Chapter 10 OSCILLATIONS

Content

- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations, resonance

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations.

(b) investigate the motion of an oscillator using experimental and graphical methods.

(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.

(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.

(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) recognise and use $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{(x_0^2 - x^2)}$.

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.

(h) describe the interchange between kinetic and potential energy during simple harmonic motion.

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and to the importance of critical damping in applications such as a car suspension system.

(j) describe practical examples of forced oscillations and resonance.

(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.

(l) show an appreciation that there are some circumstances in which resonance is useful, and other circumstances in which resonance should be avoided.

Oscillations and Waves – An Overview

Topics under Oscillations and Waves	**Chapter 10:** Oscillations **Chapter 11:** Wave Motion **Chapter 12:** Superposition
Links Between Sections and Topics	Periodic motion, where the pattern of movement repeats over time, is ubiquitous, and arises for example when objects are perturbed from a condition of stable equilibrium. While much of the motion we have considered is non-periodic, we have studied uniform circular motion, which is periodic and regular. Even in one spatial dimension, there can be complicated types of periodic motion. Nonetheless, we can gain a deep understanding of periodic motion by analysing the mathematically simplest case of free oscillations, known as simple harmonic motion (SHM). Such sinusoidally varying motion is essentially a projection of uniform circular motion, and provides a mathematical basis upon which to describe more complicated oscillations. Naturally, we revisit concepts in kinematics, dynamics, forces and energy in trying to understand SHM. When we consider a system of connected particles, the idea of single particles undergoing oscillations is the starting point that leads on to the idea of waves within the system. While we have seen how powerful the particle picture is, it turns out that the wave picture, generalised beyond classical mechanics, is equally fundamental for describing and understanding the physical universe. With waves, we move conceptually from physics of particles to the physics of continuous media. All waves are disturbances which result in oscillations. The oscillations then spread out as waves, which carry energy and can result in disturbances far away. Waves are a means of transmitting energy without the attendant transfer of matter. Remarkably, one of the many surprises of nature is that electromagnetic waves can travel through a vacuum, an example of field oscillations that do not require particles. We can also discuss wave mechanics, as waves interact, though in a qualitatively different way from how particles interact. The principle of superposition allows accurate characterisation of interaction of waves. Interference and diffraction are important wave phenomena due to the superposition of waves. However, there is actually no clear distinction between interference and diffraction. The difference in the usage of the terms is mainly historical. Many of the ideas introduced during the study of waves in this section will later be important for appreciating the limitation of classical physics in explaining the behaviour of matter on the atomic scale and understanding quantum wave-particle duality.

Applications and relevance to daily life	Oscillations and waves play important roles in engineering and nature. In nature, molecules in a solid oscillate about their equilibrium position; electromagnetic waves consist of oscillating electric and magnetic fields, and waves are present everywhere, e.g. light travelling from the Sun to Earth, water waves and sound waves. The study and control of oscillation is needed to achieve important goals in engineering, e.g. to prevent the collapse of a building due to waves created by an earthquake. Furthermore, diffraction gratings allow us to determine the frequencies of light sources ranging from lamps to distant stars. Optical engineers also create optically variable graphics (OVG) on credit cards, which incorporate diffraction grating technology, as an anti-counterfeiting measure.

Links to Core Ideas		
Systems and Interactions	**Models and Representations**	**Conservation Laws**
• A wave is a source of disturbance that can transfer energy and momentum through time and space • Interaction of electromagnetic wave with matter (e.g. reflection, refraction, diffraction, absorption, scattering)	• Simple harmonic motion of a mass characterised by a restoring force that is proportional to its displacement • Mechanical wave model • Wave nature of electromagnetic radiation • Superposition principle, which is used to explain wave phenomena (e.g. standing waves, two-source interference, diffraction) • Common representations: e.g. wavefront diagrams, displacement-time graph (characteristic of every particle), displacement-position graph (snapshot of wave in time) • Simplifying assumptions: e.g. ignore dissipative forces like friction and air resistance (negligible attenuation)	• Conservation of mechanical energy in an SHM system • The relationship between intensity and distance for a point source • The intensity distribution of a double-slit interference pattern obeys the conservation of energy

10.1 Simple Harmonic Motion (S.H.M.)

Introduction

A ***periodic*** motion is one in which an object continually retraces its path at *equal* time intervals. Many systems exhibit periodic motion. The molecules in a solid oscillate about their equilibrium positions; electromagnetic waves are characterised by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage and current vary periodically with time.

An oscillation is a special periodic motion in which the oscillator moves to and fro about an equilibrium position. This is also called harmonic motion. Simple harmonic motion is a type of such a motion.

Consider a mass attached to the end of a fixed spring (Fig. 10.1.1a), or a pendulum (Fig. 10.1.1b). In both cases, the mass and the pendulum are at their equilibrium positions. The **equilibrium position** is a position where the net force on the object is zero.

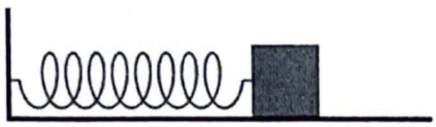


Fig. 10.1.1a

Fig. 10.1.1b

If the mass or pendulum is displaced from its equilibrium position (i.e. pulled either to the left or right), it will experience a net force that tries to restore it to its equilibrium position. This is called the **restoring force**.

The mass-spring system and pendulum systems described above are examples of oscillators undergoing harmonic motion. We will examine these two systems in greater detail later in this chapter.

For any object or particle oscillating about an equilibrium point, its motion is simple harmonic when the magnitude of the restoring force is directly proportional to the displacement of the object.

Definition

Simple harmonic motion is defined as *the motion of a particle about a fixed point such that its acceleration is proportional to its displacement from the fixed point and is always directed towards the point.*

Mathematically, $a \propto -x$

Formula

i.e. $$a = -\omega^2 x$$

where a is the acceleration,
x is the displacement from the equilibrium position,
ω^2 is a positive constant, where ω is the angular frequency of the oscillation.

Definition

Angular frequency is defined as the *rate of change of phase angle* of the oscillation, and is equal to the product of 2π and its frequency (i.e. $\omega = 2\pi f$).

The unit of ω is **radian per second (rad s^{-1})**.

The negative sign in the equation indicates that the acceleration a acts in a direction opposite to that of the displacement x.

Since the acceleration of the object is not constant, it is not possible to apply the usual kinematics equations in solving SHM problems.

Note

ⓘ Do not confuse *angular frequency* with *angular velocity*.
Even though the two have the same units of rad s^{-1} and are written with the same symbol ω, they are *not* the same. In simple harmonic motion, ω stands for angular frequency. In uniform circular motion, ω stands for angular velocity, which is the rate of change of angular displacement.

Other related quantities:

Definitions

Amplitude is the *magnitude of the maximum displacement of the particle from its equilibrium position.*

Period *T* is the *time taken to complete one oscillation.*

Frequency *f* is the *number of oscillations per unit time.*

Formulae

$$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad \omega = \frac{2\pi}{T}$$

T and f are related by the equation: $T = \frac{1}{f} \Rightarrow \omega = 2\pi f$

If T is measured in seconds, then f will be in **hertz, Hz**. (S.I. unit) where 1 Hz = 1 s^{-1}.

Characteristics of S.H.M.

Consider a particle N undergoing simple harmonic motion between points A and B, about an equilibrium position O. Throughout the motion, N experiences a restoring force, and hence acceleration, towards O.

Fig. 10.1.2 illustrates the motion of N at various instances of the motion.

The displacement *x* from point O, the velocity *v* and acceleration *a* of particle N are also indicated in Fig. 10.1.2. The direction towards the right is taken to be positive.

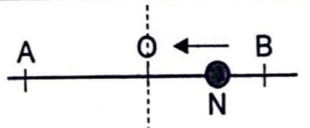
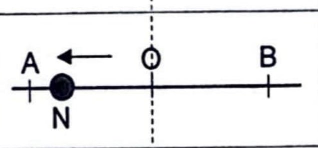
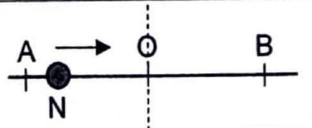
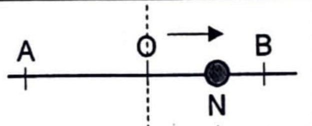
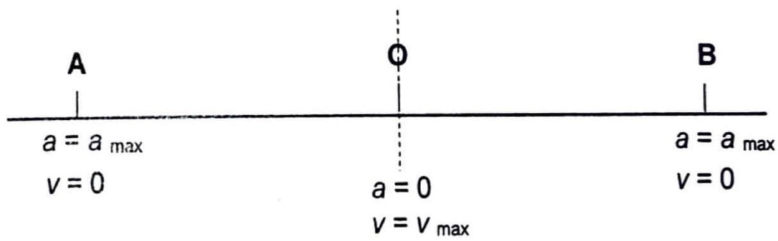
Motion	Diagram	*x*	*v*	*a*
N speeding up from B towards O		+	− ←	− ←
N slowing down from O towards A		−	− ←	+ →
N speeding up from A towards O		−	+ →	+ →
N slowing down from O towards B		+	+ →	− ←

Fig. 10.1.2

The motion of particle N has the following characteristics:

- It is symmetrical about the equilibrium position O.
- $|OA| = |OB| = x_0$, where x_0 is the **amplitude** of the motion.
- The **period** ***T*** of the oscillation is the time N takes to move from B→O→A→O→B.
- Acceleration is always directed towards O.
- The magnitude of the acceleration is zero at O and maximum at A and B.
- The speed at O is a maximum and zero at A and B.

Note



Position of N	A	O	B
acceleration of N	max.	0	max.
speed of N	0	max.	0

Displacement -Time Graph

The variation of displacement from the equilibrium position x with time t of particle N in Fig. 10.1.2, can be represented by a displacement-time graph.

If at $t = 0$, $x = +x_0$ (i.e. N is at B), the motion of N is given by $x = x_o \cos \omega t$.

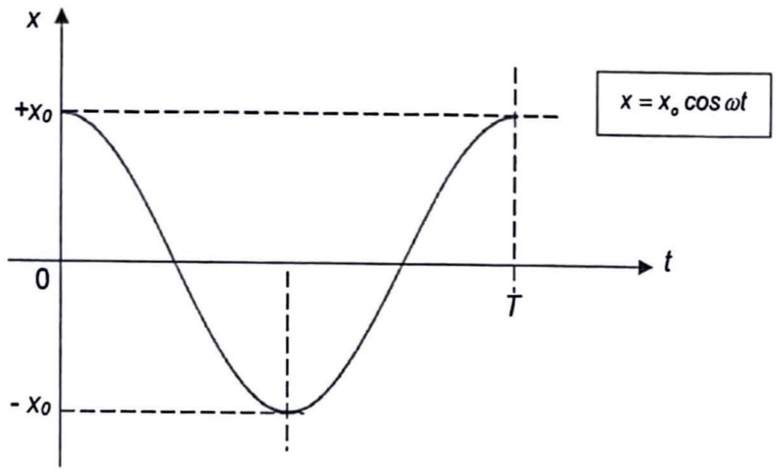


Fig. 10.1.3

If at $t = 0$, $x = 0$ (i.e. N is at O and moving towards B), the motion of N is given by $x = x_o \sin \omega t$.

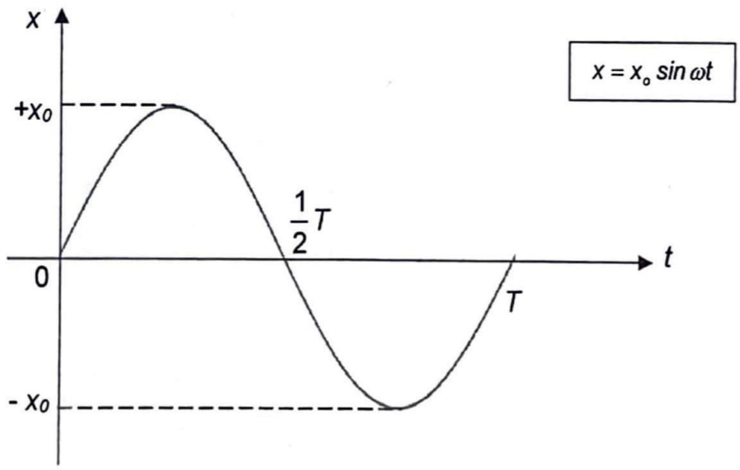


Fig. 10.1.4

Changes in x, v and a during S.H.M.

Consider the case $x = x_0$ at $t = 0$, i.e. N is at point B, displacement is given by

x* varies with *t

$$\boxed{x = x_o \cos \omega t}$$ (Refer to Fig. 10.1.5)

By calculus, we have

$$v = \frac{dx}{dt}$$
$$= \frac{d}{dt}(x_o \cos \omega t)$$

v* varies with *t

$$\boxed{v = -x_o \omega \sin \omega t}$$ (Refer to Fig. 10.1.6)

and

$$a = \frac{dv}{dt}$$
$$= \frac{d}{dt}(-x_o \omega \sin \omega t)$$

a* varies with *t

$$\boxed{a = -x_o \omega^2 \cos \omega t}$$ (Refer to Fig. 10.1.7)

Also,

$$v = -x_o \omega \sin \omega t$$
$$= -\omega x_o \sin \omega t$$
$$= -\omega x_o \left(\pm\sqrt{1-\cos^2 \omega t}\right)$$
$$= \pm\omega\sqrt{x_o^2\left(1-\cos^2 \omega t\right)}$$
$$= \pm\omega\sqrt{\left(x_o^2 - x_o^2\cos^2 \omega t\right)}$$

Formula

$$\boxed{v = \pm\omega\sqrt{\left(x_o^2 - x^2\right)}}$$ (Refer to Fig. 10.1.8)

and

$$a = -x_o \omega^2 \cos \omega t$$
$$= -\omega^2 \left(x_o \cos \omega t\right)$$

Formula

$$\boxed{a = -\omega^2 x}$$ (Refer to Fig. 10.1.9)

Graphical Illustrations The following graphs show how the displacement, velocity and acceleration of N in Fig. 10.1.2 vary with time during two complete cycles.

$x = x_o \cos \omega t$

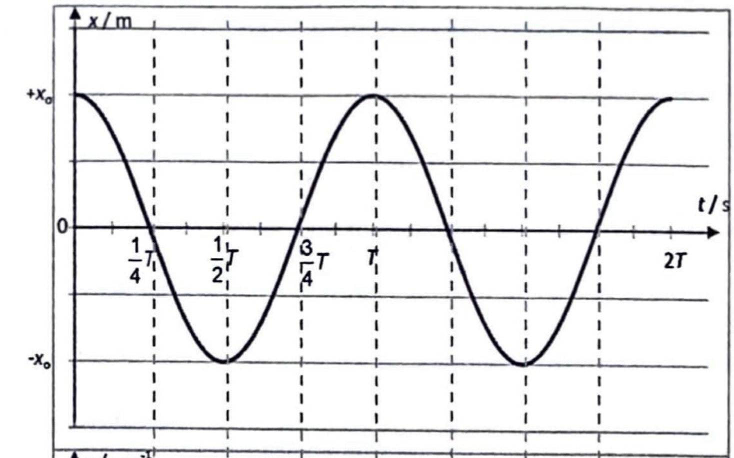


Fig. 10.1.5

$v = -x_o \omega \sin \omega t$

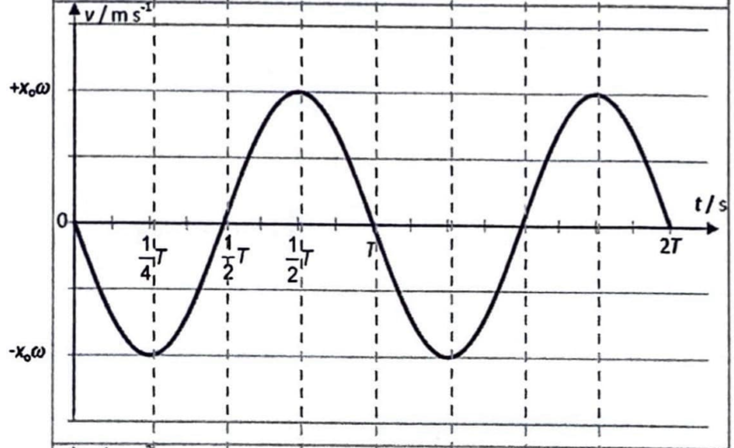


Fig. 10.1.6

$a = -x_o \omega^2 \cos \omega t$

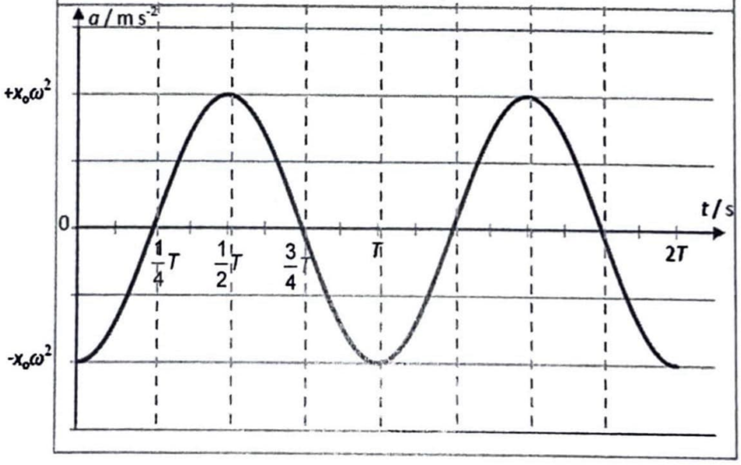


Fig. 10.1.7

1st quarter of cycle	When N is speeding up from B towards O, both *v* and *a* are directed towards O (both negative).
2nd quarter of cycle	When N is slowing down from O to A, *v* and *a* are in opposite directions: *v* is negative but *a* is positive.
3rd quarter of cycle	When N is speeding up from A towards O, both *v* and *a* are directed towards O (both positive).

