligned} -2 &= \frac{1}{x} - 3 \\ 1 &= \frac{1}{x} \\ x &= 1 \end{aligned}\right\}$$ M1 (equate $\frac{dy}{dx} = -2$ & attempt to solve for x)

$y = \ln\left(e^{-3}\right) = -3$

Coordinates of P = (1, –3) A1

(b) The normal to the curve at P meets the x-axis at Q.
Find the area of triangle OQP, where O is the origin. [3]

$y - (-3) = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}x - \frac{7}{2}$ M1 (find equation of normal)

At Q, $\frac{1}{2}x - \frac{7}{2} = 0$

$x = 7$ M1 (find x-intercept)

Area of triangle OQP

$= \frac{1}{2} \times 7 \times 3 = 10.5 \text{ units}^2$ A1

9 Atmospheric pressure is a measure of the force exerted by the mass of air on an object.

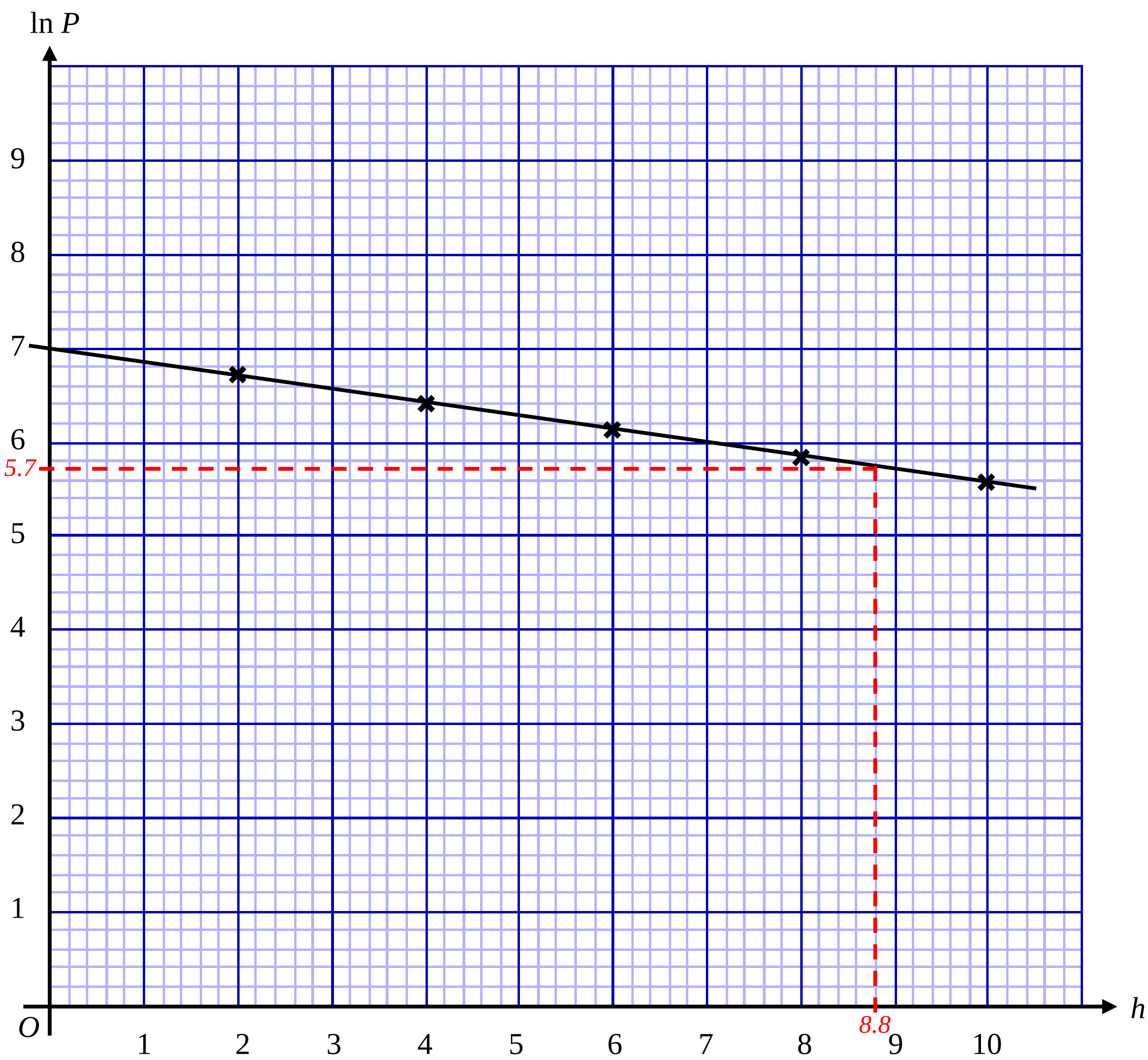
The atmospheric pressure, P millibars, exerted at the altitude h kilometres is related by the equation $P = Ae^{bh}$, where A and b are constants.

