(Pure Mathematics) Chapter 2: Inequalities

Objectives

At the end of the chapter, you should be able to:

- solve quadratic inequalities.

Content

References

1. New Additional Mathematics, Ho Soo Thong (Msc, Dip Ed), Khor Nyak Hiong (Bsc, Dip Ed)

2. New Syllabus Additional Mathematics (7th Edition), Shinglee Publishers Ptd Ltd

2.1 Definition of Inequality

An inequality is a mathematical statement involving any of '<' , '>' , '≤' and '≥'.

Examples: $2x \ge 6$, $\sin x + 4\cos x < \sqrt{\frac{2}{3}}$, $x^3 - 5x \le \frac{2}{x}$.

2.1.1 Basic Rules of Inequalities

Let $a, b, c \in \mathbb{R}$.

Basic Rules	Comments	Examples
If $a < b$, then (i) $a + c < b + c$ (ii) $a - c < b - c$	Addition and subtraction of the same number on both sides of the inequality **does not change** the inequality sign.	Given $3 < 6$, then $3 + 2 < 6 + 2$; $3 - 1 < 6 - 1$.
If $a < b$ and $c > 0$, then (i) $ac < bc$ (ii) $\frac{a}{c} < \frac{b}{c}$	Multiplication and division of the same **positive** number at both sides of the inequality **does not change** the inequality sign.	Given $3 < 5$, then $3 \times 2 < 5 \times 2$; $\frac{3}{4} < \frac{5}{4}$.
If $a < b$ and $c < 0$, then (i) $ac > bc$ (ii) $\frac{a}{c} > \frac{b}{c}$	Multiplication and division of the same **negative** number at both sides of the inequality **always change** the inequality sign.	Given $3 < 7$, then $3 \times (-2) > 7 \times (-2)$; $\frac{3}{-3} > \frac{7}{-3}$.
If $a < b$ and $b < c$, then $a < c$.	Inequalities are **transitive**.	If $-3 < 2$ and $2 < 5$, then $-3 < 5$.
If a and b are **both** positive, $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$	Taking reciprocal of the two **positive** numbers in an inequality **always changes** the inequality sign.	If $2 < 5$, then $\frac{1}{2} > \frac{1}{5}$.
If $a < b$ and $c < d$, then $a + c < b + d$ Note: $a < b$ and $c < d$ does not imply $a - c < b - d$		If $1 < 2$ and $3 < 5$, then $1 + 3 < 2 + 5$. But $1 < 2$ and $3 < 5$, $\Rightarrow 1 - 3 \not< 2 - 5$.
$ab > 0 \Leftrightarrow$ either ($a > 0$ and $b > 0$) or ($a < 0$ and $b < 0$) $ab < 0 \Leftrightarrow$ either ($a > 0$ and $b < 0$) or ($a < 0$ and $b > 0$)		

Common Mistakes	Comments	Counter Examples
$x<3$ and $y<4$ $\Rightarrow x-y<-1$	**Do not subtract inequalities** in this manner as it may lead to a sign change.	$2<3$ and $0<4$ but $2-0 \not< 3-4$
$\frac{3}{x-1}>-5$ $\Rightarrow 3>-5(x-1)$	**Do not cross multiply** unless the terms are positive, as it may lead to a sign change.	Let $x=-2$ $\frac{3}{(-2-1)}>-5$ but $3 \not> -5(-2-1)=15$
$x<2 \Rightarrow x^2<4$	**Do not square both sides** as the terms may have different signs, as it may lead to a sign change.	$-4<2$ but $(-4)^2 \not< 2^2$

Q: What do x and y represent in the table directly above?
A: They represent unknown constants (which might be positive **or** negative).

Q: What does it mean to “solve an inequality”?
A: It is to find the range of values of the quantity you are interested in (usually x) which satisfy the given inequality.

Example 1 Linear Inequality
Solve the inequality $6-3x<2$.