4th quarter of cycle	When N is slowing down from O to B, *v* and *a* are in opposite directions: *v* is positive but *a* is negative. The cycle then repeats.

The following graphs show how the velocity and acceleration of N in Fig. 10.1.2 vary with displacement.

$v = \pm\omega\sqrt{(x_0^2 - x^2)}$

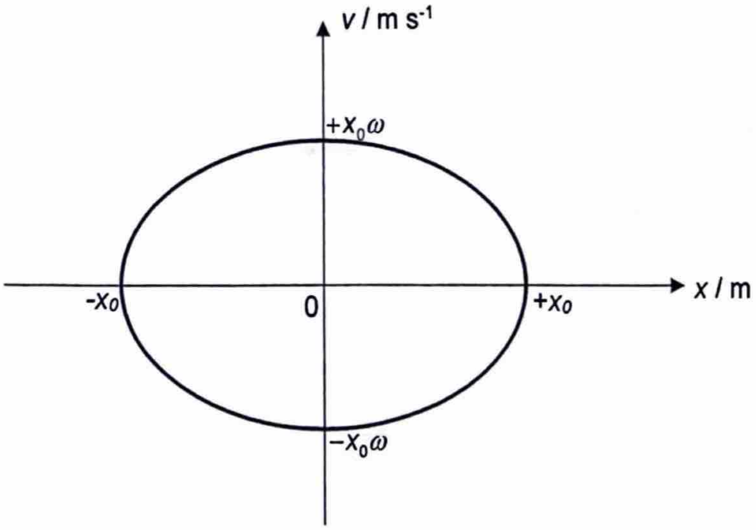


Fig. 10.1.8

$a = -\omega^2 x$

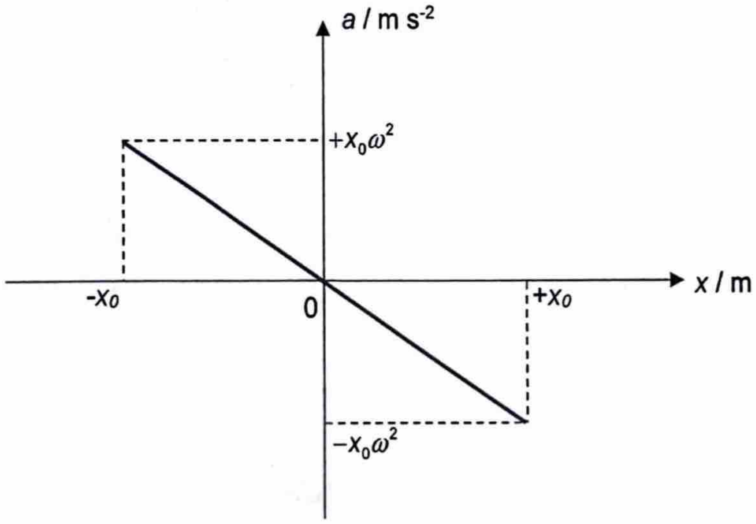


Fig. 10.1.9

Note

Fig. 10.1.9 is the graph for the defining equation for S.H.M.: $a = -\omega^2 x$
It is the *a-t* graph for any particle moving in S.H.M.

Example 1

A particle describes S.H.M. in which the displacement is given by

$$x = 0.05\cos(4\pi t)$$

where x is in metres and t in seconds.

Determine
- (a) the amplitude of the motion,
- (b) the period of the motion,
- (c) the maximum velocity of the particle,
- (d) the maximum acceleration of the particle.

Solution:

(a) amplitude, x_0 = 0.05 m

(b) $\omega = 4\pi \Rightarrow \frac{2\pi}{T} = 4\pi$

(c) $v = \pm\omega\sqrt{x_0^2 - x^2}$

Maximum velocity (when x = 0) $= \pm\omega x_0$
$= \pm(4\pi)(0.05) = \pm 0.628$ m s^{-1}

(d) $a = -\omega^2 x$

Maximum acceleration (when $x = \pm x_0$) $= \pm\omega^2 x_0$
$= \pm(4\pi)^2 (0.05) = \pm 7.90$ m s^{-2}

Example 2 (J82/II/10)

Figures (a) and (b) below show how the displacement x and the acceleration a of a body vary with time when it is oscillating with simple harmonic motion.

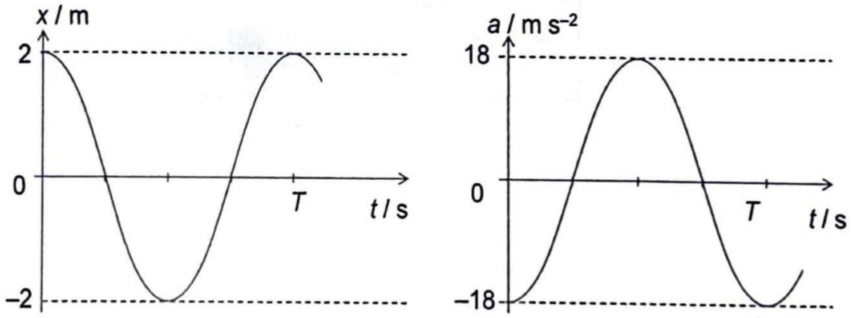


Figure (a) Figure (b)

What is the value of T?

Solution:

$$a = -\omega^2 x \Rightarrow |a_{max}| = \omega^2 x_0$$

$$18 = \left(\frac{2\pi}{T}\right)^2 (2)$$

$$T = 2\pi\sqrt{\frac{2}{18}} = 2.09 \text{ s}$$

Free Oscillations

If an object is displaced from its equilibrium position and then released, it oscillates at its *natural frequency* about the equilibrium position.

Note

A free oscillation occurs when an object oscillates with no resistive and driving forces acting on it. Its total energy and amplitude remain constant with time.

Models for S.H.M.

Two common mechanical systems are used to illustrate simple harmonic motion: spring-mass system and simple pendulum.

Spring-Mass System

Horizontal

Consider a block of mass m attached to the end of a spring of negligible mass and force constant k, with the block free to move on a horizontal frictionless surface. When the spring is neither stretched nor compressed it is at its equilibrium position as shown in Fig. 10.1.10.

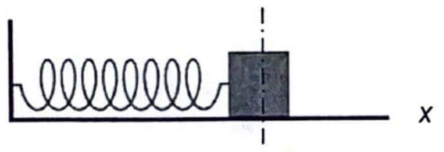


Fig. 10.1.10

The block is displaced a distance x to the right in Fig. 10.1.11.

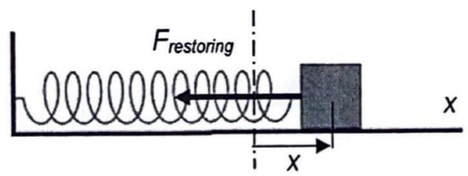


Fig. 10.1.11

The restoring force exerted towards the left by the spring on the block is

$$F_{restoring} = -kx$$

It is the resultant force acting on the block, hence by Newton's 2nd law of motion:

$$F_{restoring} = ma$$

$$-kx = ma$$

$$\boxed{a = -\left(\frac{k}{m}\right)x}$$

Formula Comparing with $a = -\omega^2 x$, $\boxed{\omega = \sqrt{\frac{k}{m}}}$ and $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$

Vertical

A vertically suspended spring of negligible mass and force constant k is stretched by an amount e when a block of mass m is hung on it and remains at rest at the equilibrium position. The block is then given an additional downward displacement y (positive direction downward) and released as shown in Fig. 10.1.12.

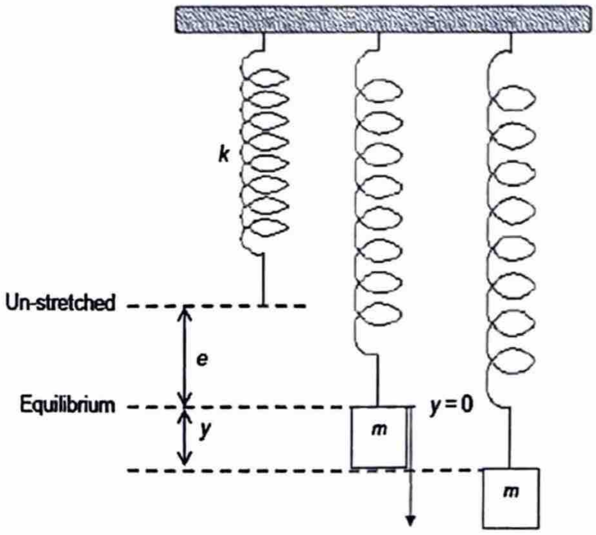


Fig. 10.1.12

The initial static equilibrium is characterised by a balance between the elastic force and the weight of the block:

$$mg = ke \qquad \text{... (1)}$$

Once the block is pulled down and released, the restoring force is the vector sum of the forces in the vertical direction.

$$\sum \vec{F}_y = \vec{F}_{restoring} = \vec{W} + \vec{F}_{spring}$$

$\vec{F}_{spring}$ is due to the total extension (e + y) of the spring at this position, i.e. $F_{spring} = k(e+y)$.

Taking downwards to be positive,

$$F_{restoring} = mg + [-k(e+y)]$$

By Newton's 2[nd] law:

$$F_{restoring} = ma$$
$$mg - ke - ky = ma$$
$$mg - mg - ky = ma \qquad ; ke = mg \text{ from (1)}$$
$$-ky = ma$$
$$\boxed{a = -\left(\frac{k}{m}\right)y}$$

Comparing with $a = -\omega^2 y$, $\boxed{\omega = \sqrt{\frac{k}{m}}}$ and $\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$

ⓘ *By comparing the horizontal and vertical spring-mass systems, it can be concluded that <u>only the mass and spring constant determine their motions</u>, i.e. the period of oscillation depends only on these quantities.*

Simple Pendulum

A simple pendulum consists of a bob suspended by a light string. The forces acting on the bob are the tension *T* in the string and the weight *mg* of the bob. When the bob is displaced by a small angle θ (<10°), it is displaced by a distance $s = L\theta$.

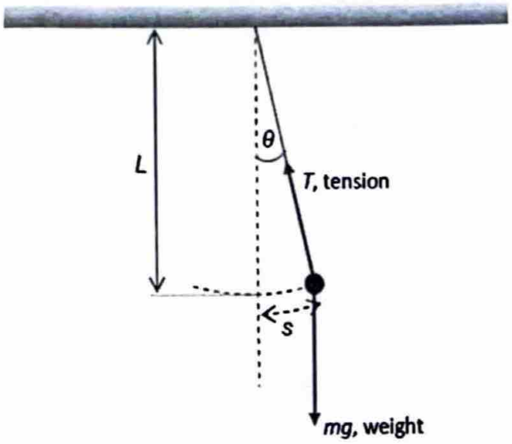


Fig. 10.1.13a

The restoring force is the tangential component of *mg*:

$$F_{restoring} = -mg\sin\theta$$

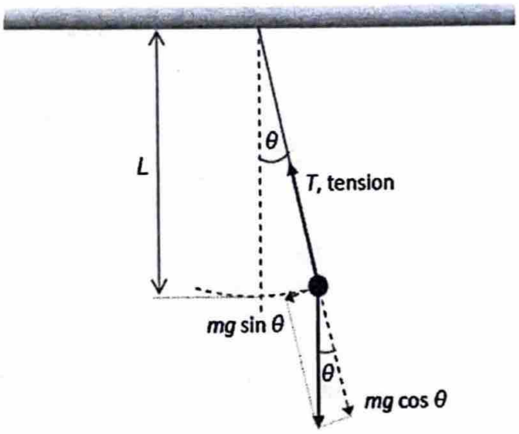


Fig. 10.1.13b

Since θ is very small, by small angle approximation,

$\sin\theta \approx \theta$ and $F_{restoring} \approx -mg\theta = -mg\dfrac{s}{L}$

By Newton's 2nd law:

$$F_{restoring} = ma$$

$$-mg\frac{s}{L} = ma$$

$$\boxed{a = -\left(\frac{g}{L}\right)s}$$

ⓘ *The motion of the simple pendulum depends only on the length of the pendulum and the acceleration due to gravity.*

Formula

Comparing with $a = -\omega^2 s$, $\boxed{\omega = \sqrt{\dfrac{g}{L}}}$ and $\boxed{T = 2\pi\sqrt{\dfrac{L}{g}}}$

Example 3 An object of mass 0.20 kg is hung from the lower end of a spring. When the object is pulled down 5.0 cm below its equilibrium position O and released, it vibrates with S.H.M. with a period of 2.0 s.

(a) What is the extension of the spring when the mass is hung at rest from the lower end of the spring?
(b) What is the speed of the mass as it passes through O?
(c) What is the magnitude of its acceleration when it is 2.5 cm above O?
(d) Through what distance will the object move in the first 0.75 s?

Solution:

(a) $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow 2.0 = 2\pi\sqrt{\frac{0.20}{k}} \Rightarrow k = 1.974 = 1.97 \text{ N m}^{-1}$

At the equilibrium position,

$$mg = ke \Rightarrow e = \frac{mg}{k} = \frac{(0.20)(9.81)}{1.974} = 0.994 \text{ m}$$

(b) $v_o = x_o\omega = (5.0 \times 10^{-2})\left(\frac{2\pi}{2.0}\right) = 0.157 \text{ m s}^{-1}$

(c) $|a| = |-\omega^2 x| = \left(\frac{2\pi}{2.0}\right)^2 (2.5 \times 10^{-2}) = 0.247 \text{ m s}^{-2}$

(d) Taking upwards and displacement above O as positive,

$$x = -5.0\cos\left(\frac{2\pi}{2.0}t\right) = -5.0\cos \pi t$$

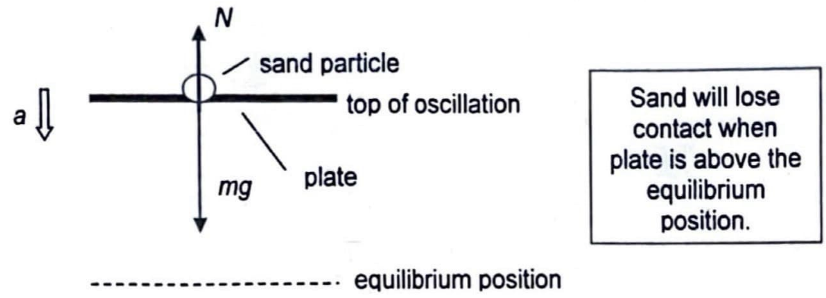
When t = 0.75 s, $x = -5.0\cos(\pi(0.75)) = 3.5$ cm

The object is 3.5 cm above the equilibrium.

Distance moved by the object in 0.75 s = 5.0 + 3.5 = 8.5 cm

Example 4 A horizontal plate is vibrating vertically in S.H.M. at a frequency of 20 Hz. What is the maximum amplitude of vibration so that fine sand on the plate always remains in contact with it?

