



EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

## MATHEMATICS

**9758/02**

Paper 2 [100 marks]

**21 September 2018**

**3 hours**

Additional Materials: Answer Paper

List of Formulae (MF26)

### READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

**Section A: Pure Mathematics [40 marks]**

- 1 (i)** Given that  $f$  is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f\left(\frac{n+1}{n}\right) + f\left(\frac{n+2}{n}\right) + \dots + f\left(\frac{2n}{n}\right) \right\}$$

is  $\int_1^2 f(x) \, dx$

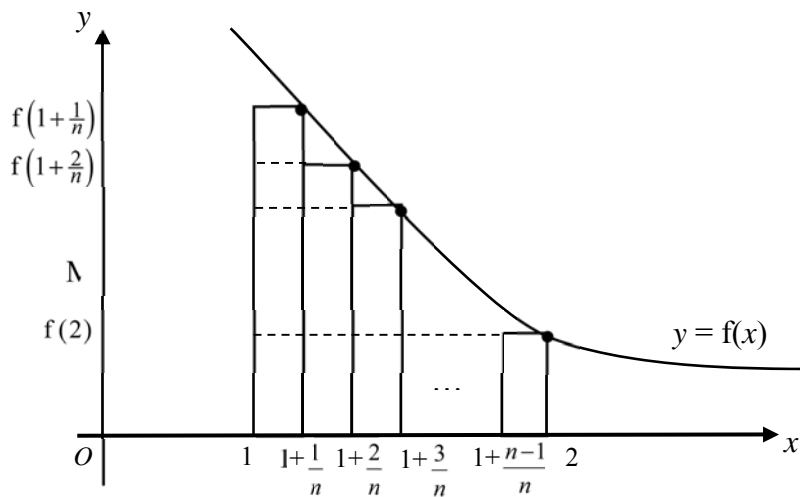
[2]

- (ii)** Hence evaluate  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{n}{n+r}$  exactly.

[3]

**Suggested Solution**

**(i)**



As shown in the diagram, the area under the curve  $y = f(x)$  from  $x = 1$  to  $x = 2$  can be approximated by the total areas of the  $n$  rectangles with width  $\frac{1}{n}$  and heights given by

$$f\left(1 + \frac{1}{n}\right), f\left(1 + \frac{2}{n}\right), f\left(1 + \frac{3}{n}\right), \dots, f\left(1 + \frac{n-1}{n}\right).$$

As  $n$  increases, the approximations will get better and approach the exact area as a limit,

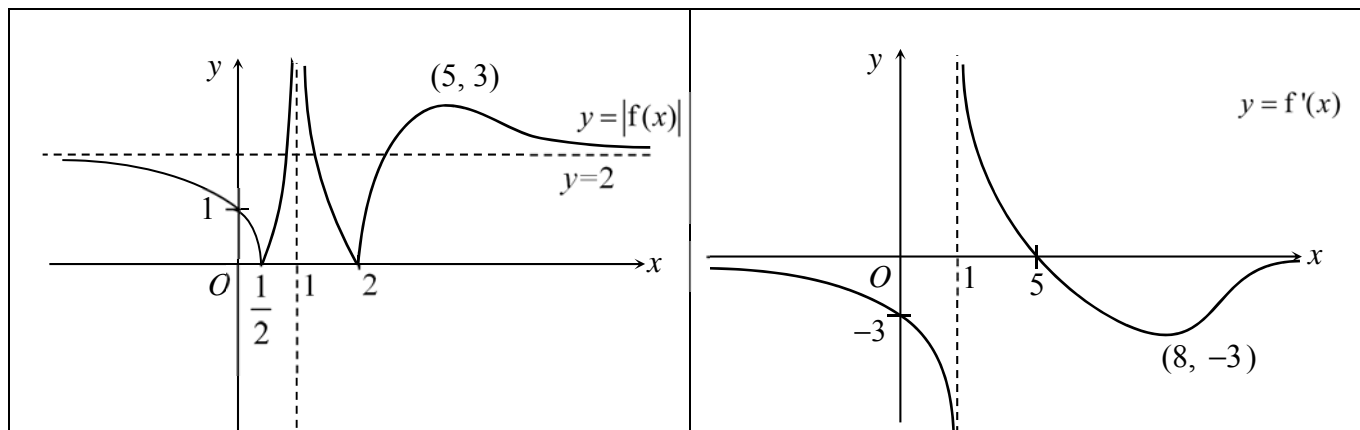
$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f\left(\frac{n+1}{n}\right) + f\left(\frac{n+2}{n}\right) + \dots + f\left(\frac{2n}{n}\right) \right\} = \int_1^2 f(x) \, dx$$

i.e. [Shown]

