MA6132
Advanced Mathematics II

Name: ______________________________ (   )   Date: ________________

Homework C: Vector Product and Planes – Part 1

1 The position vectors of points A, B and C are $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$ respectively.

Find

(i) $\mathbf{a} \times \mathbf{b}$, Ans: $\mathbf{i} - \mathbf{j} - \mathbf{k}$

(ii) $\mathbf{b} \times \mathbf{c}$, Ans: $-\mathbf{i} - \mathbf{j} + \mathbf{k}$

(iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, Ans: $-\mathbf{j} + \mathbf{k}$

(iv) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Ans: $\mathbf{i} - \mathbf{j}$

2 Relative to the origin O, the position vectors of points A and B are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively.

The point P on $\overrightarrow{AB}$ is such that $AP : PB = \lambda : 1 - \lambda$.

Show that $\overrightarrow{OP} = (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}$.

(a) Find the value of λ if $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$. Ans: $\frac{5}{18}$

(b) Find the value of λ if $\angle AOP = \angle POB$. Ans: $\frac{3}{10}$

3 The position vectors of points A and B are $\mathbf{a} = 9\mathbf{i} + 9\mathbf{j} + 9\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 14\mathbf{j} + 14\mathbf{k}$ respectively.

(a) Find the length of projection of **a** on **b**. Ans: 3

(b) Use your answer in **(a)** to resolve the vector **a** into two components, one parallel to **b** and one perpendicular to **b**.

Ans: $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $8\mathbf{i} + 11\mathbf{j} + 7$

4 Referred to the origin O, the position vectors of points A, B , C and D are

$$\mathbf{i} - \mathbf{j}, \quad \mathbf{i} + \mathbf{k}, \quad \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{j} + 2\mathbf{k}$$

respectively.

Find a

(i) parametric equation for the plane containing the points A, B and C,

Ans: $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(\mathbf{j} + \mathbf{k}) + \mu\mathbf{k}, \ \lambda, \mu \in \mathbb{R}$

(ii) vector (scalar product) equation for the plane containing the points A, B and D,

Ans: $\mathbf{r} \cdot (-\mathbf{j} + \mathbf{k}) = 1$

(iii) Cartesian equation for the plane containing the points A, C and D.

Ans: $2x + y = 1$

5 The vector equation of a plane π is $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 1$.

(i) Show that the point $A(9, 2, 6)$ lies in the plane π.

(ii) Show that the point $B(2, -5, 6)$ does not lie in the plane π, and find the shortest distance from the point B to the plane π. Ans: 7

(iii) The shortest distance from the point $C(2009, 7, m)$ to the plane π is 26. Find the possible value(s) of m. Ans: 972 or 1050

(iv) Find the coordinates of the foot of perpendicular from the point $D(4, 7, -5)$ to the plane π. Ans: (1, 1, 1)

6 The vector equations of a line l and a plane π are $\mathbf{r} = m\mathbf{i} - \mathbf{j} + \lambda\,(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, $\lambda \in \mathbb{R}$ and $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 1$ respectively.

(i) If $m = 3$, show that the line l lies in the plane π.

(ii) If $m = 6$, find the shortest distance from the line l to the plane π. Ans: 1

(iii) Find the possible value(s) of m if the shortest distance from the line l to the plane π is 7. Ans: –18 or 24

7 The equation of two lines and a plane are as follows:

l_1 : $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \alpha\,(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}),\ \alpha \in \mathbb{R}$

l_2 : $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \beta\,(m\mathbf{i} + 5\mathbf{j} - \mathbf{k}),\ \beta \in \mathbb{R}$

π : $\mathbf{r} \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 10$

(i) Find the angle between the line l_1 and the plane π. Ans: 30°

(ii) Find the possible value(s) of m if the angle between the line l_2 and the plane π is 60°. Ans: 4 or 32

Optional Question:

It is given that $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

(a) By considering the scalar product $\mathbf{a} \cdot \mathbf{b}$, prove that

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

(b) By considering the vector product $\mathbf{a} \times \mathbf{b}$, prove that

$$(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2).$$

Homework C: Vector Product and Planes – Part 2

1. For each of the following pairs of planes, find the acute angle between the planes and an equation of their line of intersection.

