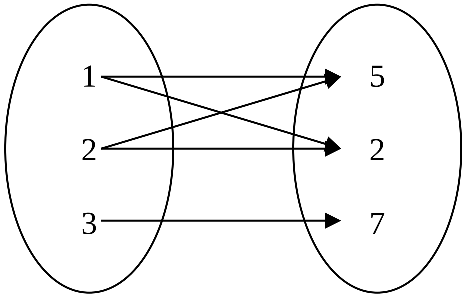
BMQ Solutions to Tutorial 4: Functions

<u>Basic Mastery Questions</u>

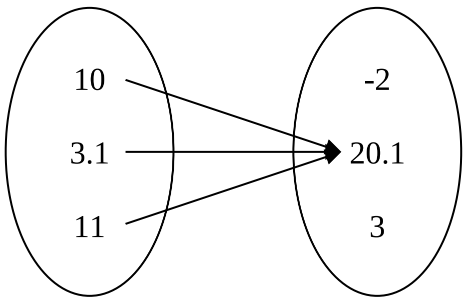
1. State, with reasons, which of the following relationships are functions.

(a)



Because $1 \mapsto 5$ and $1 \mapsto 2$
1 has no unique image.
Therefore this is not a function.

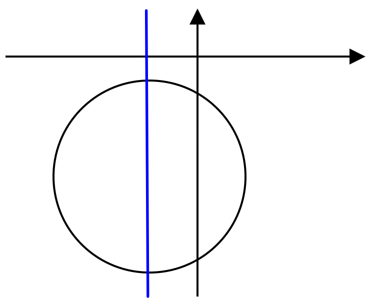
(b)



Every element in the domain has exactly one image in the range.
Therefore this is a function.

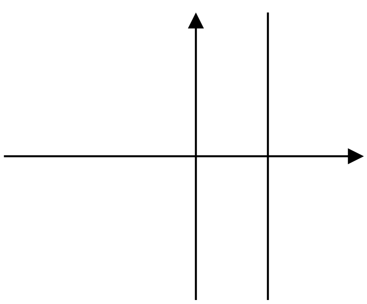
Note: Many-One relationships are still functions

(c)



A vertical line intersects the curve at 2 separate points.
Therefore, it is a one-many relationship
Therefore it is not a function.

(d)



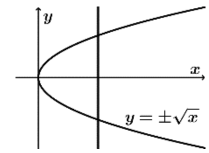
The relationship maps a number to all real numbers.
Therefore, it is a one-many relationship
Therefore it is not a function.

(e) $y = 2$

$y = 2$ is a function, since any $x \in \mathbb{R},\ x \mapsto 2$.
Therefore each element has exactly one image
Therefore it is a function.

(f) $y = \pm\sqrt{x}$

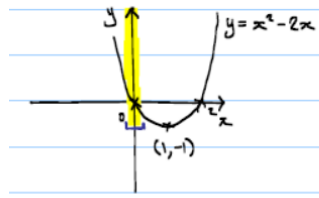
$y = \pm\sqrt{x}$ is not a function since any $x > 0$ is mapped to 2 values.
Alternatively, this graph fails the vertical line test.



2. State the range of the following functions.

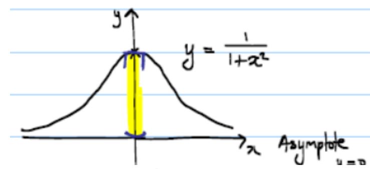
(a) $f : x \mapsto x^2 - 2x, \quad x \in \mathbb{R}$

From the graph, $R_f = [-1, \infty)$



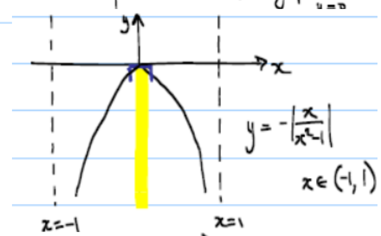
(b) $f : x \mapsto \dfrac{1}{x^2+1}, \quad x \in \mathbb{R}$

From the graph, $R_f = (0, 1]$



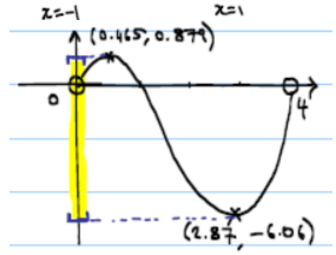
(c) $g : x \mapsto -\left|\dfrac{x}{x^2-1}\right|, \quad -1 < x < 1$

From the graph, $R_f = (-\infty, 0]$



(d) $g : x \mapsto x^3 - 5x^2 + 4x, \quad x \in (0, 4)$

From the graph (and using GC),
$R_f = [-6.06, 0.879]$



3. State, with reasons, whether the following functions have an inverse.

(a) $f : x \mapsto \dfrac{-1}{x}, \quad x \neq 0$

$f : x \longrightarrow \frac{-1}{x}, \quad x \neq 0$

Suppose $f(a) = f(b)$

$$\frac{-1}{a} = \frac{-1}{b}$$

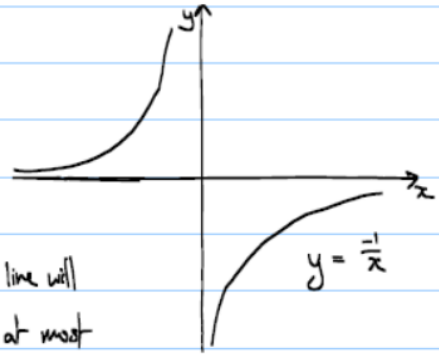
$$-a = -b$$

$$a = b$$

$\therefore$ f is one-to-one

$\therefore$ f^{-1} exists.