**(ii)**  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{n}{n+r} = \int_1^2 f(x) \, dx$  where  $f(x) = \frac{1}{x}$

$$= \int_1^2 \frac{1}{x} \, dx = [\ln |x|]_1^2 = \ln 2 - \ln 1 = \ln 2$$

2 The diagrams below show the graphs of  $y = |f(x)|$  and  $y = f'(x)$ .



On separate diagrams, sketch the graphs of:

(i)  $y = |f(2x)| + 1$  [2]

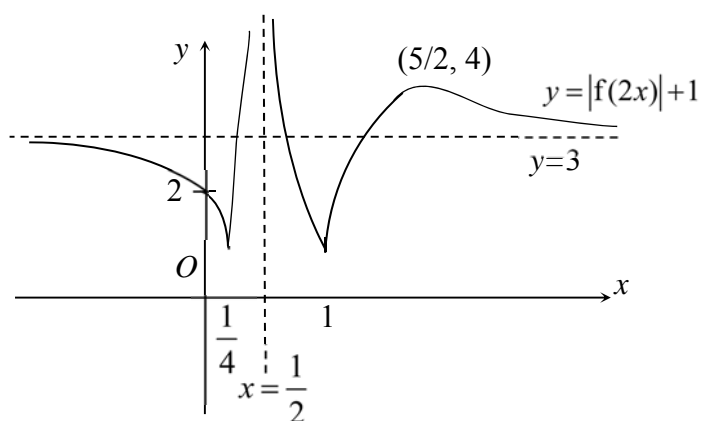
(ii)  $y = \frac{1}{f'(x)}$  [3]

(iii)  $y = f(x)$  [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

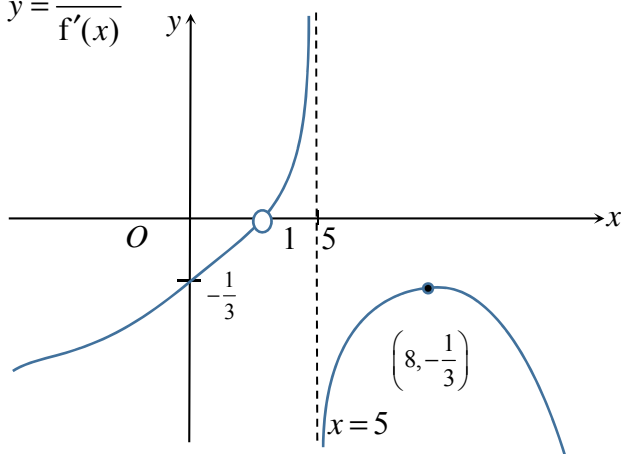
**Suggested Solution**

(i)



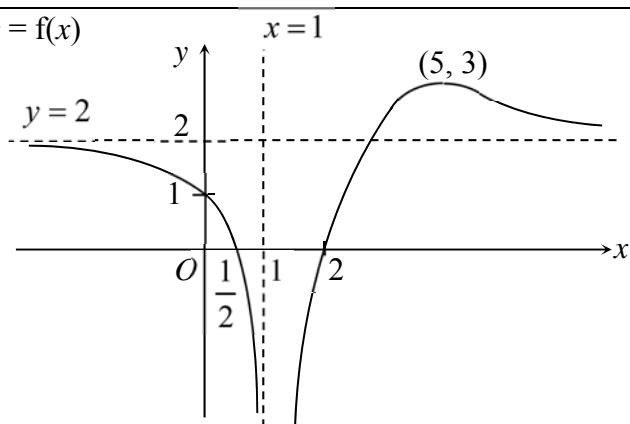
(ii)

$$y = \frac{1}{f'(x)}$$



(iii)

$$y = f(x)$$



- 3 (a) A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of  $\$P$  is deposited

to open the account and a monthly pay-out of \$ $x$  is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

- (i) Show that the amount in the account at the end of  $n$  months after the interest has been added is given by

$$P(1.002^n) - 501x(1.002^n - 1). \quad [4]$$

- (ii) Suppose a fixed monthly pay-out of \$2,000 is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

- (iii) If a principal amount of \$600,000 is placed in the account, find the number of years for which a monthly pay-out of \$2,000 per month can be sustained, leaving your answer correct to the nearest whole number. [2]

- (b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is \$ $a$ . The pay-out for each subsequent month is an increment of \$ $c$  from the pay-out of the previous month.