(i) $\mathbf{r} \cdot \begin{pmatrix} 9 \\ -3 \\ -4 \end{pmatrix} = 7$ and $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 1$

(ii) $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = -4$ and $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1$

2. Show that A(2, 3, –2) lies in the plane π_1 and π_2 whose equations are

$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} = 4$ and $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 9$ respectively. Find a vector parallel to both π_1 and π_2. Hence, deduce an equation for the line where π_1 meets π_2.

3. For each of the following pairs of planes, find an equation of their line of intersection.

(i) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(-\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$, $\lambda, \mu \in \mathbb{R}$ and

$\mathbf{r} = 2\mathbf{j} - \mathbf{k} + s(2\mathbf{i} - \mathbf{j}) + t(\mathbf{i} - \mathbf{k})$, $s, t \in \mathbb{R}$

(ii) $\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$, $\lambda, \mu \in \mathbb{R}$ and

$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix} = -5$

4. For each of the following, determine if the three planes intersect and find the point or line of intersection when it exists.

(i) $3x + 2y + z = 15$, $2x - y + 3z = 10$ and $x + y + 2z = 7$

(ii) $x - 2y - z = 6$, $2x - 4y + 2z = 15$ and $x - 2y + z = 8$

(iii) $2x + y + 2z = -2$, $6x + 5y - 2z = 6$ and $x + y - z = 4$

(iv) $2x + 2y - z = 2$, $2x - 2y + z = 2$ and $8x - 6y + 3z = 8$

5. The equations of planes π_1, π_2 and π_3 are $5x - y - 2z = 6$, $-2x + y + 3z = -4$ and $x + y + 4z = a$ respectively. Discuss the solution of these three equations when

(a) $a = -9$

(b) $a = -2$

giving a geometrical interpretation for each case.

Answers:

1. (i) 7.6°, $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$ (ii) 51.9°, $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

2. $\begin{pmatrix} 4 \\ -5 \\ 11 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 11 \end{pmatrix}$, $\lambda \in \mathbb{R}$

3. (i) $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ (ii) $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

4. (i) (4, 1, 1) (ii) No intersection (iii) No intersection (iv) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$

5. (i) No common point; each plane is parallel to the line of intersection of the other two planes.

(ii) They intersect at the line $\mathbf{r} = \begin{pmatrix} 2/3 \\ -8/3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 11 \\ -3 \end{pmatrix}$, $\lambda \in \mathbb{R}$

Quiz

The lines L_1 and L_2 meet at the point P. The line L_3 is coplanar with L_1 and L_2 and is perpendicular to L_1. Given that L_1 and L_2 are parallel to the vectors **a** and **b** respectively, show that L_3 is parallel to the vector $\mathbf{b}-\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$.

The equations of L_1 and L_2 are now known to be $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+t\begin{pmatrix}11\\10\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+s\begin{pmatrix}17\\3\\4\end{pmatrix}$ respectively, where s and t are real parameters. Find the equation of the line L_3, given that L_3 also passes through P.

The line L_4 has equation $\mathbf{r}=\begin{pmatrix}10\\-3\\1\end{pmatrix}+u\begin{pmatrix}-3\\4\\-1\end{pmatrix}$, where u is a real parameter. Determine if L_3 and L_4 are skew or intersecting.

The line L_5 is perpendicular to both L_3 and L_4. Find the acute angle between L_5 and the plane containing L_1 and L_2.

Ans: $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+\lambda\begin{pmatrix}6\\-7\\2\end{pmatrix}$; L_3 and L_4 are skew lines ; 83.9°