Alternatively



Any horizontal line will cut the curve at most once.

$\therefore$ By horizontal line test f is one-to-one

$\therefore$ f^{-1} exists.

(b) $\mathrm{g}: x \mapsto x^2 - 2x, \quad x < 0$

$\mathrm{g}: x \longmapsto x^2 - 2x, \quad x < 0$

Suppose $\quad \mathrm{g}(a) = \mathrm{g}(b)$

$a^2 - 2a = b^2 - 2b$

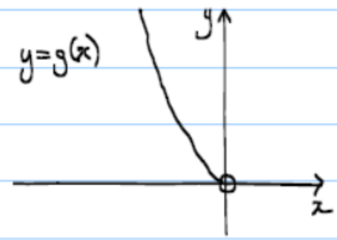
$(a-1)^2 - 1 = (b-1)^2 - 1$

$a - 1 = \pm(b-1)$ (rej $-$ve, since

$a = b \qquad a, b < 0$)

$\therefore$ g is one-to-one and g^{-1} exists.

Alternatively,



By the horizontal line test, g is one-to-one

$\therefore$ g^{-1} exists.

(c) $\mathrm{h}: x \mapsto x^2 + 2x, \quad x \geq -2$

$\mathrm{h}: x \longmapsto x^2 + 2x, \quad x \geq -2$

Then $\quad \mathrm{h}(-2) = \mathrm{h}(0) = 0$

$\therefore$ h is not one-to-one

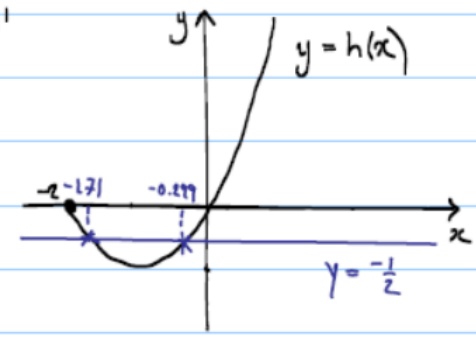
and h^{-1} does not exist.

$\because \quad \mathrm{h}(-1.71) = \mathrm{h}(-0.299) = \frac{-1}{2}$

h fails the horizontal line test.

$\therefore$ h is not one-to-one and thus h^{-1} does not exist.

Alternatively,



(d) $\mathrm{h}: x \mapsto -\ln 2x, \quad x > 0$

$h: x \longmapsto -\ln 2x, \quad x > 0$

Suppose $h(a) = h(b)$

$-\ln 2a = -\ln 2b$

$\ln 2a = \ln 2b$

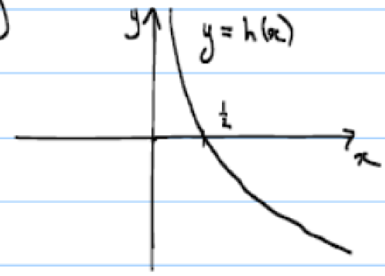
$2a = 2b$

$a = b$

$\therefore$ h is one-to-one

Thus h^{-1} exists.

Alternatively



$\because$ No horizontal line will intersect $y = h(x)$ more than once, h passes the horizontal line test.

$\therefore$ h is one-to-one and thus h^{-1} exists.

4. State, with reasons, whether the composition $g \circ f$ exists for the following functions.

a) $f: x \longmapsto -\frac{1}{x^2}, \quad x \neq 0$

$g: x \longmapsto \ln(-x), \quad x < 0$

$R_f = \mathbb{R}^- = D_g$

$\therefore$ $g \circ f$ exists

b) $f: x \longmapsto \ln(-x), \quad x < 0$

$g: x \longmapsto \frac{1}{x^2}, \quad x \neq 0$

$R_f = \mathbb{R} \nsubseteq \mathbb{R} \setminus \{0\} = D_g$

$\therefore$ $g \circ f$ does not exist.

c) $f: x \longmapsto x^3, \quad x \in (0, 1]$

$g: x \longmapsto \sqrt{x}, \quad x > 0$

$R_f = (0, 1] \subseteq \mathbb{R}^+ = D_g$

$\therefore$ $g \circ f$ exists

d) $f: x \longmapsto x, \quad x \neq 0$

$g: x \longmapsto x, \quad x < 1$

$R_f = \mathbb{R} \setminus \{0\} \nsubseteq (-\infty, 1) = D_g$

$\therefore$ $g \circ f$ does not exist.