The pay-out in the final month is \$4,000, and the total pay-out at the end of 25 complete years is \$751,500. Find the month in which the pay-out is \$2,000. [5]

### Suggested Solution

(a)(i)

| Month | Balance at the end of the month  |
|-------|--|
| 1     | $(P - x)(1.002) = P(1.002) - x(1.002)$   |
| 2     | $(P(1.002) - x(1.002) - x)(1.002)$<br>$= P(1.002)^2 - x(1.002)^2 - x(1.002)$                         |
| 3     | $(P(1.002)^2 - x(1.002)^2 - x(1.002))(1.002)$<br>$= P(1.002)^3 - x(1.002)^3 - x(1.002)^2 - x(1.002)$ |
| $n$   | $= P(1.002)^n - x(1.002)^n - x(1.002)^{n-1} - \dots - x(1.002)$                                      |

Balance after  $n$  months

$$= P(1.002)^n - x(1.002)^n - x(1.002)^{n-1} - \dots - x(1.002)$$

$$= P(1.002)^n - \frac{x(1.002)(1.002^n - 1)}{1.002 - 1}$$

$$= P(1.002)^n - 501x(1.002^n - 1) \text{ (shown)}$$

(a)(ii)

$$P(1.002)^{300} - 501(2000)(1.002^{300} - 1) \geq 0$$

$$P \geq \frac{501(2000)(1.002^{300} - 1)}{1.002^{300}} = 451761.1356$$

Minimum principal amount required is \$451762.

(a)(iii)

$$(600000)(1.002)^n - 501(2000)(1.002^n - 1) \geq 0$$

$$(1.002)^n(-402000) + 501(2000) \geq 0$$

$$(1.002)^n \leq \frac{1002000}{40200} = \frac{167}{67}$$

$$n \leq \frac{\ln\left(\frac{167}{67}\right)}{\ln 1.002} = 457.11$$

$$\text{No. of years} \leq \frac{457.11}{12} = 38.092$$

No. of years for which the pay-out can be sustained is 38.

(b)

| Month | Payout |
|-------|--------|
| 1     | a      |
| 2     | a+c    |
| 3     | a+2c   |
| N     | N      |
| 300   | 4000   |

Let \$a\$ be the pay-out in the first month.

$$a + 299c = 4000 \quad \text{--- (1)}$$

$$\frac{300}{2}(2a + 299c) = 751500$$

$$2a + 299c = 5010 \quad \text{--- (2)}$$

Solving the simultaneous equations,

$$a = 1010, \quad c = 10$$

To find the month  $n$  with a pay-out of \$2,000:

$$(1010) + (n-1)(10) = 2000$$

$$n = 100$$

The pay-out is \$2,000 in the 100<sup>th</sup> month.

Alternative solution:

Let \$a\$ be the pay-out in the first month.

$$a + 299c = 4000 \quad \text{--- (1)}$$

$$\frac{300}{2}(a + 4000) = 751500 \Rightarrow a = 1010. \text{ Hence from (1) } c = 10$$

$$\text{Consider } U_n = 1010 + (n-1)(10) = 2000 \Rightarrow n = 100$$

The pay-out is \$2,000 in the 100<sup>th</sup> month.

4 The function  $f$  is defined by

$$f(x) = \begin{cases} 4-x & \text{for } 1 \leq x < 3, \\ (x-4)^2 & \text{for } 3 \leq x < 4, \end{cases}$$

and it is given that  $f(x-3) = f(x)$  for all real values of  $x$ .

- (i) State a reason why  $f$  does not have an inverse. [1]
- (ii) Sketch the graph of  $y = f(x)$  for  $-1 < x < 6$ . [3]
- (iii) Evaluate  $f(2017)$ . [2]

The function  $g$  has domain  $[1, 4)$  and is defined by

$$g(x) = \begin{cases} 4-x & \text{for } 1 \leq x < 3, \\ (x-4)^2 & \text{for } 3 \leq x < 4. \end{cases}$$

- (iv) By sketching  $y = g(x)$  and  $y = g^{-1}(x)$  on the same diagram, state the values of  $x$  such that  $g(x) = g^{-1}(x)$ . [3]

The function  $h$ , is defined by

$$h(x) = \begin{cases} \sqrt{1-x} & \text{for } 0 \leq x < 1, \\ (x-1)^4 & \text{for } 1 \leq x \leq 3. \end{cases}$$