Solution:



The restoring force on the sand is the resultant force acting on it. It acts towards the equilibrium position.

Taking downwards as positive,

For the sand to remain in contact, $N > 0$

$$N = mg - m\omega^2 x > 0$$

$$mg > m\omega^2 x$$

$$x < \frac{g}{\omega^2}$$

$$\Rightarrow \quad x < \frac{g}{(2\pi f)^2} \quad \Rightarrow \quad x < \frac{9.81}{(2\pi(20))^2} \quad \Rightarrow \quad x < 0.00062 \text{ m}$$

Hence, the maximum amplitude of vibration is 0.62 mm for the sand to remain in contact with the plate.

10.2 Energy in Simple Harmonic Motion

Variation of Energy in S.H.M.

For a simple harmonic oscillator, where the system does no work against dissipative forces, its total energy remains constant with time.

Suppose a body of mass *m*, attached to a light spring, oscillates on a horizontal frictionless surface about the equilibrium position O:

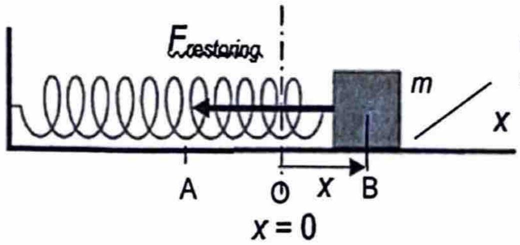


Fig. 10.2.1

Let us consider this to be an ideal oscillator, which would oscillate forever once it is set in motion. We are assuming that there is no energy loss as a result of friction or other resistance to motion.

During the cycle, there is a continual change of kinetic energy to potential energy and vice-versa. At any instant during the motion, the total energy of the system is equal to the sum of its kinetic energy and potential energy.

Variation of energy with distance from equilibrium

<u>**E_k varies with x**</u>

The **kinetic energy E_k** at a distance x from its equilibrium position is given by

$$E_k = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\pm\omega\sqrt{(x_o^2 - x^2)}\right)^2$$

Formula

$$\boxed{E_k = \frac{1}{2}m\omega^2\left(x_o^2 - x^2\right)}$$

where x_o is the amplitude of the motion.

E_p varies with x

The **potential energy E_p** at a displacement x from its equilibrium position is given by

$$E_p = \frac{1}{2}kx^2$$

$$\text{since } \omega = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega^2$$

Formula

$$\boxed{E_p = \frac{1}{2}m\omega^2 x^2}$$

Total energy E

The **total energy E** at displacement x is given by

$$E = E_k + E_p$$

$$= \frac{1}{2}m\omega^2\left(x_o^2 - x^2\right) + \frac{1}{2}m\omega^2 x^2$$

Formula

$$\boxed{E = \frac{1}{2}m\omega^2 x_o^2}$$

Note:
$E = E_{k,max} = E_{p,max}$. This is because when the body is at $x = 0$ (equilibrium position), $E_p = 0$, hence $E_k = E$ is a maximum value. When the body is at $x = \pm x_0$ (maximum displacement), $E_k = 0$, hence $E_p = E$ is maximum value.

The following graph in Fig. 10.2.2 shows the variation of E_k, E_p and E with displacement x:

Variation of E_k, E_p and E with displacement x

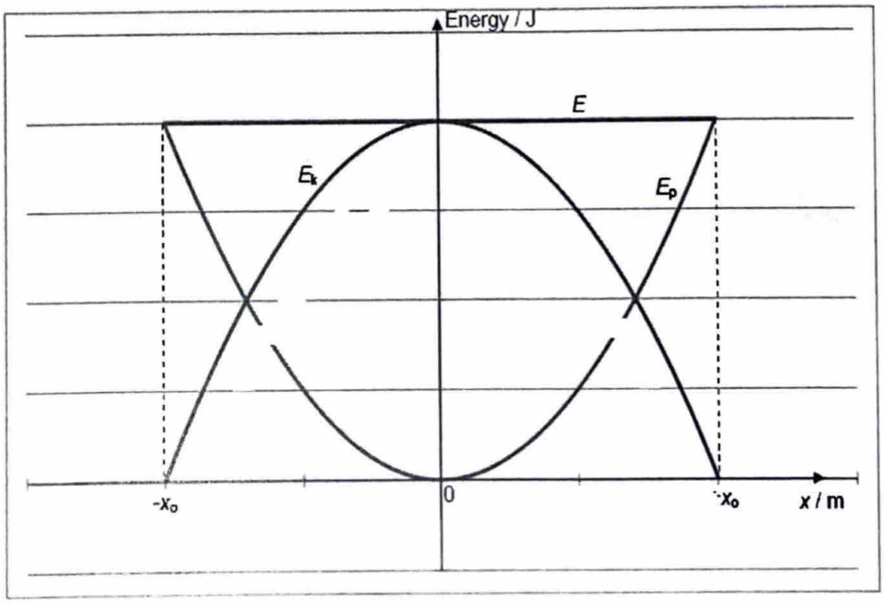


Fig. 10.2.2

Variation of energy with time

Suppose $x = x_0 \cos \omega t$ and $v = -x_0 \omega \sin \omega t$, the kinetic energy E_k and potential energy E_p at time t are given by

Kinetic Energy, E_k

$$E_k = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(-x_0 \omega \sin \omega t)^2$$
$$\boxed{E_k = \frac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t}$$

Potential Energy, E_P

$$E_p = \frac{1}{2}kx^2$$
$$= \frac{1}{2}k(x_0 \cos \omega t)^2$$
$$= \frac{1}{2}kx_0^2 \cos^2 \omega t$$
$$\boxed{E_p = \frac{1}{2}m\omega^2 x_0^2 \cos^2 \omega t}$$

E is constant with t

At any instant the **total energy, E** is given by

$$E = E_k + E_p$$
$$= \frac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t + \frac{1}{2}m\omega^2 x_0^2 \cos^2 \omega t$$
$$\boxed{E = \frac{1}{2}m\omega^2 x_0^2}$$

The graph in Fig. 10.2.4 shows the variation of E_k, E_p and E with time t.

Variation of E_k, E_p and E with time t

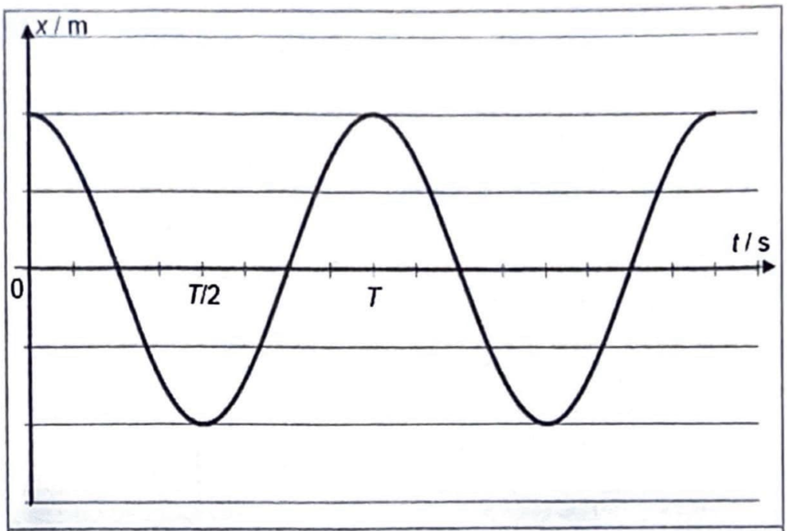


Fig. 10.2.3

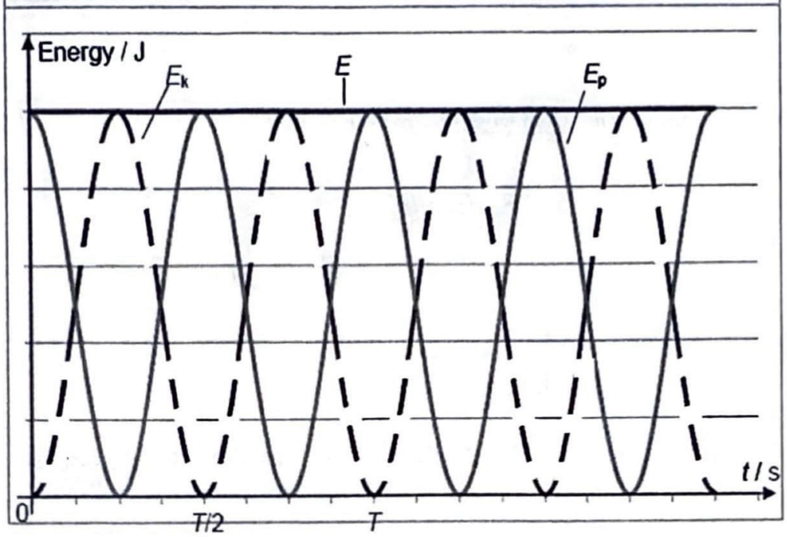


Fig. 10.2.4

ⓘ The frequency of the energy variation is *twice* that of the motion.

Example 5
Refer to Fig. 10.1.12

A mass of 0.50 kg is attached to a light spring which has a force constant of 20 N m^{-1}. The mass is displaced a distance 5.0 cm below the equilibrium position and then released. Calculate

(a) the maximum value of the potential energy of the oscillating system, assuming it is zero at the equilibrium position.

(b) the maximum velocity of the mass,

(c) the distance from the equilibrium position when the kinetic energy is one quarter of its maximum value.

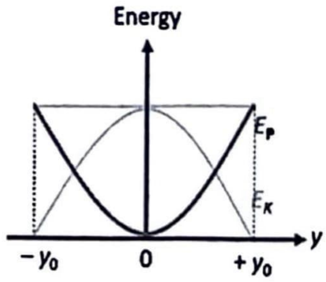
Solution:

(a) **Method 1 (using energy – displacement model for SHM)**

Maximum E_p occurs at the amplitude of oscillation.

Since $E_p = 0$ at the equilibrium, $E_P = \frac{1}{2}ky^2$

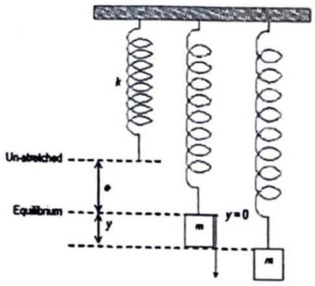
$$E_{p,\,MAX} = \frac{1}{2}k{y_0}^2 = \frac{1}{2}(20)(0.050)^2 = 0.025 \text{ J}$$



Method 2 (considering the changes in EPE and GPE)

At the equilibrium, $mg = ke$

$$e = \frac{mg}{k} = \frac{(0.50)(9.81)}{20} = 0.24525 \text{ m}$$



ⓘ The value of potential energy E_p of the vertical spring mass system depends on the changes in elastic potential energy (EPE) and gravitational potential energy (GPE).

At the mass moves from $y = 0$ to y_0, the system experiences an increase in EPE and a decrease in GPE.

At y_0, the E_p of the system is a maximum value.

$E_{P,\,MAX}$ = increase in EPE (from $y = 0$ to y_0) – decrease in GPE (from $y = 0$ to y_0)

$$= \left[\frac{1}{2}k(e+0.050)^2 - \frac{1}{2}ke^2\right] - mg(0.050)$$

$$= \left[\frac{1}{2}(20)(0.24525+0.050)^2 - \frac{1}{2}(20)(0.24525)^2\right] - (0.50)(9.81)(0.050)$$

$$= 0.025 \text{ J}$$

(b)

$$\left|E_{K,\,\mathrm{MAX}}\right| = \left|E_{P,\,\mathrm{MAX}}\right| = 0.025\ \mathrm{J}$$

$$\frac{1}{2}m{v_{MAX}}^2 = 0.025$$

$$\frac{1}{2}(0.50){v_{MAX}}^2 = 0.025$$

$$v_{MAX} = \sqrt{\frac{2(0.025)}{0.50}} = \pm 0.32\ \mathrm{m\ s^{-1}}$$

(c)

$$E_K = \frac{1}{4}E_{K,\,\mathrm{MAX}}$$

$$\frac{1}{2}m\omega^2\left({y_0}^2 - y^2\right) = \frac{1}{4}(0.025)$$

$$\frac{1}{2}(0.50)\left(\frac{20}{0.50}\right)\left(0.050^2 - y^2\right) = \frac{1}{4}(0.025)$$

$$y = 0.043\ \mathrm{m}$$

? *What does the energy-displacement graph look like for a vertical spring-mass system? Does it follow the model for SHM?* (Refer to tutorial self-practice SP10 for the energy-displacement graphs)

10.3 Damped and Forced Oscillations, Resonance

Damped Oscillations

The oscillatory motions that have been discussed so far have been for *ideal* systems.

A real oscillating system is opposed by dissipative forces, such as friction and viscous forces, which cause the amplitude of the motion to decrease with time. The system then does positive work: the energy to do this work is taken from the energy of the oscillation, and usually appears as internal energy of the surroundings and the system. Such oscillations are said to be damped.

Definition

Damping is the process whereby *energy is removed from an oscillating system.*

For a damping force which is proportional to the velocity of the mass, the decay in amplitude is *exponential*. This means that the amplitude decreases by the same fraction during each vibration.

A full mathematical analysis of damped harmonic motion shows that the frequency of the damped motion is *less than* the undamped frequency.

Degrees of Damping

The degree of damping depends on the magnitude of the retarding force (or the amount of resistance to the oscillation). In practice, the motion of an oscillator will depend on the magnitude of the damping. In certain cases, the damping may prevent the system from oscillating, and it will just return to its equilibrium position.

The following graphs show how different degrees of damping affect the displacement of an oscillating body.

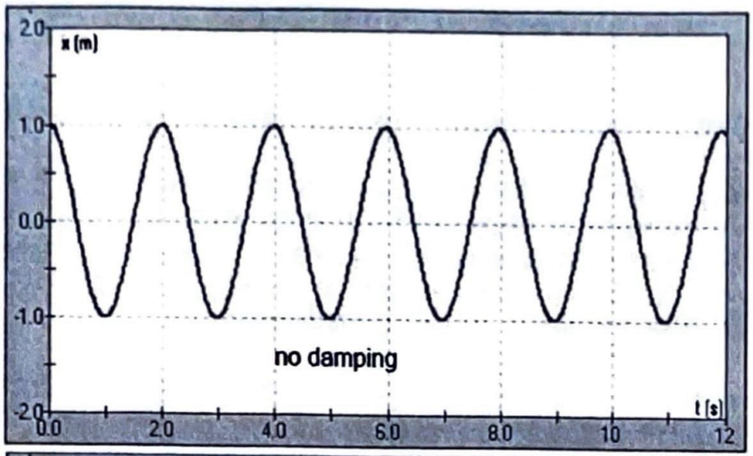


Fig. 10.3.1

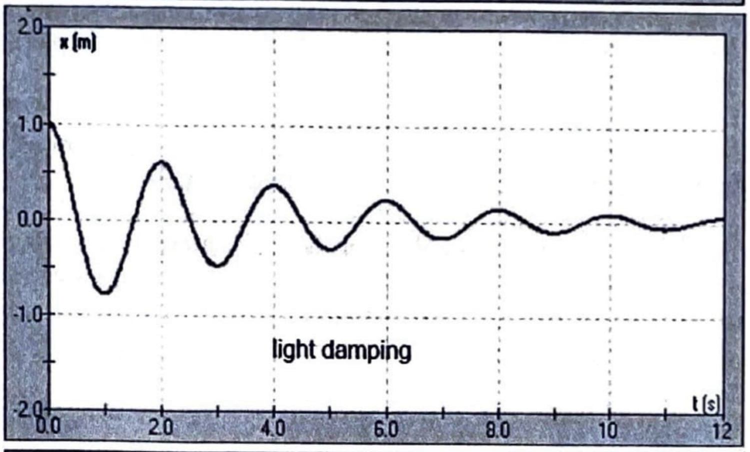


Fig. 10.3.2

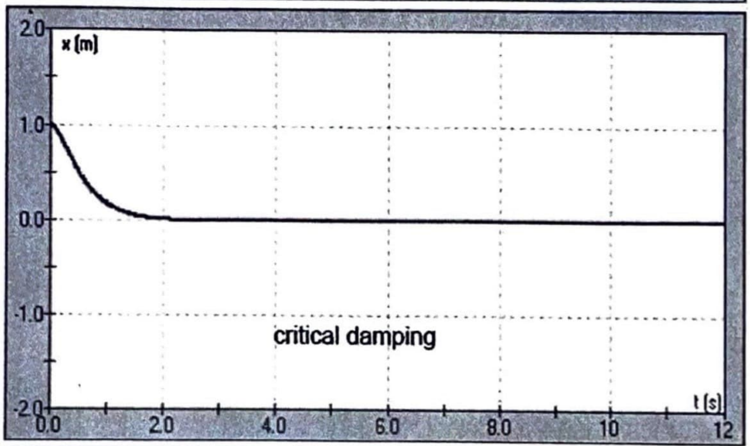


Fig. 10.3.3

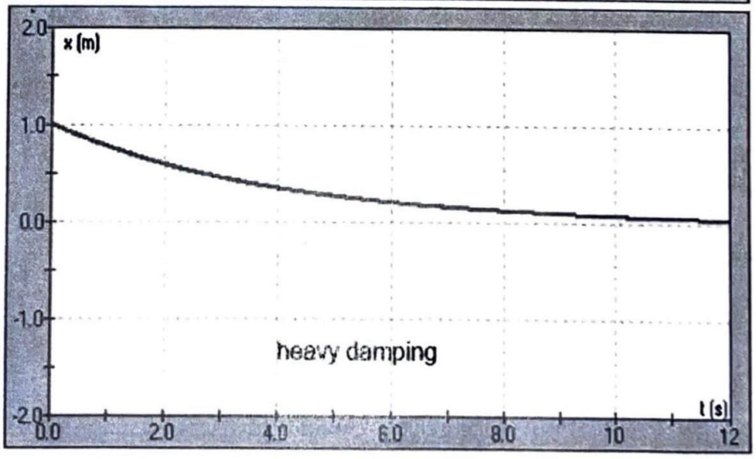


Fig. 10.3.4

Three Categories of Damping

Light Damping results in oscillations whereby the amplitude decays exponentially with time. The frequency of oscillations is slightly smaller than the undamped frequency. (Fig. 10.3.2)