The following table shows the mean atmospheric pressure at various altitudes.

h (kilometres)	2	4	6	8	10
P (millibars)	810	595	446	340	262

(a) Plot $\ln P$ against h and draw a straight line graph to illustrate the information. [2]

h	2	4	6	8	10
$\ln P$	6.70	6.39	6.10	5.83	5.57



B1 (plot at least 3 correct points) B1 (best fit line)

(b) Express the equation $P = Ae^{bh}$ in a form that will yield the straight line graph in **part (a)**.
Hence explain how the graph may be used to determine the value of *A* and of *b*. [3]

$P = Ae^{bh}$
$\ln P = \ln Ae^{bh}$
$\ln P = \ln A + \ln e^{bh}$
$\ln P = bh + \ln A$ B1

The value of *A* can be determined by finding the **vertical intercept** of the graph. B1

The value of *b* can be determined by finding the **gradient** of the graph. B1

(c) Use your graph to estimate the atmospheric pressure, to the nearest millibar, when an object is at sea level. [1]

At sea level, $h = 0$,
$\ln P = 7$
$P = e^7 = 1096.633 \approx 1097$ millibars (nearest whole) B1

(d) The atmospheric pressure at the summit of Mount Everest is 300 millibars.
Use your graph to estimate the altitude of Mount Everest. [1]

When $P = 300$,
$\ln P = \ln 300 = 5.70$

From the graph,
$h = 8.8$ km B1

10 A patient's blood pressure, $\mathrm{P}(t)$ in mmHg, can be modelled by the function

$$\mathrm{P}(t) = 22\cos(2.5\pi t) + 116,$$

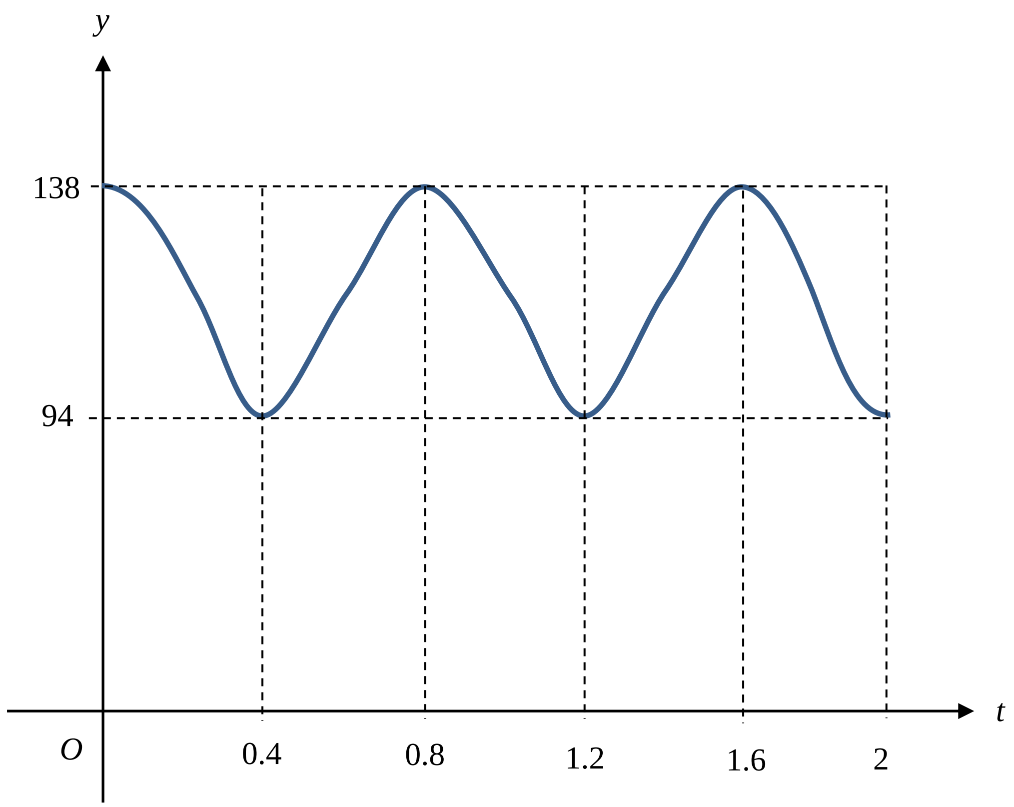
where t is the time in seconds.