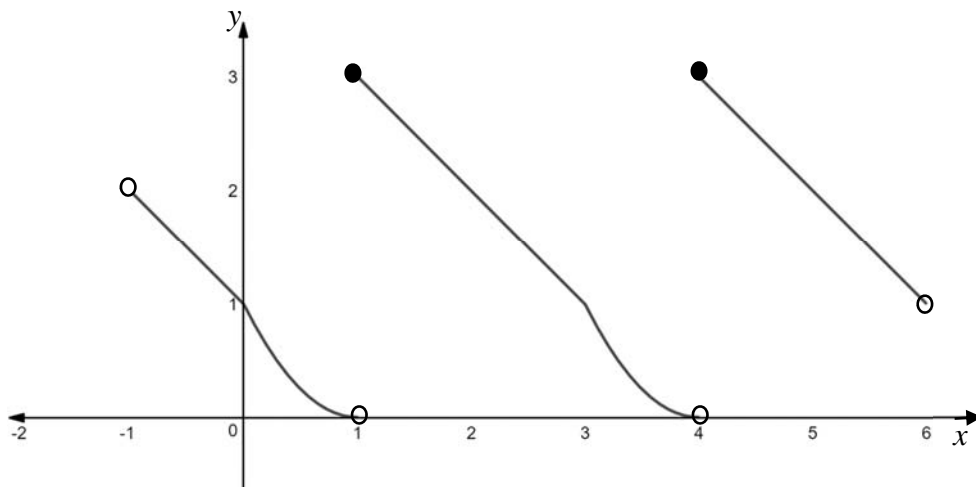
- (v) Explain why  $hg^{-1}$  doesn't exist. [1]
- (vi) Given that  $hg$  exist, define  $hg$  in similar form as function  $h$ . [2]
- (vii) Find the range of  $hg$ . [2]

### Suggested Solution

(a)(i) Since  $f(x-3) = f(x)$ , we can easily find 2 values of  $x$  (for example  $f(-3) = f(0)$ ) to show that it's not 1-1, hence no inverse.

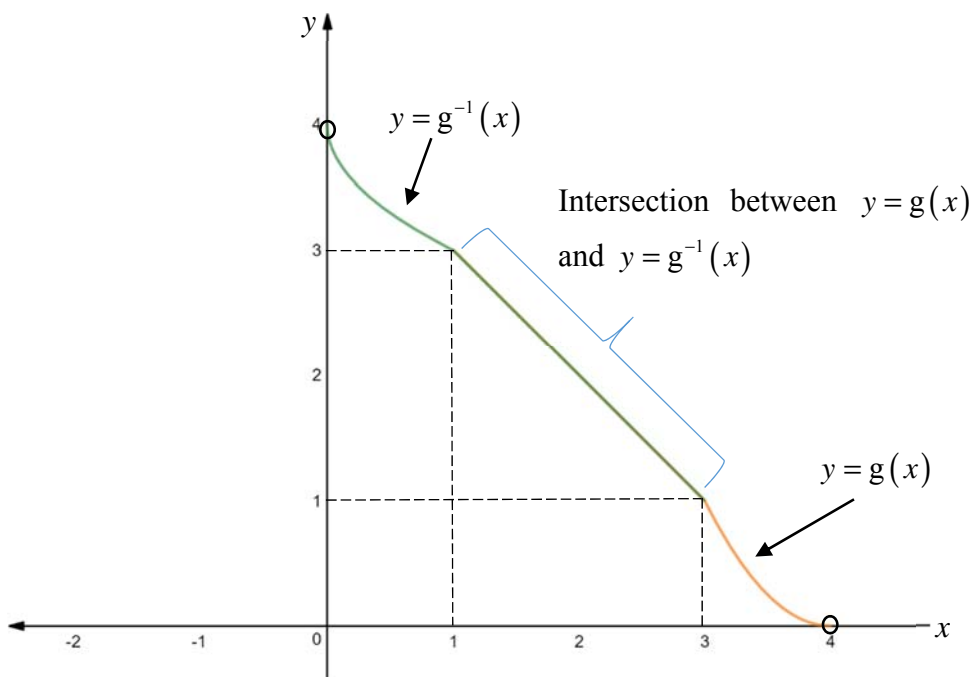


(ii)



(iii)  $f(2017) = f(2014) = f(2011) = L = f(1) = 3$

(iv)



From the graph, the values of  $x$  such that  $g(x) = g^{-1}(x)$  is  $1 \leq x \leq 3$ .

(v)  $R_{g^{-1}} = [1, 4] \not\subset [0, 3] = D_h$ ,  $hg^{-1}$  doesn't exist

(vi)  $hg(x) = \begin{cases} (3-x)^4 & \text{for } 1 \leq x < 3, \\ \sqrt{1-(x-4)^2} & \text{for } 3 \leq x < 4. \end{cases}$

(vii) Range of  $hg = [0, 16]$

**Section B: Probability and Statistics [60 marks]**

- 5** Two fair 4-sided dice each has its faces labelled '1', '2', '3' and '4'. The two dice are thrown and the absolute difference in score on their bottom faces is denoted by  $X$ .
- (i) Find  $P(X = x)$  for all possible values of  $x$ . [2]
- (ii) Find  $E(X)$  and  $\text{Var}(X)$ . [2]