Critical Damping results in no oscillation and the system returns to the equilibrium position in the shortest time. (Fig. 10.3.3)

Heavy Damping results in no oscillation and the system takes a long time to return to its equilibrium position. (Fig. 10.3.4)

Energy Variation of Damped Oscillations

Light Damping

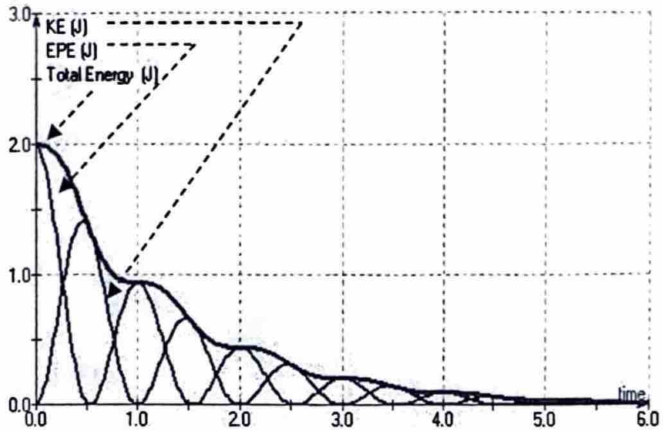


Fig. 10.3.5

Critical Damping

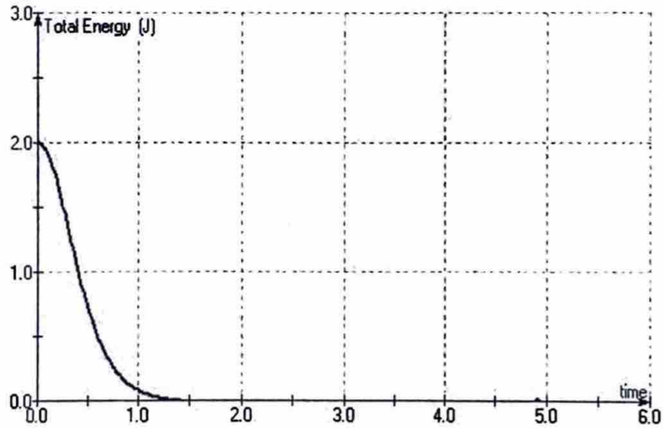


Fig. 10.3.6

Heavy Damping

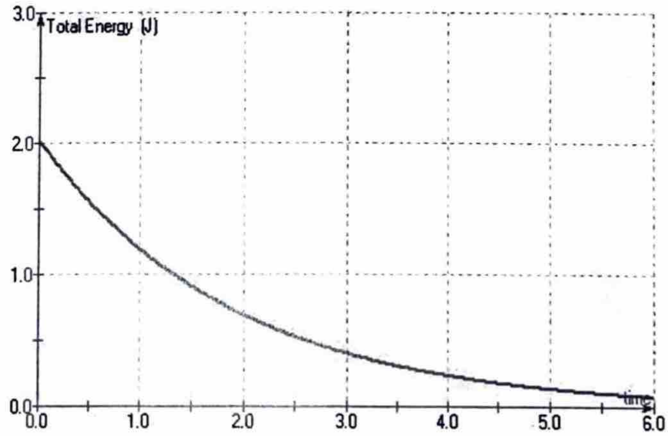


Fig. 10.3.7

Importance of Critical Damping: Car Suspension System

The degree of damping of a mechanical system is important. Too little damping results in a large number of oscillations: too much damping leads to there being too long a time when the system cannot respond to further disturbances.

This is illustrated well by the trouble which car manufacturers take with the suspension of cars. The suspension is the link between the wheels and axles of a car and the body and the passengers, and consists of a spring which is damped by a shock absorber.

Without the suspension system, the wheels' vertical motion, due to road imperfections (e.g. a bump), is transferred to the car frame, which moves upwards, and the tires can lose contact with the road completely. Then, under the downward force of gravity, the tires can slam back onto the road surface.

The suspension system will absorb the energy of the vertically accelerated wheel, allowing the frame and body to ride nearly undisturbed while the wheels and tires follow the bump in the road.

Diagram from HowStuffWorks website)

A shock absorber consists of a piston that moves in a cylinder containing a viscous fluid. Holes on the piston allow it to move up and down in a damped manner and the amount of damping is adjusted so that the suspension system is close to the condition of critical damping.

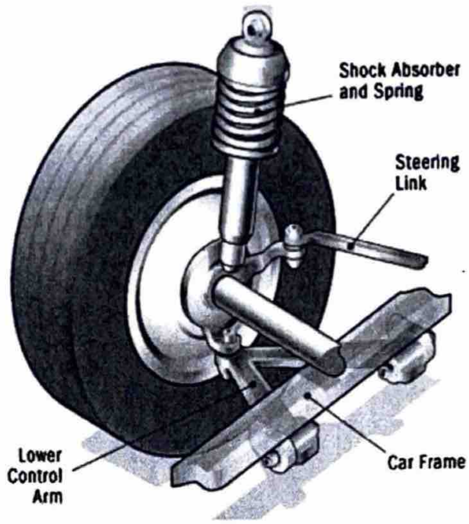


Fig. 10.3.8

A good suspension system is one in which the damping is critical or slightly under critical as this results in a comfortable ride and also leaves the car ready to respond to further bumps in the road quickly.

Fig. 10.3.9 shows that by the time the car has reached P the shock absorbing system is ready for the drop in road surface. After Q, it is ready for another bump.

The oscillation of the spring of a car suspension is critically damped when it goes over a bump; the passengers in the car quickly and smoothly regain equilibrium.

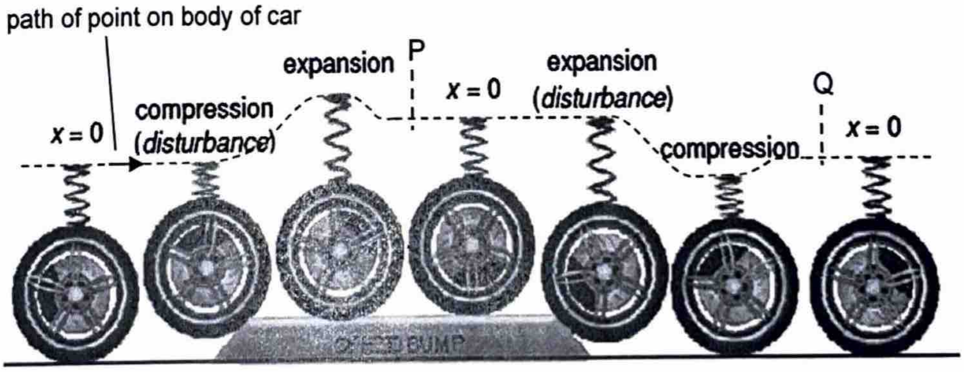


Fig. 10.3.9

Without the shock absorber, a car spring, after a compression, will extend and release
the energy it absorbs from the rise of a bump at an uncontrolled rate. The spring will continue to bounce at its natural frequency until all of the energy originally put into it is used up. A suspension built on springs alone would make for an extremely bouncy and uncomfortable ride and, depending on the terrain, an uncontrollable car.

A heavily damped shock absorbing system would still have a compressed spring by the time P is reached and so would not be able to respond to the sudden drop in road surface. So long as there are bumps on a road then these must have an effect on a passenger in a car. The shock-absorbing system can only reduce the forces applied. It cannot eliminate them because, clearly, in the above diagram, the passenger must rise and drop eventually by the height of the bump.

Instruments such as analogue balances and electrical meters are also designed to be critically damped so that the pointer comes quickly to the correct position in the shortest possible time without oscillating.

Forced Oscillation and Resonance

Since all macroscopic mechanical oscillations are damped, energy is continually being lost from the system. If we wish to maintain the vibrations at constant amplitude, then energy must be supplied at the rate at which energy is being dissipated to the surroundings and within the system. A force must therefore be applied to oppose the damping forces.

Note

Forced oscillations are produced when a body is subjected to a periodic external driving force and is made to oscillate at the frequency of the driving force, which may not be its natural frequency.

The device / machine providing this periodic driving force is known as the driver.

Demonstration Using Barton's Pendulums

Fig. 10.3.10 shows a setup of Barton's pendulums. It consists of a number of very light pendulums (made from paper cones) of varying length (A, B, C, D and E) and one pendulum with a heavy bob (X). All the pendulums are suspended from the same string. The massive pendulum X, called the driver pendulum, is made to oscillate and the motion of the rest of the pendulums are observed.

The setup is used to demonstrate what happens when a system is made to vibrate at some frequency other than its own natural frequency of vibration.

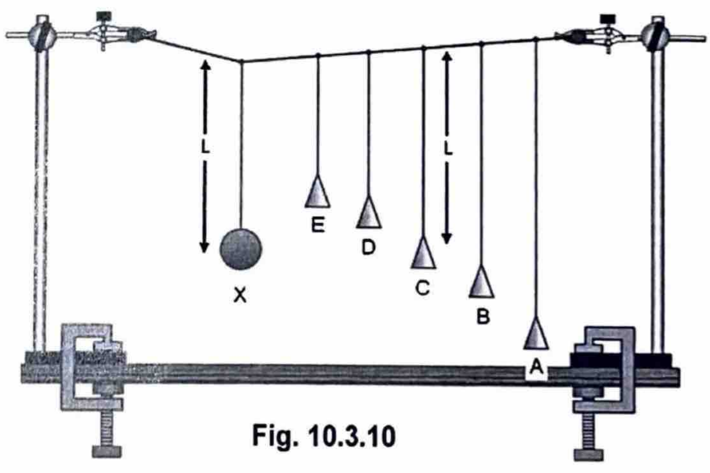


Fig. 10.3.10

Forced Oscillations

The motion of the Barton's pendulums can be divided into two distinct sections. Initially, it is very chaotic; the pendulums tend to oscillate at their own natural frequency (determined by their length) while the driving pendulum tries to make them all oscillate at its own frequency. Gradually, the driving pendulum wins and the pendulums are all forced to oscillate at a frequency which is not the same as their own natural frequency. Energy is being transferred from the driver pendulum to the driven pendulums. This is an example of *forced oscillations.*

Resonance

The Barton's pendulums experiment shows that the forced vibrations are at the maximum when the natural frequency of the driven system is equal to the frequency of the driving oscillator. Pendulum C, which has the same length and thus has the same natural frequency as pendulum X, is observed to oscillate with the largest amplitude. This is an example of *resonance*. At resonance, maximum energy is being transferred by way of the string from the driving system (X) to the driven system (C)

Variation of Amplitude of a Forced Oscillation with Driving Frequency

Fig. 10.3.11 is a frequency response graph which shows how the amplitude x_0 of a forced oscillation depends on the driving frequency *f* when the system is damped at different degrees.

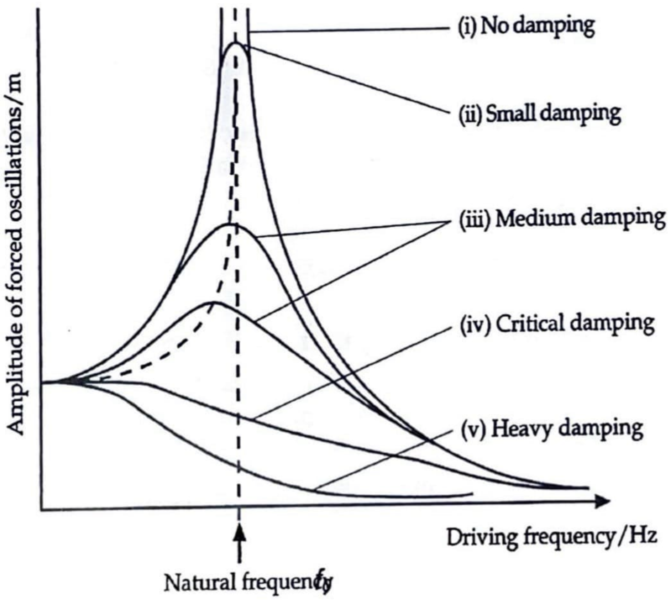


Fig. 10.3.11

For a forced oscillation, when conditions are *steady*, the following observations are made:

- The amplitude of a forced oscillation depends upon:
  1. the damping of the system,
  2. the relative values of the driving frequency *f* and the natural frequency f_0 of the system (i.e. how far *f* is from f_0).
- The oscillations with largest amplitude (i.e. resonance) occur when *f* is approximately equal to f_0.

The sharpness of resonance is determined by the degree of damping:

1. When there is no damping, the amplitude of resonance becomes infinite. (Fig. 10.3.11(i))
2. When damping is light, the amplitude is large but falls off rapidly when the driving frequency of the body differs slightly from the natural frequency of the body. The resonance is sharp. (Fig. 10.3.11(ii))
3. When the degree of damping increases, the amplitude at resonance decreases. The curve falls off gradually and maximum amplitude occurs at a frequency that is lower than the natural frequency of the body. (Fig. 10.3.11(iii))
4. When damping is critical or heavy, the resonance is flat. (Fig. 10.3.11(iv),(v))

Definition

Resonance occurs when a system *responds at maximum amplitude to an external driving force. This occurs when the frequency of the driving force is equal to the natural frequency of the driven system.*

Circumstances in which resonance is useful

Tuning a radio receiver (Electrical resonance)

The electrons in a radio receiving aerial are forced to vibrate by the radio wave passing the aerial. When we tune the receiver, we are making the natural frequency of the electrical circuit equal to the frequency of the signal. Hence the tuning circuitry uses resonance to isolate and amplify the signal of the required frequency.

Increasing the intensity of a note produced by a string in a musical instrument (Acoustic resonance)

This is done by coupling the vibrating string to a resonator. The air inside a cavity (e.g. a guitar body) and the material of the instrument (the thin wooden body of the guitar) all vibrate producing much greater vibrations in the surrounding air than would be produced by the string alone.

Magnetic resonance

Energy from strong oscillating magnetic fields is used to cause the nuclei of atoms to oscillate and emit radio frequency signals. In any given molecule there will be many resonant frequencies, and whenever resonance occurs energy is absorbed. The pattern of energy absorption can be used to detect the presence of particular molecules within any specimen and biochemists are using the technique to study complex molecules and the part they play in biological processes.

Magnetic resonance is also being used instead of X-rays as an imaging system (MRI) in the medical field. The radio frequency signals emitted are made to encode position information by varying the magnetic field. The contrast between different tissues is determined by the rate at which excited atoms return to the equilibrium state. One major advantage of magnetic resonance used in this way is that no ionising radiation is involved.