The systolic pressure (highest pressure) occurs when the heart beats, and the diastolic pressure (lowest pressure) occurs when the heart is at rest between beats.

(a) State the amplitude and period of $\mathrm{P}(t) = 22\cos(2.5\pi t) + 116$. [2]

Amplitude = 22	B1
Period = $\dfrac{2\pi}{2.5\pi} = 0.8$	B1

(b) Sketch the graph of $y = \mathrm{P}(t)$ for $0 \le t \le 2$. [2]



B1 (correct shape) B1 (correct amplitude & period)

(c) The pulse rate is the number of times a heart beats per minute.
A normal resting pulse rate should be between 60 to 100 beats per minute.
Show that the patient's pulse rate is normal. [2]

Since the duration of 1 heart beat is 0.8 sec,	M1
Patient's pulse rate = $\frac{60}{0.8} = 75$ beats per minute Hence the patient's pulse rate is normal.	AG1

(d) According to health guidelines, someone with systolic pressure above 140 mmHg or diastolic pressure above 90 mmHg has high blood pressure and should see a doctor.
Determine whether the patient needs to see a doctor. Justify your answer. [1]

Since the **diastolic pressure** (94 mmHg) is **above 90 mmHg**, the patient has high blood pressure and **should see the doctor**.	B1

11 A particle moves in a straight line so that t seconds after passing through a fixed point O, its velocity v m/s is given by $v = 5\cos\left(\frac{t}{2}\right)$. Find

(a) the initial velocity of the particle, [1]

Initial velocity = $5\cos\left(\frac{0}{2}\right) = 5$ m/s B1

(b) the value of t, in terms of π, when the particle first comes to instantaneous rest, [3]

At instantaneous rest, $5\cos\left(\frac{t}{2}\right) = 0$ M1

$\frac{t}{2} = \cos^{-1}(0) = \frac{\pi}{2}$ M1

$t = \pi$ s A1

(c) the distance travelled by the particle in the first 5 seconds, after passing through O. [4]

(a)

$s = \int 5\cos\left(\frac{t}{2}\right)\mathrm{d}t = \dfrac{5\sin\left(\frac{t}{2}\right)}{\frac{1}{2}} + c$ M1

$s = 10\sin\left(\frac{t}{2}\right) + c$

When $t = 0$, $s = 0$, $\Rightarrow c = 0$ M1

When $t = \pi$, $s = 10\sin\left(\frac{\pi}{2}\right) = 10$

When $t = 5$, $s = 10\sin\left(\frac{5}{2}\right) = 5.984$

M1(seen either one)

Distance $= 10 + (10 - 5.984)$

$= 14.016$

≈ 14.0 m A1

12 A curve has the equation $y = 3 + \left(\frac{x}{2} - 1\right)^4$. The point (p, q) is the stationary point on the curve.