**Suggested Solution**

(i) Consider the number of outcomes with the respective differences out of 16:

$$P(X = 0) = \frac{4}{16} = \frac{1}{4}$$

$$P(X = 1) = \frac{6}{16} = \frac{3}{8}$$

$$P(X = 2) = \frac{4}{16} = \frac{1}{4}$$

$$P(X = 3) = \frac{2}{16} = \frac{1}{8}$$

(ii)

$$E(X) = (0)\left(\frac{1}{4}\right) + (1)\left(\frac{3}{8}\right) + (2)\left(\frac{1}{4}\right) + (3)\left(\frac{1}{8}\right) = 1.25$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$\begin{aligned} &= (0)^2\left(\frac{1}{4}\right) + (1)^2\left(\frac{3}{8}\right) + (2)^2\left(\frac{1}{4}\right) + (3)^2\left(\frac{1}{8}\right) - (1.25)^2 \\ &= 0.9375 \end{aligned}$$

- 6** (a) Two events  $A$  and  $B$  occur with probabilities  $a$  and  $b$  respectively.  
Let  $c = P(A \cap B)$ .
- (i) Find the minimum value of  $c$ , in terms of  $a$  and  $b$ , if  $a + b > 1$ . [2]
- (ii) Find the minimum value of  $c$  if  $a = 0.6$  and  $b = 0.2$ . [1]
- (b) For independent events  $A$  and  $B$ , prove that the events  $A'$  and  $B'$  are also independent. [3]

**Suggested Solution**

(a)

By considering  $P(A \cup B) \leq 1$

$$P(A \cap B') + P(A \cap B) + P(B \cap A') \leq 1$$

$$\text{Let } P(A \cap B) = c$$

Then we have

$$a - c + c + b - c \leq 1$$

$$c \geq a + b - 1$$

Note: Can also use  $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1 \Rightarrow a + b - c \leq 1 \Rightarrow c \geq a + b - 1$

Thus, minimum possible value of  $P(A \cap B)$  is  $a + b - 1$

when  $P(A) = 0.6$  and  $P(B) = 0.2$ , [note that  $P(A) + P(B) < 1$ ].

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cap B) = 0.8 - P(A \cup B)$$

minimum possible value of  $P(A \cap B)$  is 0

[i.e. when  $P(A \cup B) = P(A) + P(B)$ ].

(b)

$$P(A' \cap B') = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - P(A) - P(B) + P(A)P(B) \text{ (since A, B are indept)}$$

$$= P(A') - P(B)[1 - P(A)]$$

$$= P(A') - P(B)[P(A')]$$

$$= P(A')[1 - P(B)]$$

$$= P(A')P(B')$$

Thus  $A'$  and  $B'$  are independent as well.

7 When Mr. Lee sends a text message to any of his students over the weekend, he gets a reply, on average, about 6 out of 10 times. On a particular weekend, Mr. Lee sends a text message to 25 students.

(i) State, in the context of this question, two assumptions needed to model the number of students that reply by a binomial distribution. [2]

(ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]

Assume now that these assumptions do in fact hold. Stating the distribution clearly,

(iii) find the probability that at least half of the students reply. [3]

It is given instead that the probability of a randomly chosen student replying is  $p$ . Find the least value of  $p$  such that there is at least a 90% chance of more than 20 students replying to the text message. [3]

### Suggested Solution

(i) A student replying has a constant probability of 0.6.

The event of a student replying is independent of another student. OR Whether a student replies is independent of another student.

(ii) Some students are more likely to reply a message. Hence the assumption of equal probability might not hold.

(iii)

Let  $X$  be the number of students that reply, out of 25.

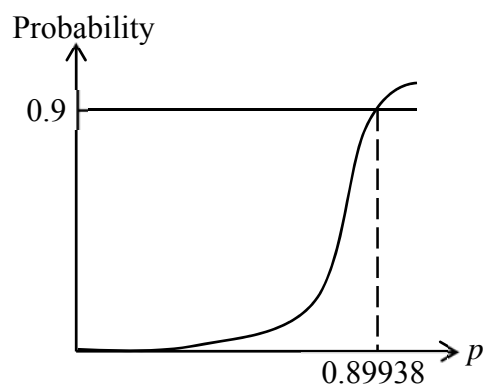
$$X \sim B(25, 0.6)$$

$$\begin{aligned} \text{Required probability} &= P(X \geq 13) \\ &= 1 - P(X \leq 12) \\ &= 0.84623 \quad (5\text{sf}) \\ &= 0.846 \quad (3\text{sf}) \end{aligned}$$

(iv) Let  $Y$  be the number of students that reply, out of 25. Let  $p$  be the new probability of a student replying.

$$Y \sim B(25, p)$$

$$\text{Given that } P(Y > 20) \geq 0.9 \text{ (or } P(Y \leq 20) \leq 0.1)$$



Therefore, the least probability of a student replying is 0.899 [accept 0.900].