Circumstances in which resonance should be avoided

However, resonance is not always useful.

All mechanical structures have one or more natural frequencies, and if a structure is subjected to a strong external driving force that matches one of these frequencies, resonance is said to occur and the resulting oscillations of the structure may rupture it.

At Angers, France in 1850, a French infantry battalion was marching over a suspension bridge when it collapsed, resulting in the deaths of 220 men. Since that time, it has been common practice to order soldiers to break step when crossing a bridge. The soldiers' marching caused sufficient vibration and twisting to break the bridge.

A more modern bridge disaster occurred in 1940 when wind-induced oscillations caused the collapse of the Tacoma Narrows Bridge in the U.S. state of Washington. The bridge's natural mode of vibration coupled with the wind forces, produced unstable oscillations with increased amplitude that were beyond the strength of the suspender cables.

Resonance was also the cause for the collapse of some buildings during a major earthquake in Mexico in 1985. Many intermediate-height buildings collapsed because their natural frequency matched that of the seismic waves, whereas taller or shorter buildings were unaffected.

A more mundane example of resonance is the way in which the bodywork of a bus can vibrate violently at a particular engine speed.

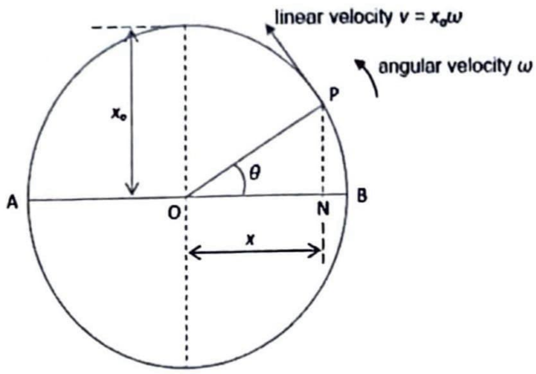
As such, engineers have to carry out elaborate vibration tests on model structures of, for example, bridges, buildings and aeroplanes before they are satisfied that the design features will prevent extremely large amplitudes from building up in the system.

USEFUL S.H.M. EQUATIONS

At $t = 0$ s	$x = x_0$	$x = 0$
Displacement w.r.t. time	$x = x_0 \cos \omega t$	$x = x_0 \sin \omega t$ (given in formulae list)
Velocity w.r.t. time	$v = -x_0 \omega \sin \omega t$ $= -v_0 \sin \omega t$	$v = x_0 \omega \cos \omega t$ $= v_0 \cos \omega t$
Acceleration w.r.t. time & displacement (defining equation for S.H.M.)	$a = -x_0 \omega^2 \cos \omega t$ $= -\omega^2 x$	$a = -x_0 \omega^2 \sin \omega t$ $= -\omega^2 x$
Velocity w.r.t. displacement	$v = \pm \omega \sqrt{(x_0^2 - x^2)}$ (given in formulae list)	
Maximum velocity	$v_{max} = \pm \omega x_0$	
Maximum acceleration	$a_{max} = \pm \omega^2 x_0$	
Kinetic energy	$E_k = \frac{1}{2} m \omega^2 (x_0^2 - x^2)$	
Potential energy (for total P.E. = 0 at equilibrium)	$E_p = \frac{1}{2} m \omega^2 x^2$	
Total energy	$E = E_k + E_p = E_{k,max} = E_{p,max}$ $= \frac{1}{2} m \omega^2 (x_0^2 - x^2) + \frac{1}{2} m \omega^2 x^2$ $= \frac{1}{2} m \omega^2 x_0^2$	
Period, frequency, angular frequency	$T = \frac{1}{f}$ $\omega = \frac{2\pi}{T} = 2\pi f$	
Spring-mass system	$\omega = \sqrt{\frac{k}{m}}$ $T = 2\pi \sqrt{\frac{m}{k}}$	
Simple pendulum	$\omega = \sqrt{\frac{g}{L}}$ $T = 2\pi \sqrt{\frac{L}{g}}$	

APPENDIX

Relationship between Uniform Circular Motion and S.H.M.



Point P moves in a circle of radius x_o at a steady angular velocity ω. N is the projection of P to the diameter AOB of the circle. As P moves steadily round the circle, N moves to and fro along AOB.

The centripetal acceleration of P is $x_o\omega^2$, directed towards O.

Assume that $t = 0$ when $\theta = 0$ (i.e. $t = 0$ when $x = x_o$ or when the point N is at B).
After a time t,

$$\theta = \omega t$$

$$x = x_o \cos\theta = x_o \cos\omega t$$

The acceleration of N is the component of the acceleration of P parallel to AB:

$$a = -x_o\omega^2 \cos\theta$$

The negative sign indicates that the acceleration is directed towards O.
We can write

$$\begin{aligned} a &= -x_o\omega^2 \cos\theta \\ &= -x_o\omega^2 \cos\omega t \\ &= -\omega^2 (x_o \cos\omega t) \\ &= -\omega^2 x \end{aligned}$$

Thus N is in S.H.M.

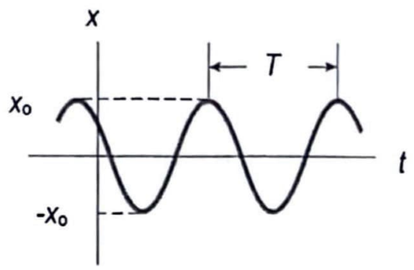
The period of N (time taken for N to go from B to A and back again) is given by

$$T = \frac{2\pi}{\omega}$$

The model shows that when a point moves in a uniform circular motion, the projection of that point to the diameter of the circle moves in S.H.M.

Phase Constant

In general, $x = x_0 \cos(\omega t + \phi)$



ϕ is the *phase constant* or the initial phase angle.

The phase of the motion is the quantity $(\omega t + \phi)$.

x_0 and ϕ are determined uniquely by the position and velocity of the particle at $t = 0$. E.g. if the particle is at $x = x_0$ at $t = 0$, then $\phi = 0$.

Phase Difference

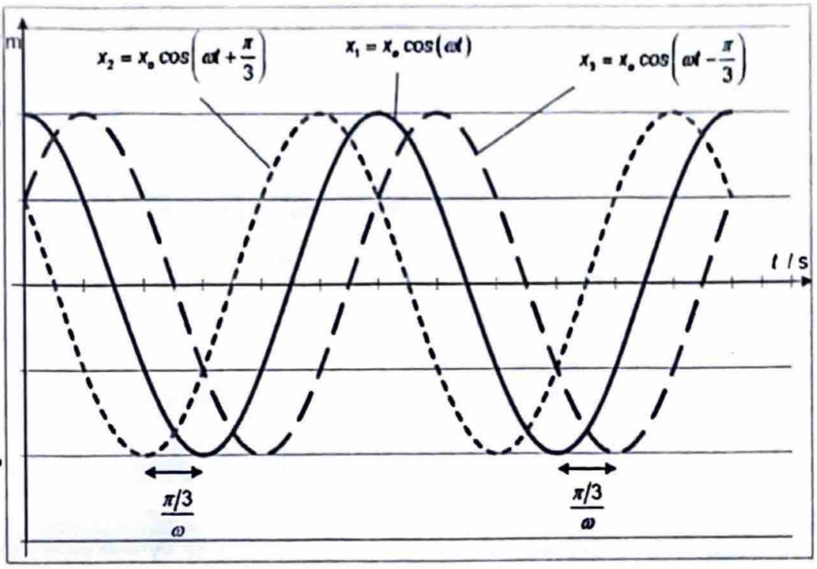
A graph of $x = x_0 \cos(\omega t + \phi)$ is the graph of $x_0 \cos \omega t$ displaced *to the left* by a time interval $\frac{\phi}{\omega}$.

The motion described by $x = x_0 \cos(\omega t + \phi)$ is *not in phase* with that described by $x_0 \cos \omega t$. It is out of phase by angle ϕ (radian) or time $\frac{\phi}{\omega}$. The plus sign indicates that this motion *leads* by time $\frac{\phi}{\omega}$ and so the graph is displaced to the left.

If the motion was described by $x = x_0 \cos(\omega t - \phi)$, the graph would be displaced to the right. This motion would be said to *lag* by time $\frac{\phi}{\omega}$.

Phase difference between two oscillators is the fraction of a complete oscillation by which one is ahead of the other. It can be expressed as a fraction of an oscillation, or, more usually, as an angle, measured in radians.

Example

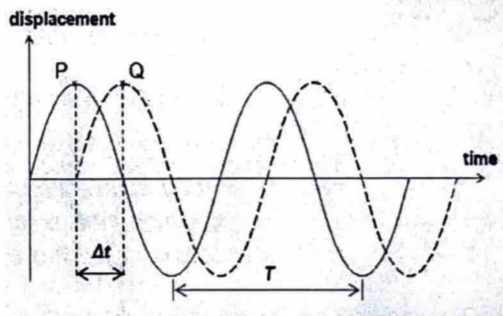


The motion described by $x_2 = x_0 \cos\left(\omega t + \frac{\pi}{3}\right)$ is not in phase with that described by $x_1 = x_o \cos \omega t$. It is out of phase by $\frac{\pi}{3}$ radian or time $\frac{\pi/3}{\omega}$. The plus sign indicates that x_2 *leads* x_1 by time $\frac{\pi/3}{\omega}$ = T/6 and so the graph is displaced to the *left*. The motion described by $x_3 = x_o \cos\left(\omega t - \frac{\pi}{3}\right)$ is the graph of x_1 displaced to the *right*. x_3 is said to *lag* x_1 by time $\frac{\pi/3}{\omega}$ = T/6.

The phase difference ϕ between two waveforms P and Q having the same period can be calculated using

$$\frac{\Delta t}{T} = \frac{\phi}{2\pi}$$

$$\Rightarrow \boxed{\phi = \frac{2\pi}{T}\Delta t}$$



Example

Refer to the graphs in Fig. 10.1.5, Fig. 10.1.6 and Fig. 10.1.7. What are the *time difference* and the *phase difference* between (i) *v* and *x*, (ii) *v* and *a*, and (iii) *a* and *x*?

<u>Solution:</u>

(i) The time difference is *T*/4, the phase difference is $\pi/2$ rad, and *v* leads *x*.

(ii) The time difference is *T*/4, the phase difference is $\pi/2$ rad, and *a* leads *v*.

(iii) The time difference is *T*/2, the phase difference is π rad, and *x* and *a* are in 'antiphase'.

RAFFLES INSTITUTION
YEAR 5-6 PHYSICS DEPARTMENT

Tutorial 10 OSCILLATIONS

Self-Check Questions

S1 What do you understand by the terms displacement, amplitude, period, frequency and angular frequency of a simple harmonic motion?

S2 Express the period T in terms of frequency f and angular frequency ω.

S3 Define *simple harmonic motion.* State the defining equation of simple harmonic motion.

S4 Write down a solution for a simple harmonic oscillator which starts its motion from the equilibrium position. How do you express its velocity and acceleration in terms of time?

S5 Draw graphs to show the changes in displacement, velocity and acceleration with respect to time of the oscillator in **S4**.

S6 Draw graphs to show how the velocity and acceleration of the oscillator in **S4** vary with displacement.

S7 Describe the variation between kinetic and potential energy with time during simple harmonic motion of a mass attached to a horizontal spring. Assume the mass is moving on a frictionless surface. What is the frequency of the energy variation as compared with that of the vibration itself?

S8 Give one practical example of a lightly damped oscillation. Why is critical damping in a car suspension system important?

S9 What do you understand by forced oscillations and resonance?

S10 Sketch a set of graphs, using the same axes, to show how the amplitude of forced oscillation varies with driving frequency for very light, moderate and heavy damping. Explain the features of your graphs.

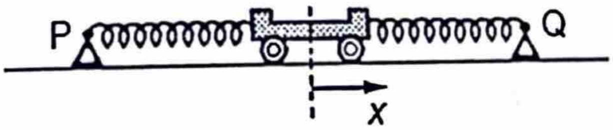
S11 Describe two examples of resonance, one in which this phenomenon is useful and the other in which it is a nuisance.

Self-Practice Questions

SP1 A trolley of mass 2 kg with free-running wheels is attached to two fixed points P and Q by two springs under tension as shown in the figure below.

The trolley is displaced a small distance (0.05 m) towards Q by a force of 10 N and is then released. The equation of the subsequent motion is $\ddot{x} = -\omega^2 x$, where x is the displacement from the equilibrium position. What is the constant ω^2?

(Note: velocity = $\dot{x}$, acceleration = $\ddot{x}$)



A 0.25 rad² s⁻²

B 1.0 rad² s⁻²

C 4.0 rad² s⁻²

D 100 rad² s⁻²

E 400 rad² s⁻²

J82/II/9

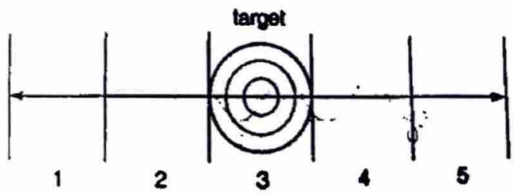
SP2 In which of the following lists are all three quantities constant when a particle moves in undamped simple harmonic motion?

A	acceleration	force	total energy
B	amplitude	angular frequency	acceleration
C	angular frequency	acceleration	force
D	force	total energy	amplitude
E	total energy	amplitude	angular frequency

J85/I/8 ; J92/I/9

SP3 In a fairground shooting game, a gun fires at a moving target. The gun fires by itself at random times. The player has to point the gun in a fixed direction, and the target moves from side to side with simple harmonic motion.

At which region should the player take a fixed aim in order to score the greatest number of hits?



A 3

B either 1 or 5

C either 2 or 4

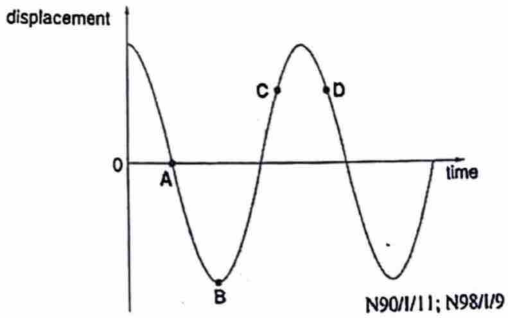
D any of 1, 2, 3, 4 and 5

J90/I/11 ; N95/I/9

SP4 The diagram shows the graph of displacement against time for a body performing simple harmonic motion.

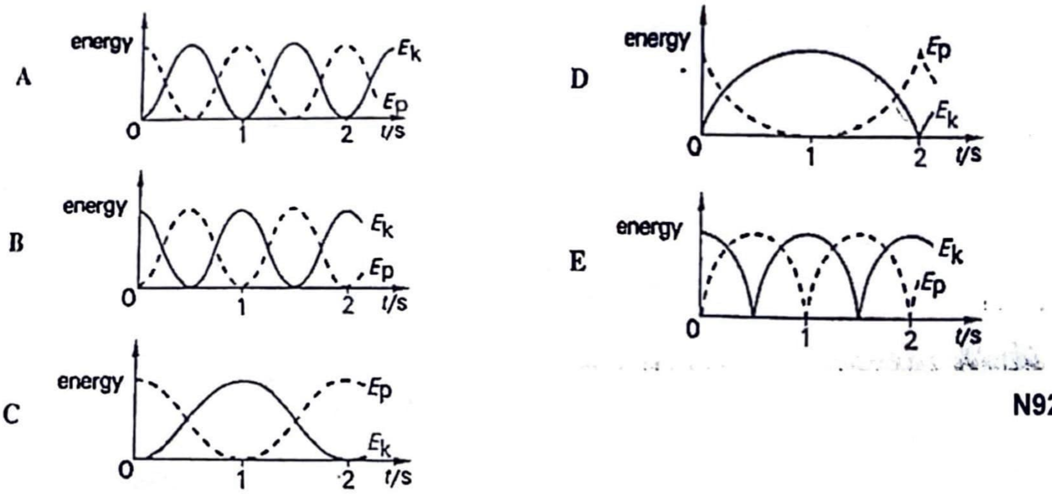
At which point are the velocity and acceleration in opposite directions?

N90/I/11 ; N98/I/9



SP5 The bob of a simple pendulum of period 2 s is given a small displacement and then released at time $t = 0$.