Solution:

$6-3x<2$

$-3x<2-6$

$-3x<-4$

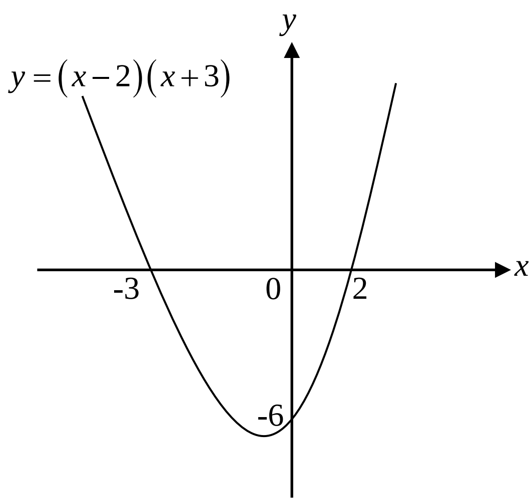
$x>\frac{-4}{-3}$

$x>1\frac{1}{3}$

2.2 Solving Quadratic Inequalities

Consider the quadratic inequality $(x-2)(x+3)<0$. The range of values of x which satisfies the inequality can be found from the quadratic curve $y=(x-2)(x+3)$ as follows:

For $(x-2)(x+3)<0$ (i.e. y is *negative*), we choose the interval for which the curve is *below* the x-axis.
$(x-2)(x+3)<0 \Rightarrow -3<x<2$



For $(x-2)(x+3)>0$ (i.e. y is *positive*), we choose the interval for which the curve is *above* the x-axis.
$(x-2)(x+3)>0 \Rightarrow x<-3$ or $x>2$

Note that $(x-2)(x+3)=0 \Rightarrow x=2$ or $x=-3$.

For $(x-2)(x+3)\le 0$ (i.e. $y\le 0$), we choose the interval for which the curve is *on* and *below* the x-axis.
$(x-2)(x+3)\le 0 \Rightarrow -3\le x\le 2$

For $(x-2)(x+3)\ge 0$ (i.e. $y\ge 0$), we choose the interval for which the curve is *on* and *above* the x-axis.
$(x-2)(x+3)\ge 0 \Rightarrow x\le -3$ or $x\ge 2$

A guide to solving quadratic inequalities:

1. Collect all the terms to one side of the inequality so that the other side is zero.
2. Find the x-intercepts of the graph by factorization or by using the quadratic formula.
3. Sketch the quadratic function and identify the required region.
4. Solve.

Example 2

Find the exact solution set of the following inequalities.

(a) $\left(x-1\right)^2 > 9$ (b) $2+3x-2x^2 \le 0$

(c) $(x+3)(x+4) \le 6$ (d) $x^2 < 1+2x$

(e) $x^2 + 7 > 4x$ (f) $x^2 + 2x + 5 < 0$

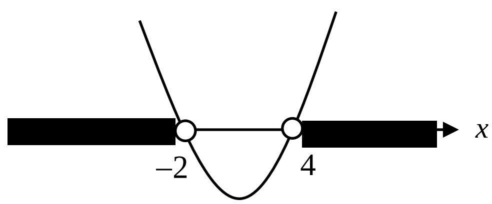
Solution:

(a) $\left(x-1\right)^2 > 9$

$\left(x-1\right)^2 - 3^2 > 0$

$\left(x-1-3\right)\left(x-1+3\right) > 0$

$(x-4)(x+2) > 0$



$x < -2$ or $x > 4$

The solution set is $\{x \in \mathbb{R} : x < -2 \quad \text{or} \quad x > 4\}$ or $\left(-\infty, -2\right) \cup \left(4, \infty\right)$

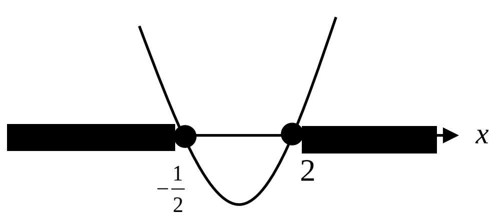
Common mistake:

Note that $(x-4)(x+2) > 0$ does **NOT** mean $x > 4$ or $x > -2$.

(b) $2+3x-2x^2 \le 0$

$\left(2-x\right)\left(1+2x\right) \le 0$

$\left(x-2\right)(1+2x) \ge 0$



$x \le -\dfrac{1}{2}$ or $x \ge 2$

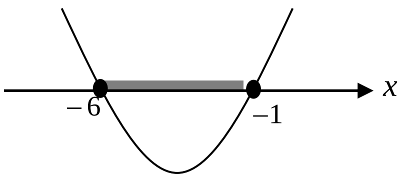
The solution set is $\{x \in \mathbb{R} : x \le -\dfrac{1}{2} \text{ or } x \ge 2\}$ or $(-\infty, -\dfrac{1}{2}] \cup [2, \infty)$