- 8 A teacher, Mr. Ku, suspects that the average time a student spends on his or her mobile phone per day is  $\mu_0$  minutes. He selected a random sample of 97 students in the school who own mobile phones and recorded the amount of time each student spent on his or her phone in a randomly selected day. The results are displayed in the table below.

|   |    |    |    |    |     |     |
|---|----|----|----|----|-----|-----|
| Time spent per day<br>(to nearest minute) | 60 | 65 | 72 | 90 | 110 | 180 |
| Number of people                          | 11 | 20 | 32 | 18 | 10  | 6   |

- (i) Calculate unbiased estimates of the population mean and variance of the time a student spends on his or her mobile phone per day. [2]

The null hypothesis that the average time a student spends on his or her mobile phone per day is  $\mu_0$  minutes is tested, at 5% level of significance, against the alternative hypothesis that the average time a student spends on his or her mobile phone per day differs from  $\mu_0$  minutes.

- (ii) Determine the range of values of  $\mu_0$  for which the null hypothesis is rejected. [5]
- (iii) Explain, in the context of this question, the meaning of ‘at 5% level of significance’. [1]
- (iv) If the null hypothesis in (ii) is rejected at 5% significance level, can we reject the null hypothesis at 1% level of significance? Explain your answer. [1]

| Suggested Solution   |
|--|
| <p>(i) Using a GC,</p> <p>The unbiased estimate of population mean is <math>83.134 \approx 83.1</math> (3sf)</p> <p>The unbiased estimate of population variance is <math>29.011^2 \approx 841.64 \approx 842</math> (3sf)</p>   |
| <p>(ii) Let <math>X</math> be the time a student spent on their mobile phone per day.</p> <p>Let <math>\mu</math> be the population mean time a student spent on their mobile phone per day.</p> <p>Test: <math>H_0 : \mu = \mu_0</math> vs <math>H_1 : \mu \neq \mu_0</math></p> <p>Perform a 2-tail test and 5% level of significance</p> <p>Under <math>H_0</math>, since <math>n=97</math> is large, by Central Limit Theorem, <math>\bar{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)</math></p> <p>approximately, with unknown <math>\mu_0</math> and <math>s^2 = 841.64</math></p> <p>At a 5% (2-tail), the Critical Region is <math>\{ z  &gt; 1.96\}</math></p> |

Since  $H_0$  is rejected,  $|z - value| > 1.96$

$z\text{-value} < -1.96$  or  $z\text{-value} > 1.96$

$$\frac{83.134 - \mu_0}{\sqrt{841.64/97}} < -1.96 \quad \text{or} \quad \frac{83.134 - \mu_0}{\sqrt{841.64/97}} > 1.96$$
$$\mu_0 > 88.907 \approx 88.9 \quad \text{or} \quad \mu_0 < 77.361 \approx 77.4$$

**(iii)** 5% level of significance means that there is a 0.05 probability of concluding that the average time a student spent on their mobile phone per day differs from  $\mu_0$  when in actual fact it did not.

**(iv)** Test:  $H_0 : \mu = \mu_0$  vs  $H_1 : \mu \neq \mu_0$

Since we reject  $H_0$  at 5% level of significance,

$$p\text{-value} = p < 0.05$$

But that doesn't imply  $p < 0.01$ . Hence, it's inconclusive.

- 9 The table below gives the world record time, in seconds, for the 100 metre race for the various years.

|           |      |      |      |      |      |      |      |      |      |
|-----------|------|------|------|------|------|------|------|------|------|
| Year, $x$ | 1993 | 1994 | 1996 | 1999 | 2002 | 2005 | 2007 | 2008 | 2009 |
| Time, $t$ | 9.87 | 9.85 | 9.84 | 9.79 | 9.78 | 9.77 | 9.74 | 9.69 | 9.58 |

- (i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

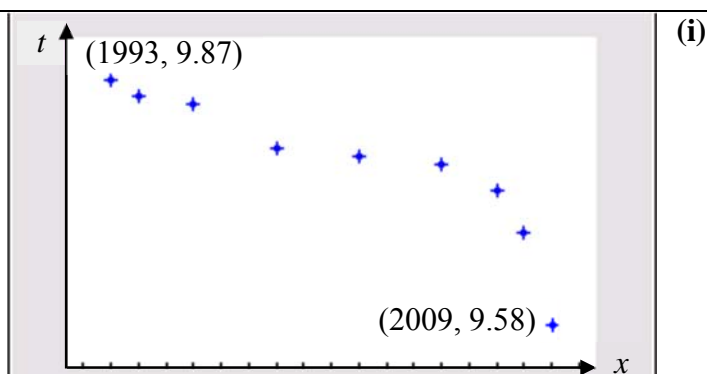
It is thought that the world record time  $t$  can be modelled by one of the following formulae

$$t = a + bY \quad \text{or} \quad t = c + dY^3$$

where  $a, b, c, d$  are constants and  $Y = (x - 2000)$ .