(a) Determine the coordinates of the stationary point (p, q). [4]

$$y = 3 + \left(\frac{x}{2} - 1\right)^4$$

$$\frac{dy}{dx} = 4\left(\frac{x}{2} - 1\right)^3 \bullet \frac{1}{2} = 2\left(\frac{x}{2} - 1\right)^3$$ M1 (find 1st derivative)

Let $\frac{dy}{dx} = 0$,

$$2\left(\frac{x}{2} - 1\right)^3 = 0$$

$$\frac{x}{2} - 1 = 0$$ M1 (equate to zero and attempt to find x)

$$x = 2$$

$$\Rightarrow y = 3$$

Stationary point = $(2, 3)$ A1, A1 (correct pair of coordinates)

(b) **(i)** Justify whether y is increasing or decreasing for values of x less than p. [2]

For $x < 2$,

$$\left(\frac{x}{2} - 1\right)^3 < 0$$

$$\frac{dy}{dx} = 2\left(\frac{x}{2} - 1\right)^3 < 0$$

M1 (use $\left(\frac{x}{2} - 1\right)^3 < 0$ to show dy/dx > 0)

Therefore, y is **decreasing** when $x < 2$. A1

(ii) Hence infer whether y is increasing or decreasing for values of x greater than p. [1]

For $x > 2$,

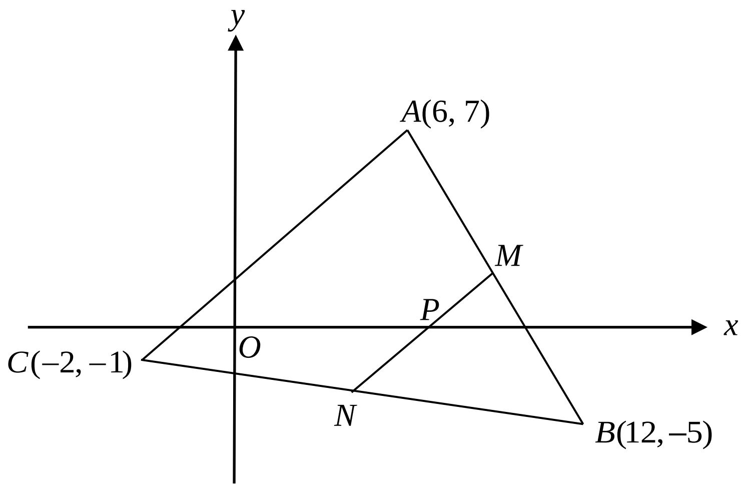
$$\frac{dy}{dx} = \left(\frac{x}{2} - 1\right)^3 > 0$$

Therefore, y is **increasing** when $x > 2$. B1

(c) What do the results of **part (b)** imply about the stationary point? [1]

The stationary point is a **minimum point**. B1

13 Solutions to this question by accurate drawing will not be accepted.



The diagram above shows a triangle ABC with vertices at $A(6, 7)$, $B(12, -5)$ and $C(-2, -1)$.
M and N are the mid-points of AB and BC respectively. The line MN cuts the x-axis at P.

(a) Find the coordinates of P. [4]

$M = \left(\frac{12+6}{2}, \frac{-5+7}{2}\right) = (9,1)$ and $N = \left(\frac{12+(-2)}{2}, \frac{-5+(-1)}{2}\right) = (5,-3)$ M1

gradient of $MN = \frac{1-(-3)}{9-5} = 1$ M1 (apply gradient formula)

Let $P = (x, 0)$, gradient of $NP = \frac{0-(-3)}{x-5} = 1$ M1 (find x)

$\Rightarrow x = 8$

$\therefore P = (8,0)$ A1

(b) Find the ratio $AC : MN$. [1]

$AC : MN = 2:1$ B1

(c) Find the area of the quadrilateral *ACNM*. [2]

Area of trapezium *ACNM* $= \frac{1}{2}\begin{vmatrix} 6 & -2 & 5 & 9 & 6 \\ 7 & -1 & -3 & 1 & 7 \end{vmatrix}$

$= \frac{1}{2}[-6+6+5+63-(-14)-(-5)-(-27)-6]$ M1

$= 54 \text{ units}^2$ A1

(d) Explain why quadrilateral *ACNM* is a trapezium. [2]

By midpoint theorem, $AC // MN$	**<u>OR</u>**	$\text{gradient}_{AC} = \text{gradient}_{MN} = 1$ $\Rightarrow AC // MN$	M1

Since quadrilateral *ACNM* has **one pair of parallel sides**, it is a trapezium. AG1