(c) $(x+3)(x+4) \le 6$

$x^2+7x+12-6 \le 0$

$x^2+7x+6 \le 0$

$(x+6)(x+1) \le 0$



$-6 \le x \le -1$

The solution set is $\{x \in \mathbb{R} : -6 \le x \le -1\}$ or $[-6, -1]$

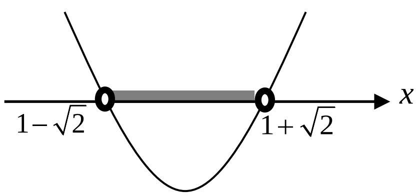
(d) $x^2 < 1+2x$

$x^2-2x-1<0$

$(x-1)^2-2<0$

$(x-1)^2-(\sqrt{2})^2<0$

$(x-1-\sqrt{2})(x-1+\sqrt{2})<0$



$1-\sqrt{2} < x < 1+\sqrt{2}$

The solution set is $\{x \in \mathbb{R} : 1-\sqrt{2} < x < 1+\sqrt{2}\}$

Alternatively, we can solve it this way:

Let $x^2-2x-1=0$.

Then

$$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-1)}}{2(1)}$$

$\vdots$

$=1 \pm \sqrt{2}$

$x^2 < 1+2x$

$x^2-2x-1<0$

$\left[x-\left(1+\sqrt{2}\right)\right]\left[x-\left(1-\sqrt{2}\right)\right]<0$

$(x-1-\sqrt{2})(x-1+\sqrt{2})<0$

(Then sketch the curve on the left and write the answer)

(e) $x^2 + 7 > 4x$

$x^2 - 4x + 7 > 0$

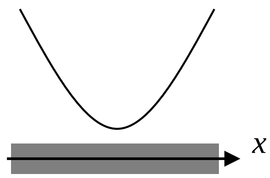
$(x-2)^2 - 2^2 + 7 > 0$

$(x-2)^2 + 3 > 0$

Since $(x-2)^2 \geq 0$,

$(x-2)^2 + 3 \geq 3 > 0$ for all real values of *x*.

Thus, all real values of *x* will satisfy the inequality given in the question.



The solution set is $\mathbb{R}$

Alternatively, we can solve it this way:
Let $x^2 - 4x + 7 = 0$.
Discriminant
$= (-4)^2 - 4(1)(7) = -12 < 0$
Since coefficient of $x^2 > 0$ and discriminant < 0, therefore
$x^2 - 4x + 7 > 0$ for all real values of *x*.
Thus, all real values of *x* will satisfy the inequality given in the question.
The solution set is $\mathbb{R}$

(f) $x^2 + 2x + 5 < 0$

$(x+1)^2 - 1^2 + 5 < 0$

$(x+1)^2 + 4 < 0$

Since $(x+1)^2 \geq 0$,

$(x+1)^2 + 4 \geq 4 > 0$ for all real values of *x*.

Thus, there is no real value of *x* that satisfy the inequality given in the question.
The solution set is $\varnothing$

Alternatively, we can solve it this way:
Let $x^2 + 2x + 5 = 0$.
Discriminant
$= 2^2 - 4(1)(5) = -16 < 0$
Since coefficient of $x^2 > 0$ and discriminant < 0, therefore
$x^2 + 2x + 5 > 0$ for all real values of *x*.
Thus, there is no real value of *x* that satisfy the inequality given in the question.
The solution set is $\varnothing$

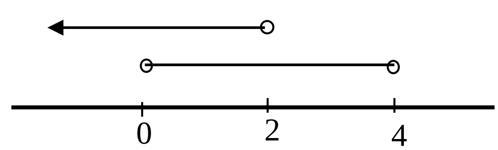
2.3 Solving Simultaneous Inequalities

In inequalities, $a < x < b$ means $x > a$ **and** $x < b$.

"and" in this context has the meaning of "intersection" which is represented by $\cap$.

The simultaneous inequalities

$0 < x < 4$ **and** $x < 2$ give $0 < x < 2$.



"or" in inequalities has the meaning of "union" which is represented by $\cup$.

The simultaneous inequalities
$0 < x < 4$ **or** $x < 2$ give $x < 4$.