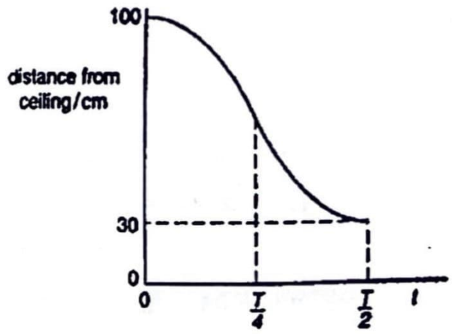
Which diagram shows the variations with time of the bob's kinetic energy E_k and its potential energy E_p?



N92/I/9

SP6 A mass hanging from a spring suspended from a ceiling is pulled down and released. The mass then oscillates vertically with simple harmonic motion of period T. The graph shows how its distance from the ceiling varies with time t.

What can be deduced from this graph?



A The amplitude of the oscillation is 70 cm.

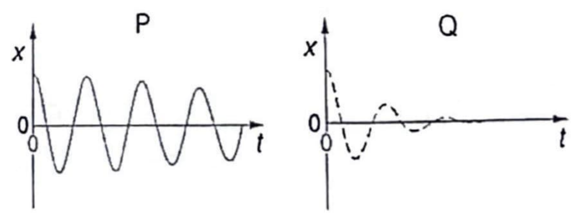
B The kinetic energy is a maximum at $t = \frac{T}{2}$.

C The restoring force on the mass increases between $t = 0$ and $t = \frac{T}{4}$.

D The speed is a maximum at $t = \frac{T}{4}$.

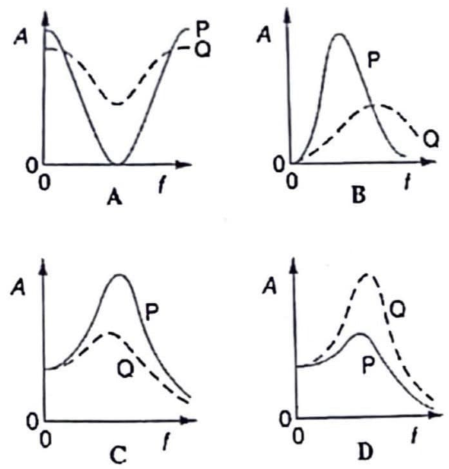
J2000/I/9

SP7 Two objects P and Q are given the same intial displacement and are then released. The graphs show the variation with time t of their displacements x.



P and Q are then subjected to driving forces of the same constant amplitude and of variable frequency *f*.

Which graph represents the variation with *f* of the amplitudes *A* of P and of Q?



J86/I/8 ; N93/I/7 ; J99/I/9

SP8 The displacement of a particle P which moves with simple harmonic motion can be described by the expression

$$x = (0.05) \sin 8\pi t$$

where *x* is in metres and *t* in seconds.

(a) What is the amplitude of the motion?

(b) What is the frequency of the motion?

(c) How long does it take for the particle to complete one oscillation?

(d) What is the velocity of the particle as it passes through its equilibrium position, and at the extreme end of the swing?

(e) What is the maximum acceleration of the particle during its motion?

Another particle Q also moves with simple harmonic motion of the same frequency. However, the motion of Q *lags* that of P by π/2 rad and the amplitude of Q is twice that of P.

Draw, using the same axes, the displacement-time graphs for motions of P and Q.

Write an equation to describe how the displacement of Q varies with time.

SP9 A light spring stretches 0.150 m when a 0.300 kg mass is hung from its lower end. The mass is pulled down 0.100 m below this equilibrium point and released. Determine

(a) the spring constant, [1]

(b) the amplitude of the oscillation, [1]

(c) the maximum velocity, [1]

(d) the magnitude of velocity when the mass is 0.050 m from equilibrium, [1]

(e) the magnitude of the maximum acceleration of the mass. [1]

SP10 Discuss the energy changes which take place when a mass suspended from a spring is pulled downwards and released, such that it oscillates vertically.

Discussion Questions

D1 **(a)** **[J95/III/3 part]**

In one harbour, the equation for the depth h of water is

$$h = 5.0 + 3.0\sin\frac{2\pi t}{45600},$$

where h is given in metres and t is the time in seconds.

(The angle $\frac{2\pi t}{45600}$ is in radians.) For this harbour, calculate

(i) the maximum depth of water, [1]

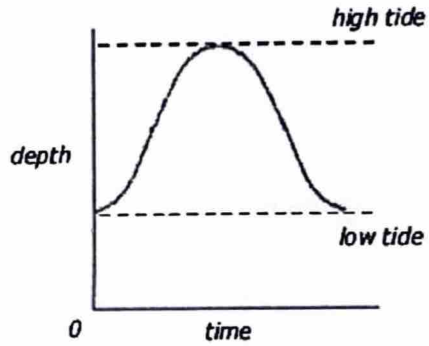
(ii) the minimum depth of water, [1]

(iii) the time interval between high- and low-water, [2]

(iv) two values of t at which the water is 5.0 m deep, [2]

(v) the length of time for each tide during which the depth of water is more than 7.0 m. [4]

(b) **[J91/I/9]**

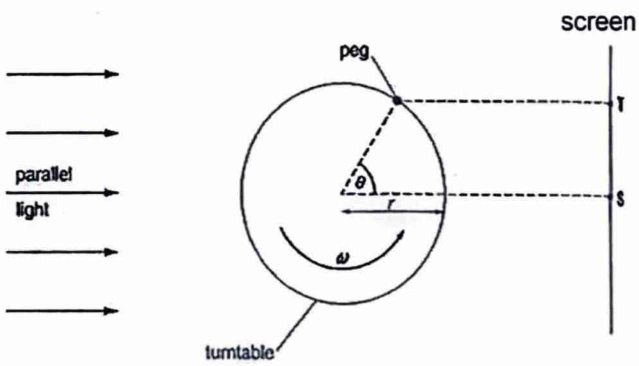


The rise and fall of water in a harbour is simple harmonic. The depth varies between 1.0 m at low tide and 3.0 m at high tide. The time between successive low tides is 12 hours.

A boat which requires a minimum depth of water of 1.5 m, approaches the harbour at low tide. How long will the boat have to wait before entering? [2]

D2 **[N96/II/2]**

A vertical peg is fixed to the rim of a horizontal turntable of radius r, rotating with a constant angular speed ω as shown in the figure.



Parallel light is incident on the turntable so that the shadow of the peg is observed on a screen which is normal to the incident light. At time t = 0, θ = 0 and the shadow of the peg is seen at S. At some later time t, the shadow is seen at T.

(a) **(i)** Write down an expression for θ in terms of ω and t. [1]
(ii) Derive an expression for the distance ST in terms of r, ω and t. [1]

(b) By reference to your answer to **(a)(ii)**, describe the motion executed by the shadow on the screen. [1]

(c) The turntable has a radius r of 20 cm and an angular speed ω of 3.5 rad s^{-1}. Calculate, for the motion of the shadow on the screen,
(i) the amplitude, [1]
(ii) the period, [1]
(iii) the speed of the shadow as it passes through S, [2]
(iv) the magnitude of the acceleration of the shadow when the shadow is instantaneously at rest. [2]

D3 **[N03/II/4]**
(a) Define *simple harmonic motion.* [2]

(b) A horizontal metre rule is clamped at one end. The free end oscillates vertically as shown in Fig. 3.1

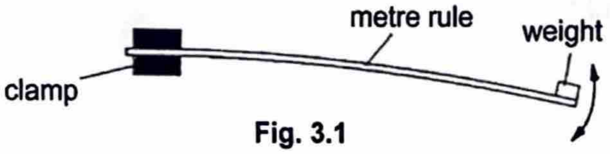


Fig. 3.1

Fig. 3.2 shows the variation with time t of the velocity v of a point at the free end of the rule.

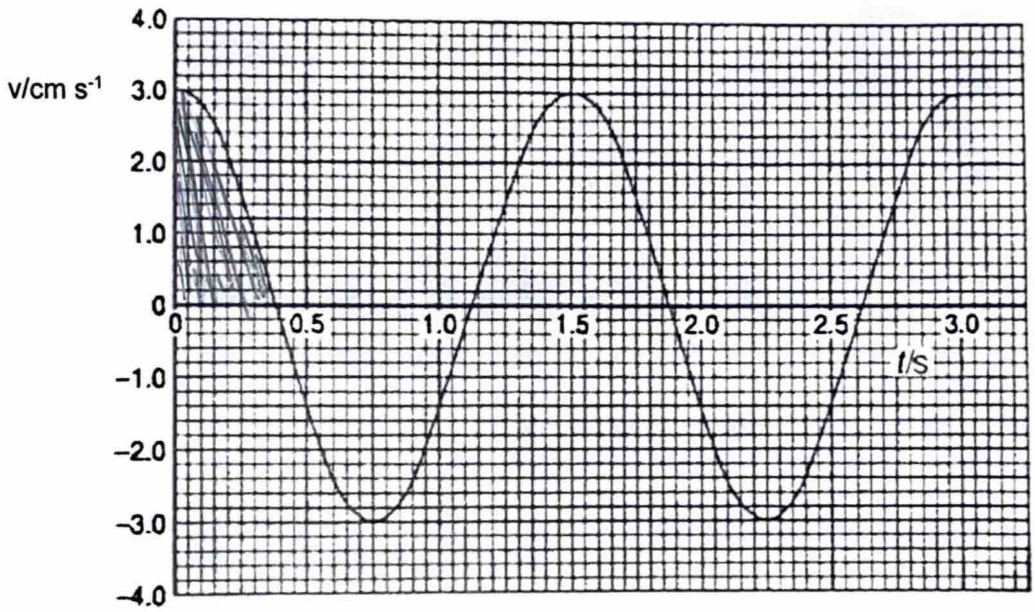


Fig. 3.2

(i) On Fig. 3.2, shade an area that represents the amplitude of the oscillations of the free end of the rule. [1]

(ii) Determine, for these oscillations,
1. the frequency,
2. the amplitude. [4]

(iii) On the axes of Fig. 3.3, sketch a graph to show the variation with displacement *d* of the velocity *v* of the end of the rule. Mark a scale on the d-axis. [2]

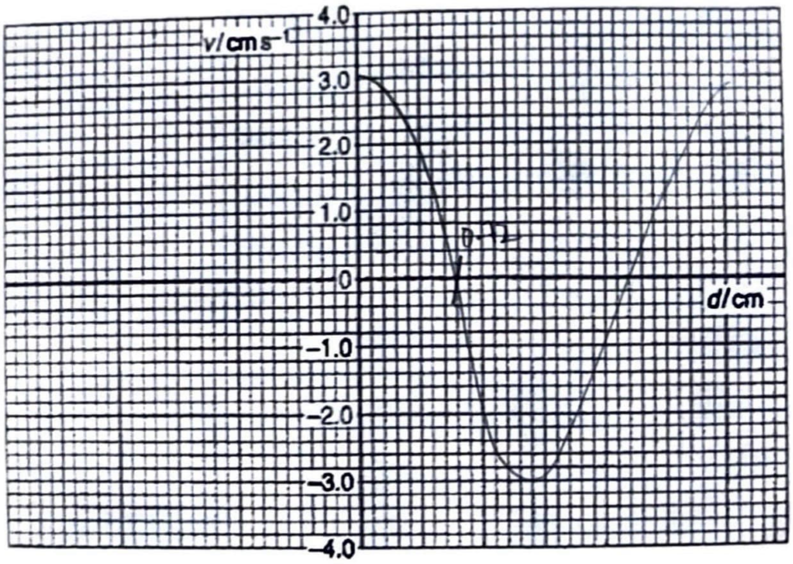
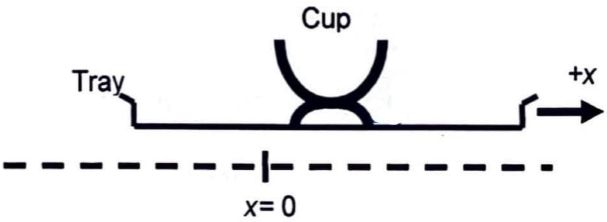


Fig. 3.3

D4 A tray, holding an empty cup, is moved horizontally back and forth in simple harmonic motion. At one instant of time, the tray is displaced to the right of the equilibrium position ($x = 0$) as indicated by the arrow shown in the figure below.



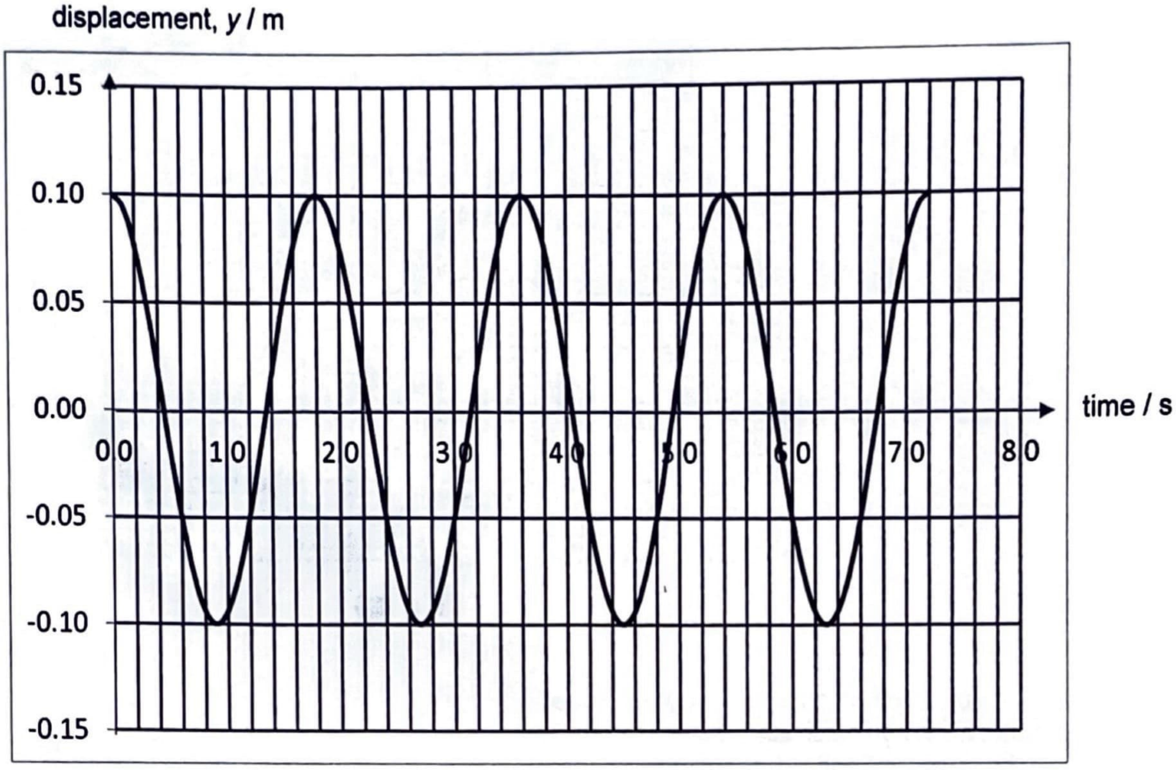
(a) Draw the frictional force *F* acting on the cup for the instant of time shown. [1]

(b) Write an equation for *F* in terms of the mass *m* of the cup, the angular frequency ω of the motion, and the displacement *x* of the tray. [1]

(c) Given that the maximum value of *F* is half the weight of the cup, explain why the cup will be observed to slip if the frequency of oscillation increases beyond a certain value. [2]

(d) If the amplitude of the motion is 0.050 m, calculate the maximum possible frequency such that the cup does not slip. [2]

D5 An object undergoing simple harmonic motion has displacement y, as shown.
Use the graph to determine the amplitude, period and angular frequency of this oscillation.