- (ii) Using the value of the product moment correlation coefficient, explain which of  $t = a + bY$  and  $t = c + dY^3$  is the better model. [3]
- (iii) Using the better model found in part (ii), find the equation of a suitable regression line. Use it to estimate the world record time in the year 2018. Comment on the reliability of this estimate. [3]
- (iv) Deduce, with justification, the product moment correlation coefficient between  $t$  and  $x$ . [1]

### Suggested Solution



(i)

- (ii) For  $t = a + bY$ ,  $r = -0.89351 \approx -0.894$  (3sf)

For  $t = c + dY^3$ ,  $r = -0.97163 \approx -0.972$  (3sf)

Since  $-0.972$  is closer to  $-1$  than  $-0.894$ ,  $t = c + dY^3$  is the better model.

- (iii) Using GC,  $t = 9.7985 - (2.5279 \times 10^{-4})(x - 2000)^3$

$$\begin{aligned} \text{When } x = 2018, t &= 9.7985 - (2.5279 \times 10^{-4})(2018 - 2000)^3 \\ &= 8.3242 \approx 8.32 \quad (3\text{sf}) \end{aligned}$$

The estimate obtained by extrapolation is not reliable since  $x = 2018$  lies outside the range of the sample data. ( $1993 \leq x \leq 2009$ )



(iv) Since the PMCC is not affected by linear transformation ( $mx+C$  with  $m > 0$ ),  
 $r = -0.89351 \approx -0.894$  (3sf)

**10** In this question, you should state clearly the distribution of any random variables that you define.

The volume,  $S$ , in ml, of perfume in a randomly chosen small bottle has mean 20 and variance  $\sigma^2$ .

(i) If  $\sigma = 15$ , explain why  $S$  may not be appropriately modelled by a normal distribution. [2]

It is now assumed that  $S$  follows a normal distribution

(ii) Given that 6.68% of the small bottles contains more than 23 ml of perfume, find the value of  $\sigma$ . [2]

For the rest of the question, the volume of perfume,  $S$ , in ml, in a randomly chosen small bottle follows the distribution  $N(20, 4)$  and the volume of perfume,  $L$ , in ml, in a randomly chosen large bottle follows the distribution  $N(100, 25)$ .

(iii) Calculate the probability that 6 randomly chosen small bottles and 9 randomly chosen large bottles contain a total volume of at least 1 litre of perfume. [3]

(iv) Calculate the probability that the volume of perfume in a randomly chosen large bottle differs from 6 times the volume of perfume in a randomly chosen small bottle by more than 25ml. [3]

(v) State, in this context, an assumption needed for your calculations in parts (iii) and (iv). [1]

### Suggested Solution

(i) Let  $S$  be the volume of perfume in a randomly chosen small bottle.

$$S \sim N(20, \sigma^2)$$

$$\text{When } \sigma = 15, P(S < 0) = 0.0912 \quad (3\text{sf})$$

This means that there is a significant chance (9.12%) that a small bottle has a negative volume of perfume. Hence, a normal distribution with  $\mu = 20, \sigma = 15$  is not appropriate.

[Accept 99.7% including between (-25 to 65) which includes -ve volume. ]

(ii)  $S \sim N(20, \sigma^2)$

Given  $P(S > 23) = 0.0668$

$$P\left(Z > \frac{23-20}{\sigma}\right) = 0.0668$$

Using GC,  $\frac{23-20}{\sigma} = 1.5001$

$$\begin{aligned}\sigma &= \frac{3}{1.5001} \\ &= 1.9999 \quad (5\text{sf}) \\ &= 2.00 \quad (3\text{sf})\end{aligned}$$

(iii)  $S \sim N(20, 2^2)$

Let  $L$  be the volume of perfume in a randomly chosen large bottle.

$$L \sim N(100, 5^2)$$

Let  $T = S_1 + L + S_6 + L_1 + L + L_9$

$$\begin{aligned}E(T) &= E(S_1) + L + E(S_6) + E(L_1)L + E(L_9) \\ &= 6E(S) + 9E(L) \\ &= 6(20) + 9(100) \\ &= 1020\end{aligned}$$

$$\begin{aligned}\text{Var}(T) &= \text{Var}(S_1) + L + \text{Var}(S_6) \\ &\quad + \text{Var}(L_1)L + \text{Var}(L_9) \\ &= 6\text{Var}(S) + 9\text{Var}(L) \\ &= 6(4) + 9(25) \\ &= 249\end{aligned}$$