Exercise 1

1. Find the exact solution set of the following inequalities.

(a) $(x-2)^2 < 4$

(b) $x^2 \le \frac{7x}{2} + 36$

(c) $4x + 1 - x^2 < 0$

(d) $\frac{5x^2 - 2x - 4}{x^2 + 1} \le 4$

(e) $x^2 - 2x + 5 < 0$

(f) $x^2 - 2x + 5 > 0$

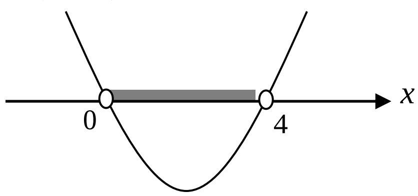
Solution:

(a) $(x-2)^2 < 4$

$(x-2)^2 - 2^2 < 0$

$(x-2-2)(x-2+2) < 0$

$(x-4)x < 0$



$0 < x < 4$

The solution set is $\{x \in \mathbb{R} : 0 < x < 4\}$

(b) $x^2 \le \frac{7x}{2} + 36$

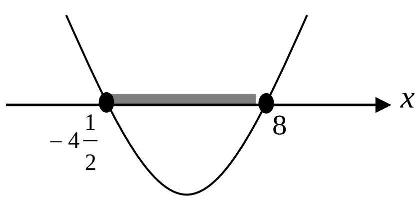
$2x^2 - 7x - 72 \le 0$

$(2x+9)(x-8) \le 0$

Consider $2x^2 - 7x - 72 = 0$

$x = \frac{-(-7) \pm \sqrt{49 - 4(2)(-72)}}{2(2)}$

$x = \frac{7 \pm 25}{4} \Rightarrow x = -4\frac{1}{2}$ or $x = 8$



$-4\frac{1}{2} \le x \le 8$

The solution set is $\left\{x \in \mathbb{R} : -4\frac{1}{2} \le x \le 8\right\}$

(c) $4x+1-x^2<0$

$x^2-4x-1>0$

$(x-2)^2-2^2-1>0$

$(x-2)^2-5>0$

$(x-2)^2-\left(\sqrt{5}\right)^2>0$

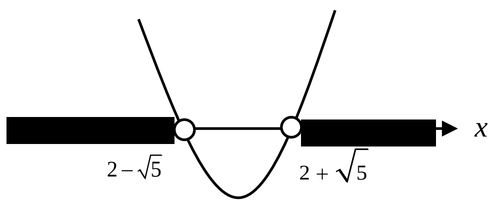
$\left(x-2-\sqrt{5}\right)\left(x-2+\sqrt{5}\right)>0$

$(x-\left(2+\sqrt{5}\right))(x-\left(2-\sqrt{5}\right))>0$

Consider $x^2-4x-1=0$

$x=\dfrac{-(-4)\pm\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$

$x=\dfrac{4\pm\sqrt{20}}{2}=\dfrac{4\pm2\sqrt{5}}{2}=2\pm\sqrt{5}$



$x<2-\sqrt{5}$ or $x>2+\sqrt{5}$

The solution set is $\left\{x\in\mathbb{R}: x<2-\sqrt{5} \text{ or } x>2+\sqrt{5}\right\}$

(d) $\dfrac{5x^2-2x-4}{x^2+1}\leq 4$

Since $x^2+1>0,\quad x\in\Re$

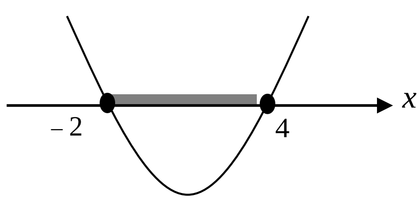
$5x^2-2x-4\leq 4\left(x^2+1\right)$

$5x^2-2x-4-4\left(x^2+1\right)\leq 0$

$5x^2-2x-4-4x^2-4\leq 0$

$x^2-2x-8\leq 0$

$(x-4)(x+2)\leq 0$



The solution set is $\left\{x\in\mathbb{R}: -2\leq x\leq 4\right\}$

(e) $x^2-2x+5<0$

$(x-1)^2+4<0$

Since $(x-1)^2\geq 0$, $(x-1)^2+4\geq 4>0$ for all real values of x.

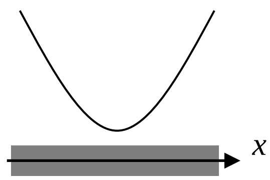
Thus, there is no real value of x that satisfy the inequality given in the question.
The solution set is $\varnothing$

(f) $x^2 - 2x + 5 > 0$

$(x-1)^2 + 4 > 0$

Since $(x-1)^2 \ge 0$, $(x-1)^2 + 4 \ge 4 > 0$ for all real values of x.

Thus, all real values of x will satisfy the inequality given in the question.