State, for each of the following, a time at which the oscillating object has

(a) maximum positive velocity, [1]

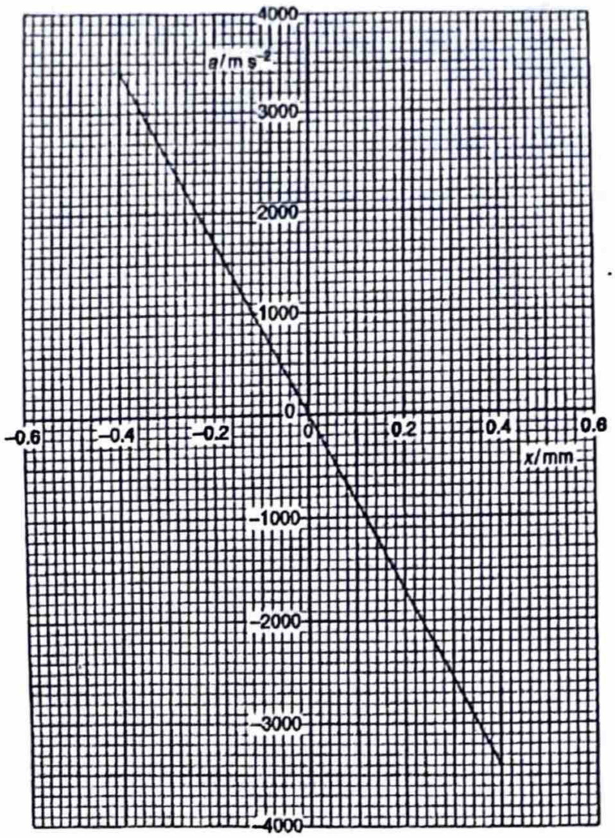
(b) maximum positive acceleration, [1]

(c) maximum negative acceleration, [1]

(d) maximum kinetic energy, [1]

(e) maximum potential energy. [1]

D6 **[N06/II/3]**



The figure shows the variation with displacement *x* of the acceleration *a* of a particle P attached to the cone of a loudspeaker.

(a) Use the figure to

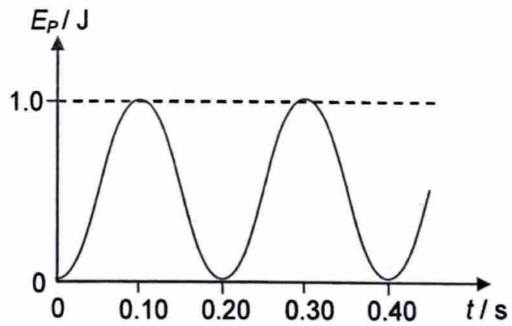
(i) explain why the motion of particle P is simple harmonic, [2]

(ii) show that the frequency of oscillations of particle P is 460 Hz. [2]

(b) **(i)** The magnitude of the gradient of the line in the figure is *G*. Show that, for a particle of mass *m* oscillating with amplitude *A*, its maximum kinetic energy E_{MAX} is given by $E_{MAX} = \frac{1}{2}mGA^2$. [3]

(ii) Determine E_{MAX} for particle P of mass 2.5×10^{-3} kg. [2]

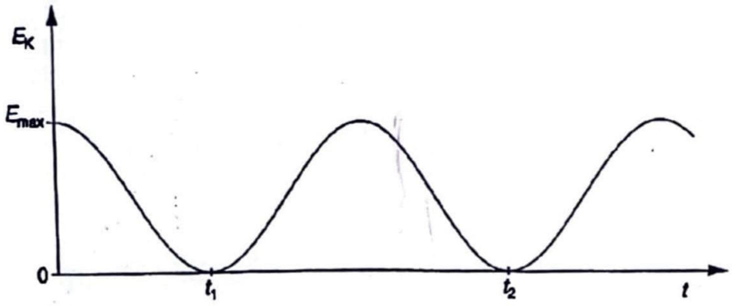
D7 **(a)** An object undergoes simple harmonic motion with an amplitude of 0.30 cm. The graph shows the variation of its potential energy E_P with time *t*.



What is the maximum acceleration and mass of the object? [4]

(b) **[N15/III/8 part]**
A longitudinal sound wave is travelling through a gas causing oscillations of gas molecules that are simple harmonic.

(i) The gas molecules, each of mass 5.3×10^{-26} kg, are vibrating at a frequency of 835 Hz and have an amplitude of vibration of 610 nm.
The variation with time *t* of the vibrational kinetic energy E_K of a molecule is shown in the figure below.



Determine, for one vibrating molecule,

1. the period *T* of the vibrations, [2]
2. the time interval $(t_2 - t_1)$, [1]
3. the maximum speed v_{max}, [2]
4. the maximum vibrational kinetic energy E_{max}. [2]

(ii) By reference to the speed of sound in a gas at room temperature, comment on your answer in **(i)3**. [2]
(Given: speed of sound in gas at room temperature is 340 m s^{-1}.)

D8 **[N13/III/7 part]**
(a) A tube, sealed at one end, has a uniform area of cross-section *A*. Some sand is placed in the tube so that it floats upright in a liquid of density ρ, as shown in Fig. 8.1.

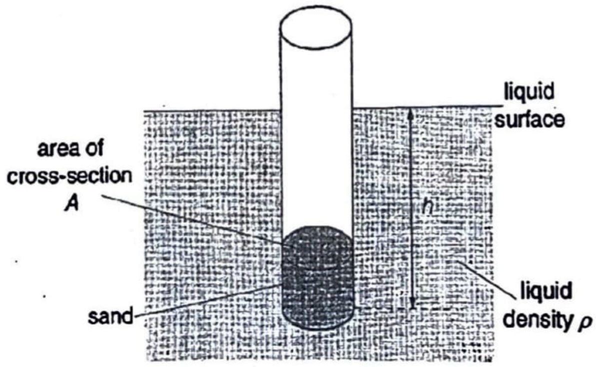


Fig. 8.1

The total mass of the tube and the sand is *m*.
The tube floats with its base a distance *h* below the surface of the liquid.

Derive an expression relating *m* to *h*, *A* and ρ. Explain your working. [3]

(b) The tube in **(a)** is displaced vertically and then released.
For a displacement x, the acceleration a of the tube is given by the expression

$$a = -\left(\frac{\rho A g}{m}\right)x$$

where g is the acceleration of free fall.

(i) Explain why the expression leads to the conclusion that the tube is performing simple harmonic motion. [3]

(ii) The tube has total mass m of 32 g and the area A of its cross-section is 4.2 cm^2. It is floating in liquid of density ρ of 1.0×10^3 kg m^{-3}.

Show that the frequency of oscillation of the tube is 1.8 Hz. [3]

(c) The tube in **(a)** is now placed in a different liquid.

The tube oscillates vertically. The variation with time t of the vertical displacement x of the tube is shown in Fig. 8.2.

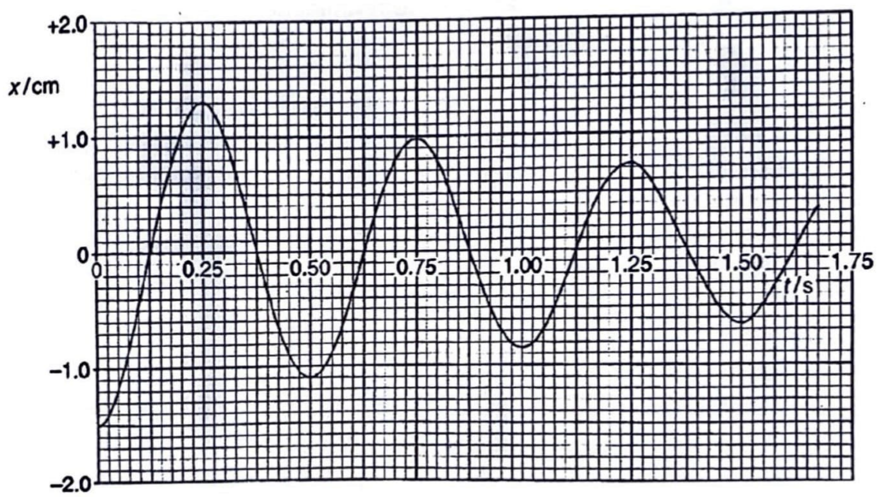


Fig. 8.2

(i) Use Fig. 8.2 to

1. determine the frequency of oscillation of the tube, [2]
2. calculate the density of the liquid. [2]

(ii) 1. Suggest two reasons why the amplitude of the oscillation decreases with time. [2]
2. Calculate the decrease in energy of the oscillation during the first 1.0 s. [3]

D9 **[J97/II/2]**

Fig. 9.1 illustrates a mass which can be made to vibrate vertically between two springs. The vibrator itself has constant amplitude. As the frequency is varied, the amplitude of vibration of the mass is seen to change as shown in Fig. 9.2.

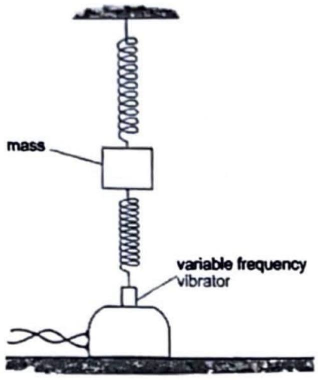


Fig. 9.1

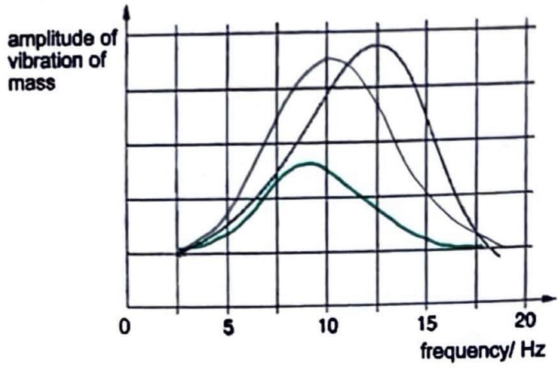


Fig. 9.2

(a) Name the phenomenon which is illustrated in Fig. 9.2. [1]

(b) For the mass vibrating at maximum amplitude, calculate
(i) the angular frequency, [2]
(ii) the period. [2]

(c) A light piece of card is fixed to the mass with its plane horizontal. On Fig. 9.2, draw a line to show the variation with frequency of the amplitude of vibration of the mass. [2]

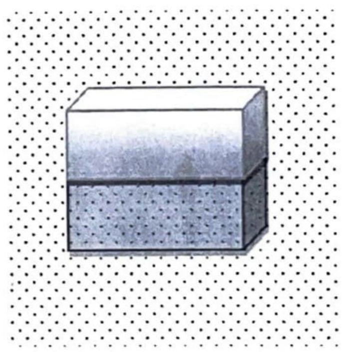
(d) State one situation in which the phenomenon illustrated in Fig. 9.2 is used to advantage. [1]

D10 **[N94/II/2]**

A block of wood of mass *m* floats in still water as shown in the figure. When the block is pushed down into the water, without totally submerging it, and is then released, it bobs up and down in the water with a frequency *f* given by the expression:

$$f = \frac{1}{2\pi}\sqrt{\frac{28}{m}}$$

where *f* is measured in Hz and *m* in kg.



Surface water waves of speed 0.90 m s^{-1} and wavelength 0.30 m are then incident on the block. These cause resonance in the up-and-down motion of the block.

(a) Explain what is meant by the term *resonance*. [2]

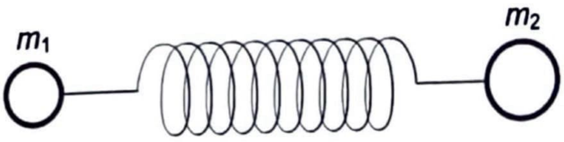
(b) Calculate
(i) the frequency of the water waves, [1]
(ii) the mass of the block. [2]

(c) Describe and explain what happens to the amplitude of the vertical oscillations of the block after the following changes are made independently:
(i) water waves of larger amplitude are incident on the block, [2]
(ii) the distance between the wave crests increases, [2]
(iii) the block has absorbed some water. [2]

Note: for a wave, $v = f\lambda$ where v is the speed, f is the frequency and λ is the wavelength

Challenging Questions

C1 The figure below shows an isolated oscillatory system. Two bodies of mass m_1 and m_2 are joined by a light spiral spring. Each body oscillates along the axis of the spring, which obeys Hooke's law in both extension and compression.



(a) The bodies move in opposite directions and the centre of mass of the system is stationary. Explain why the periods of oscillations of both bodies are the same.

(b) Show that when the body on the left moves through a distance x, the change in length of the spring is:

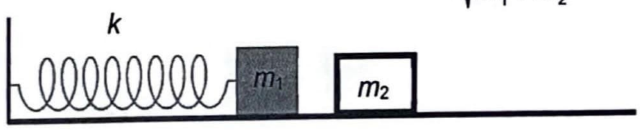
$$x\left(1+\frac{m_1}{m_2}\right)$$

(c) Hence, show that its period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m_1 m_2}{(m_1+m_2)k}}$$

where k is the spring constant of the spring.

C2 Two masses slide on a frictionless table. Mass m_1, but not m_2, is fastened to a spring. If now m_1 and m_2 are pushed to the left so that the spring is compressed a distance x, show that the amplitude of the oscillation of m_1 after the spring system is $x\sqrt{\dfrac{m_1}{m_1+m_2}}$.



Answers

D1 **(a)(i)** 8.0 m **(ii)** 2.0 m **(iii)** 22800 s **(iv)** 0, 22800 s, 45600 s **(v)** 12200 s
(b) 2.0 hours

D2 **(c)(i)** 20 cm **(ii)** 1.80 s **(iii)** 0.700 m s^{-1} **(iv)** 2.45 m s^{-2}

D3 **(b)(ii) 1.** 0.6667 Hz **2.** 0.72 cm

D4 **(d)** 1.58 Hz

D5 **(a)** 1.35 s, 3.15 s, 4.95 s
(b) 0.90 s, 2.70 s, 4.50 s
(c) 0.00 s, 1.80 s, 3.60 s
(d) 0.45 s, 1.35 s, 2.25 s, 3.15 s, 4.05 s , 4.95 s
(e) 0.00 s, 0.90 s, 1.80 s, 2.70 s, 3.60 s, 4.50 s

D6 **(b)(ii)** 1.7×10^{-3} J

D7 **(a)** 0.74 m s^{-2}, 900 kg
(b)(i) 1. 1.20×10^{-3} s **2.** 5.99×10^{-4} s **3.** 3.20×10^{-3} m s^{-1} **4.** 2.71×10^{-31} J

D8 **(c)(i) 1.** 2.0 Hz **2.** 1.23×10^{3} kg m^{-3} **(ii) 2.** 3.86×10^{-4} J

D9 **(b)(i)** 78.5 rad s^{-1} **(ii)** 0.0800 s

D10 **(b)(i)** 3.0 Hz **(ii)** 0.079 kg

Tutorial 10 Oscillations Suggested Solutions

S1 displacement, x – distance in a specific direction from the equilibrium position

amplitude, x_o – the magnitude of maximum displacement from equilibrium position

period, T – time taken to complete one oscillation

frequency, f – number of oscillations per unit time

angular frequency, ω – rate of change of phase angle, it is equal to the product of 2π and frequency

S2 $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

S3 Simple harmonic motion is defined as the motion of a particle about a fixed point such that its acceleration is proportional to its displacement from the fixed point and is always directed towards the point.

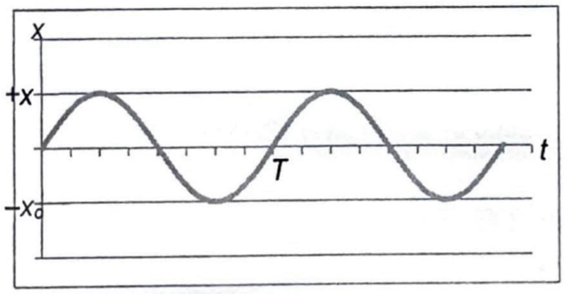
The defining equation is $a = -\omega^2 x$.