$$T \sim N\left(1020, (\sqrt{249})^2\right)$$

Required probability =  $P(T > 1000)$

$$= 0.89750 \quad (5\text{sf})$$

$$= 0.898 \quad (3\text{sf})$$

|   |  |
|---|--|
| <p>(iv)</p> $E(6S - L) = 6E(S) - E(L)$ $= 6(20) - 100$ $= 20$ $\text{Var}(6S - L) = 6^2 \text{Var}(S) + \text{Var}(L)$ $= 36(4) + 25$ $= 169$ $6S - L \sim N(20, 13^2)$ <p>Required probability = <math>P( 6S - L  &gt; 25)</math></p> $= 1 - P(-25 < 6S - L < 25)$ $= 0.35053 \quad (5\text{sf})$ $= 0.351 \quad (3\text{sf})$ |  |
| <p>(v) An assumption needed is that the distribution of volumes of perfume in <b>ALL</b> (between and within <math>S</math> and <math>L</math>) are independent of one another.</p>   |  |

- 11** A bag contains 5 cards with the letter 'A', 3 cards with the letter 'B' and 2 cards with the letter 'C'.

A A A A A B B B C C

The 10 cards are arranged at random in a row to form a letter sequence. For example, AABBCAAACB is a possible letter sequence.

- (i) Find the number of possible letter sequences. [1]
- (ii) Find the number of possible letter sequences if no two 'B's are next to each other **and** no two 'C's are next to each other. [4]
- (iii) Find the probability that the first two letters are identical given that the second letter is **not** an 'A'. [4]

The 10 cards are now arranged at random in a circle.

- (iv) Find the probability that no two 'A's are next to each other. [3]

| Suggested Solution  | Comments |
|---|----------|
| (i) Number of different letter sequences<br>$= \frac{10!}{5!3!2!} = 2520$   |          |
| (ii) Number of different letter sequences<br>= (Number with no adjacent 'B's) – (Number with no adjacent 'B's, but with adjacent 'C's)<br>= (No. of ways to permute 7 units – 5As and 2Cs, and slot in 3Bs in between the 7 units) – (No. of ways to permute 6 units – 5As and CC, and slot in 3Bs in between the 6 units)<br>$= \left( \frac{7!}{5!2!} \cdot \binom{8}{3} \right) - \left( \frac{6!}{5!} \cdot \binom{7}{3} \right)$ $= 966$ |          |
| (iii) $P(1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ letter identical} \mid 2^{\text{nd}} \text{ letter is not 'A'})$<br>$= \frac{P(BB \text{ or } CC)}{P(AA' \text{ or } A'A')}$ $= \frac{P(BB) + P(CC)}{P(AA') + P(A'A')}$  |          |

$$= \frac{\left(\frac{3}{10}\right)\left(\frac{2}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{1}{9}\right)}{\left(\frac{5}{10}\right)\left(\frac{5}{9}\right) + \left(\frac{5}{10}\right)\left(\frac{4}{9}\right)}$$

$$= \frac{8}{45}$$

Alternative Solution:

$P(1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ letter identical} \mid 2^{\text{nd}} \text{ letter is not 'A'})$

$$= \frac{P(BB \text{ or } CC)}{P(2^{\text{nd}} \text{ letter is } B \text{ or } C)}$$

$$= \frac{\frac{\text{No. of permutations starting with } BB \text{ or } CC}{\text{Total no. of permutations}}}{\frac{\text{No. of permutations with 2nd letter } B \text{ or } C}{\text{Total no. of permutations}}}$$

$$= \frac{\left(\frac{8!}{5!2!} + \frac{8!}{5!3!}\right)}{\left(\frac{9!}{5!2!2!} + \frac{9!}{5!3!}\right)}$$

$$= \frac{8}{45}$$

(iv) Treat all the As as distinct ( $A_1, A_2, A_3, A_4, A_5$ ), Bs as distinct ( $B_1, B_2, B_3$ ) and Cs as distinct ( $C_1, C_2$ ).

Probability

$$= \frac{\text{No. of ways to permute 10 cards in a circle with all As separate}}{\text{No. of ways to permute 10 cards in a circle}}$$

$$= \frac{\text{No. of ways to permute } B_1 B_2 B_3 C_1 C_2 \text{ in a circle, and slot in the As}}{\text{No. of ways to permute 10 cards in a circle}}$$

$$= \frac{\left(\frac{5!}{5}\right)(5!)}{\left(\frac{10!}{10}\right)} = \frac{1}{126}$$

Alternative Solution:

Treat 5 As as identical, 3 Bs as identical and 2 Cs as identical. Since all cards are used, each outcome is equally probable.

Probability

$$= \frac{\text{No. of ways of arranging 3Bs and 2Cs in a circle, and slot in the 5As}}{\text{No. of ways of arranging 5As, 3Bs and 2Cs in a circle}}$$

$$= \frac{\left( \frac{5!}{5 \cdot 3!2!} \right)}{\left( \frac{10!}{10 \cdot 5!3!2!} \right)} = \frac{1}{126}$$