The solution set is $\mathbb{R}$

2. Find the range of values of x for which $x^2 - 5x + 1$ lies between -5 and 15 inclusive.

Solution:

$-5 \le x^2 - 5x + 1 \le 15$

$-5 \le x^2 - 5x + 1$ and $x^2 - 5x + 1 \le 15$

$x^2 - 5x + 6 \ge 0$ and $x^2 - 5x - 14 \le 0$

$(x-2)(x-3) \ge 0$ $\quad$ $(x-7)(x+2) \le 0$

$x \le 2$ or $x \ge 3$ $\quad$ $-2 \le x \le 7$

The intersection of the above two inequalities gives $-2 \le x \le 2$ or $3 \le x \le 7$

3. Find the range of values of k for which the equation $x^2 + 2kx + 3k + 4 = 0$ has real roots.

Solution:

Since $x^2 + 2kx + 3k + 4 = 0$ has real roots,

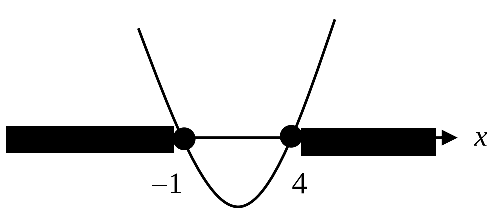
Discriminant ≥ 0

$(2k)^2 - 4(1)(3k+4) \ge 0$

$4k^2 - 12k - 16 \ge 0$

$k^2 - 3k - 4 \ge 0$

$(k-4)(k+1) \ge 0$



$k \le -1$ or $k \ge 4$

4. Find the range of possible values of the constant k if the quadratic function $kx^2+4k-kx-2x$ is always (i) positive (ii) negative.

Solution:

(i) Since the $kx^2+4k-kx-2x$ is always positive, the following 2 conditions must be met:

(1) Discriminant < 0 and (2) $k>0\cdots(2)$

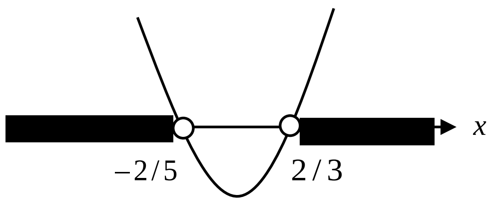
$(-k-2)^2-4(k)(4k)<0$

$k^2+4k+4-16k^2<0$

$-15k^2+4k+4<0$

$15k^2-4k-4>0$

$(5k+2)(3k-2)>0\cdots(1)$



$k<-\dfrac{2}{5}$ or $k>\dfrac{2}{3}$

Combining (1) and (2) gives $k>\dfrac{2}{3}$

(ii) Since the $kx^2+4k-kx-2x$ is always negative, the following 2 conditions must be met:

(1) Discriminant < 0 and (2) $k<0$ --- (2)

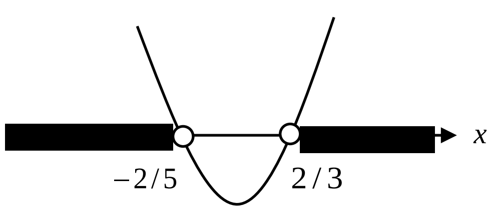
$(-k-2)^2-4(k)(4k)<0$

$k^2+4k+4-16k^2<0$

$-15k^2+4k+4<0$

$15k^2-4k-4>0$

$(5k+2)(3k-2)>0\cdots(1)$



$k<-\dfrac{2}{5}$ or $k>\dfrac{2}{3}$

Combining (1) and (2) gives $k<-\dfrac{2}{5}$.

5. The curve C has equation $x^2+(y-1)^2=4$. Find the range of values of k such that the line $y=x+k+1$ intersects C exactly twice.

Solution:

Substitute $y=x+k+1$ into $x^2+(y-1)^2=4$,

$x^2+(x+k)^2=4$

$x^2+x^2+2kx+k^2=4$

$2x^2+2kx+k^2-4=0$

Since line cuts C twice, Discriminant > 0

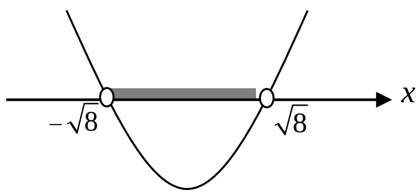
$(2k)^2-4(2)(k^2-4)>0$

$4k^2-8k^2+32>0$

$-4k^2+32>0$

$k^2-8<0$

$(k+\sqrt{8})(k-\sqrt{8})<0$



$-\sqrt{8}<k<\sqrt{8}$

Practise Questions

1. Find the exact solution set of the following inequality

$$x^2 - 3 > 6x$$

Solution:

$x^2 - 3 > 6x$

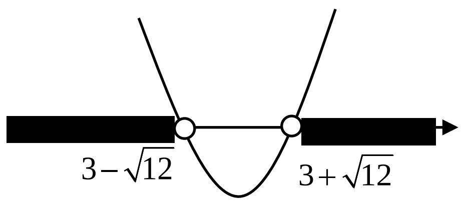
$x^2 - 6x - 3 > 0$

$(x-3)^2 - 3^2 - 3 > 0$

$(x-3)^2 - 12 > 0$

$(x-3)^2 - \left(\sqrt{12}\right)^2 > 0$

$(x - 3 - \sqrt{12})(x - 3 + \sqrt{12}) > 0$



$x > 3+\sqrt{12}$ or $x < 3-\sqrt{12}$

The solution set is $\{x \in \mathbb{R} : x > 3+\sqrt{12} \text{ or } x < 3-\sqrt{12}\}$

> Alternatively, we can solve it this way:
> Let $x^2 - 6x - 3 = 0$.
> Then
> $$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)}$$
> $\vdots$
> $= 3 \pm \sqrt{12}$
> $x^2 - 3 > 6x$
> $x^2 - 6x - 3 > 0$
> $\left[x - \left(3+\sqrt{12}\right)\right]\left[x - \left(3-\sqrt{12}\right)\right] > 0$
>
> (Then sketch the curve on the left and write the answer)

2. [ACJC/H1 Promo/2017/Q1]
Find, algebraically, the range of values of k for which $(2-k) + 2kx + x^2 > 0$ for all real values of x.

Solution:

Since $x^2 + 2kx + (2-k) > 0$,

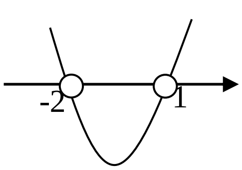
Discriminant < 0 and coefficient of $x^2 = 1 > 0$

$(2k)^2 - 4(1)(2-k) < 0$

$4k^2 + 4k - 8 < 0$

$k^2 + k - 2 < 0$

$(k-1)(k+2) < 0$



$-2 < k < 1$

3. [NYJC/H1 Promo/2017/Q1]
Find the range of values of k for which $kx^2+(k+1)x+(k+1)$ is never negative for all real values of x.

Solution:

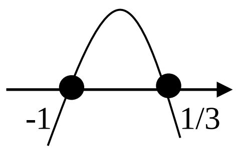
Since $kx^2+(k+1)x+(k+1)\geq 0$,

Discriminant ≤ 0 and coefficient of $x^2 = k > 0$

$(k+1)^2 - 4(k)(k+1) \leq 0$ and $k > 0$

$(k+1)(k+1-4k) \leq 0$

$(k+1)(1-3k) \leq 0$



$k \leq -1$ or $k \geq \frac{1}{3}$

Combining $k \leq -1$ or $k \geq \frac{1}{3}$ and $k > 0$ gives $k \geq \frac{1}{3}$

4 [DHS/H1 Promo/2017/Q2]
The line $y=(2k+1)x+k$ intersects the curve $y=1-\frac{k}{x}$, where k is a non-zero constant. Without using a calculator, find the exact set of values of k.

Solution:

At point(s) of intersection,

$(2k+1)x+k = 1-\frac{k}{x}$

$\Rightarrow (2k+1)x^2+(k-1)x+k=0$

Since line and curve may intersect at least once,
Discriminant ≥ 0

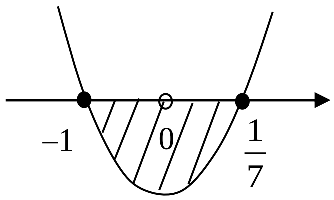
$(k-1)^2 - 4k(2k+1) \geq 0$

$k^2 - 2k + 1 - 8k^2 - 4k \geq 0$

$-7k^2 - 6k + 1 \geq 0$

$7k^2 + 6k - 1 \leq 0$

$(7k-1)(k+1) \leq 0$



$\therefore$ set of values = $\{ k \in \mathbb{R} : -1 \leq k \leq \frac{1}{7}, k \neq 0 \}$

Summary

Guide to solving quadratic inequalities:

1. Collect all the terms to one side of the inequality so that the other side is zero.
2. Find the x-intercepts of the graph by factorization or by using the quadratic formula.
3. Sketch the quadratic function and identify the required region.
4. Solve.

Checklist

I am able to:

☐ solve quadratic inequalities