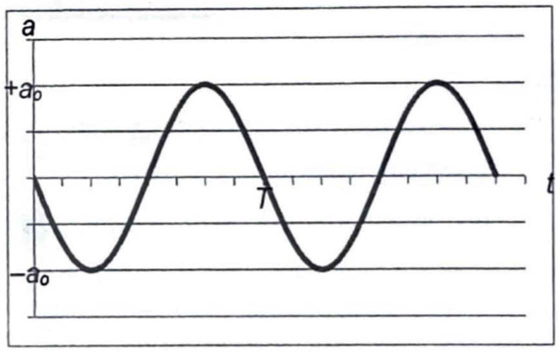
S4 $x = x_o \sin \omega t$

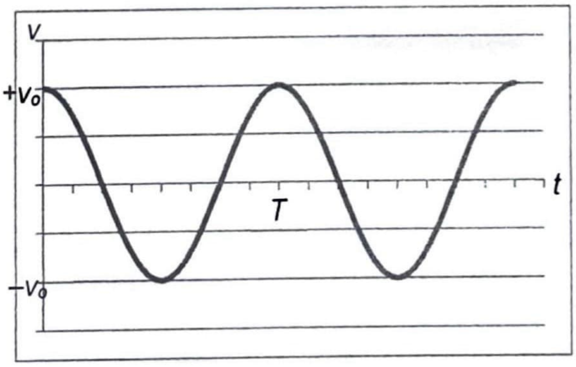
$v = x_o \omega \cos \omega t = v_o \cos \omega t$, where $v_o = x_o \omega$ is the maximum velocity

$a = -x_o \omega^2 \sin \omega t = -a_o \sin \omega t$, where $a_o = x_o \omega^2$ is the maximum acceleration

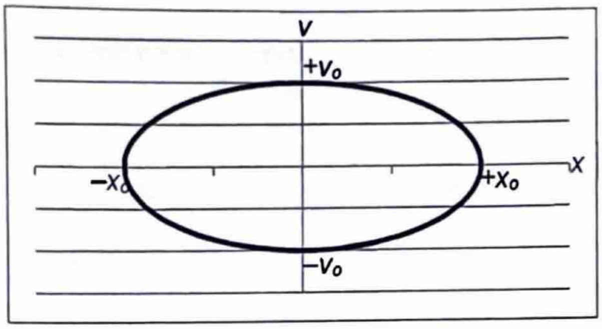
S5

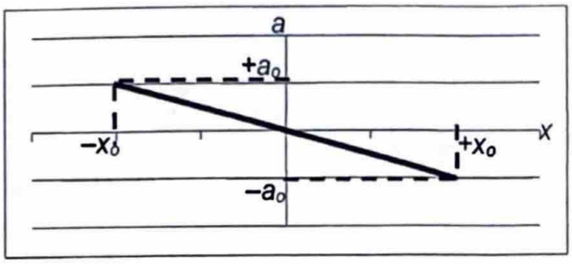




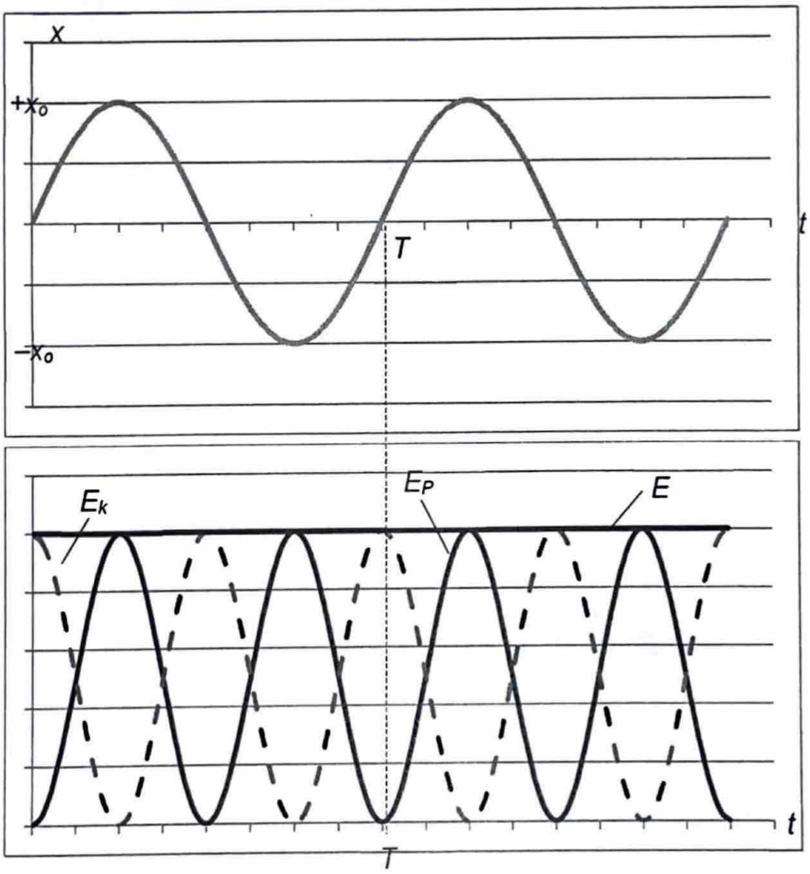


S6





S7



There is a continual change of energy from kinetic energy E_k to potential energy E_P and *vice-versa*. E_k is greatest and E_P is zero as the mass passes through the equilibrium position.

As the mass approaches the endpoints, its E_k decreases (E_P increases).

As the mass approaches the equilibrium point, its E_k increases (E_P decreases).

The frequency of energy variation is <u>twice</u> that of the oscillation itself.
(At any instant during the motion, the total energy E of the system is constant and equal to the sum of E_k and E_P. Note that $E = \frac{1}{2}m\omega^2 x_0{}^2 = \frac{1}{2}m(2\pi f)^2 x_0{}^2$, is proportional to $x_o{}^2$ or f^2.)

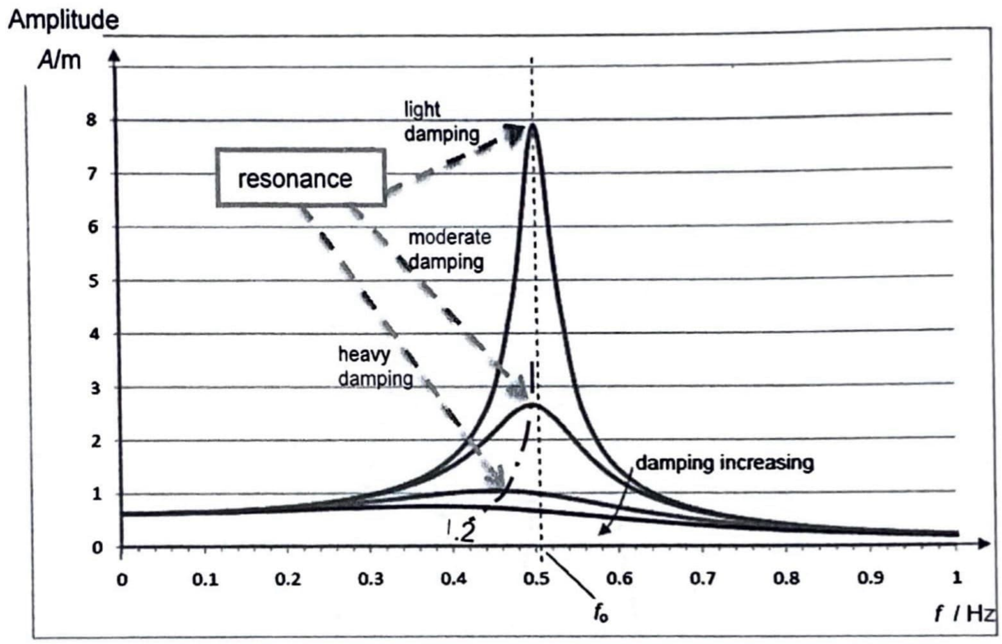
S8 A swinging pendulum or vibrating tuning fork are examples of lightly damped oscillations.

The suspension system of a car includes shock absorbers and springs. When a car goes over a bump, the springs will be displaced from their equilibrium lengths. The viscous oil in the shock absorbers provide damping to enable the springs to smoothly and quickly return to their equilibrium lengths without oscillating up and down. The springs are critically damped. This will reduce the discomfort of the passengers. Without critical damping, the body of the car will oscillate up and down after going over a bump, which is undesirable.
(*Refer to lecture notes on the car suspension system.*)

S9 Forced oscillations are produced when a body is subjected to a periodic external driving force.

Resonance occurs when a system responds at maximum amplitude to an external driving force. This occurs when the frequency of the driving force is equal to the natural frequency of the driven system.

S10



When conditions are *steady*, the amplitude of a forced oscillation depends upon the damping of the system and the relative values of the driving frequency f and the natural frequency f_o of the system. Oscillations with the largest amplitude (i.e. resonance) occur when f is approximately equal to f_o.

The sharpness of resonance is determined by the degree of damping.

When damping is light, the amplitude is large but falls off rapidly when the driving frequency of the body differs slightly from the natural frequency of the body. The resonance is sharp.

When damping is moderate, the amplitude at resonance decreases. The curve falls off gradually and maximum amplitude occurs at a frequency that is lower than the natural frequency of the body.

When damping is heavy, the resonance is flat.

S11 *Refer to lecture notes on circumstances in which resonance is useful or should be avoided.*

SP1 Trolley is performing S.H.M. since it is given that its acceleration, $a = \ddot{x} = -\omega^2 x$.
At maximum displacement x_0,

$$F_{max} = ma_{max}$$

$$a_{max} = \frac{F_{max}}{m}$$

$$\omega^2 = \frac{F_{max}}{mx_0} = \frac{10}{2(0.05)} = 100 \text{ rad}^2 \text{ s}^{-2}$$

Answer: D

SP2 In an undamped S.H.M., total energy of the oscillating system is a constant though kinetic and potential energies are constantly being transformed from one form to the other.
Since total energy and maximum energy is constant, amplitude is also constant.
Period of the oscillation is also constant, hence angular frequency is also constant.
Answer: E

SP3 Player should aim at the regions where the moving target is moving the slowest i.e. at the extreme ends of the S.H.M. (Target is momentarily stationary at the amplitude of the oscillation.)
Answer: B

SP4 At point C, the body's velocity is positive (gradient positive) and since its displacement positive, its acceleration is negative ($a = -\omega^2 x$).
Answer: C

SP5 At $t = 0$, kinetic energy of the bob is zero and potential energy is maximum. This will happen again at $t = 1$ s and $t = 2$ s when the bob is at the points of maximum displacement.
Hence the frequency of the energy variations is twice that of the oscillation itself (i.e. the period of the energy variations is half that of the oscillation itself).
Answer: A

SP6 From $t = 0$ to $t = T/4$ (a quarter through its oscillation), the speed of the mass increases from zero to a maximum at the equilibrium position and acceleration/restoring force decreases from a maximum to zero. The amplitude is thus (100 –30) / 2 = 35 cm.
At t = T/2, the mass is at its maximum displacement from the equilibrium position. It is momentarily at rest. So speed and kinetic energy is zero.
Answer: D

SP7 P experiences a smaller damping force than Q because Q comes to rest in a shorter time. Hence under forced oscillations, P should exhibit larger amplitudes and a larger resonance frequency.
Answer: C

SP8 Compare $x = (0.05)\sin 8\pi t$ with $x = x_o \sin \omega t$.

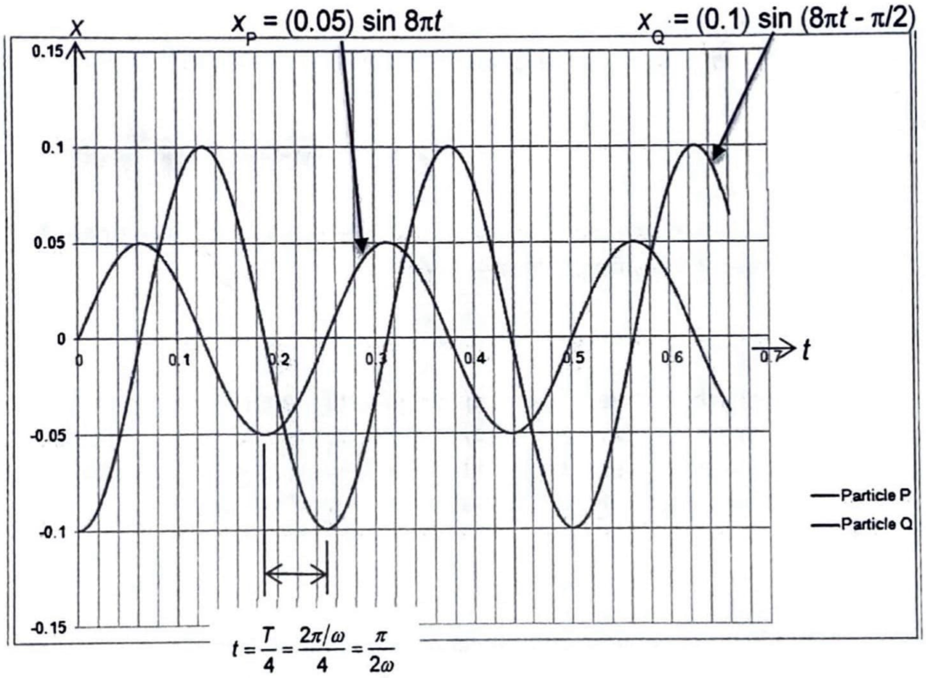
(a) amplitude, $x_o = 0.05$ m

(b) frequency, $f = \omega / 2\pi = 8\pi / 2\pi = 4$ Hz

(c) period, $T = 1/f = ¼$ s $= 0.25$ s

(d) velocity of the particle as it passes through its equilibrium position,
$|v_o| = \omega x_o$
$= (8\pi)(0.05 \text{ m})$
$= 1.26 \text{ m s}^{-1}$;
velocity of the particle at the extreme end of the swing = 0 m s^{-1}

(e) maximum acceleration of the particle during its motion,
$|a_o| = \omega v_o$
$= (8\pi)(8\pi)(0.05 \text{ m})$
$= 31.6 \text{ m s}^{-2}$



SP9 (a) $mg = ke \Rightarrow (0.300)(9.81) = k(0.150) \Rightarrow k = 19.6 \text{ N m}^{-1}$

(b) $y_o = 0.100$ m

(c) $v_o = y_o\omega = y_o\left(\sqrt{\frac{k}{m}}\right) = (0.100)\left(\sqrt{\frac{19.62}{0.300}}\right) = 0.808 \text{ m s}^{-1}$

(d) $v = \omega\sqrt{(y_o^2 - y^2)} = \sqrt{\frac{19.62}{0.300}}\sqrt{(0.100^2 - 0.050^2)} = 0.700 \text{ m s}^{-1}$

(e) $a_o = y_o\omega^2 = (0.100)\left(\frac{19.62}{0.300}\right) = 6.54 \text{ m s}^{-2}$

SP10 Assume that the spring is not compressed during the oscillation of the mass, i.e. the mass is not pulled down a distance which is more than the equilibrium extension.

Take the lowest position of the oscillating mass to be the zero gravitational potential energy level.

At the lowest position of the oscillation, the gravitational potential energy and kinetic energy of the system is at a minimum while its elastic potential energy is at a maximum.

As the mass moves upwards, elastic potential energy is converted into gravitational potential energy and kinetic energy.

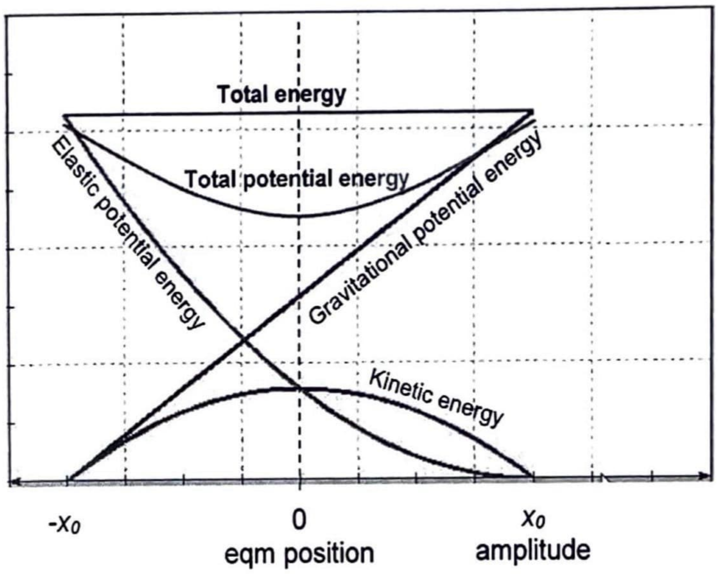
When the mass is at the centre of its oscillation, the system has maximum kinetic energy.

As it continues to move upwards, kinetic and elastic potential energy of the system is converted into gravitational potential energy.

At the highest position of the oscillation, the system has the highest gravitational potential energy, lowest elastic potential energy and zero kinetic energy.

The interchange between these three forms of energy continues as the mass oscillates but there will be energy lost in the form of internal energy due to air resistance.

Assuming the spring is displaced elastically by a distance equal to the equilibrium extension, the following graph showing the variation of energies with displacement will be obtained.



(Refer to notes Example 5)
If the spring is displaced elastically by a distance equal to the equilibrium extension and the zero level for gravitational potential energy is taken to be midway between the un-stretched and equilibrium positions instead, the following graph will be obtained.