TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

H2 FURTHER MATHEMATICS — 9649/01

Paper 1

12 SEPTEMBER 2024

3 hours

Additional material: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

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1 The matrix **A** is given by $\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 0.5\end{pmatrix}$.

(a) Form a conjecture for $\mathbf{A}^n$ in the form $\begin{pmatrix}1 & a-b^{n-1}\\ 0 & b^n\end{pmatrix}$, for $n\in\mathbb{Z}^+$, where a and b are real constants to be determined. [2]

(b) Using mathematical induction, prove the conjecture formed in part **(a)**. [4]

2 Let $x=\tan\theta$.

(a) Use De Moivre's Theorem to express $\tan 5\theta$ in terms of x, showing your working clearly. [4]

(b) Hence, without using a calculator, find the exact value of $\tan^2\left(\frac{\pi}{5}\right)$. [3]

3 The positive numbers x_n satisfy the relation

$$\left(x_{n+1}\right)^2=3x_n-1,\quad \text{for } n\in\mathbb{Z}^+.$$

(a) Find the values of α and β, where $\alpha<\beta$, such that when $x_1=\alpha$ or $x_1=\beta$, the sequence remains constant. [2]

(b) By considering $\left(x_{n+1}\right)^2-\left(x_n\right)^2$, or otherwise, show algebraically that if $\alpha<x_1<\beta$, then the sequence increases and converges to β. You may assume that the sequence converges. [5]

4 **(a)** Express $\dfrac{1}{1-x}$ as a series, where $|x|<1$. [1]

The Fibonacci sequence is given as $u_{n+2}=u_{n+1}+u_n$ for $n\in\mathbb{Z},\ n\geq 0$ with initial conditions $u_0=0$ and $u_1=1$.

Let $\mathrm{f}(x)=\sum_{r=0}^{\infty}u_r x^r$. You may assume that $\mathrm{f}(x)$ converges.

(b) By considering $\mathrm{f}(x)-x\mathrm{f}(x)-x^2\mathrm{f}(x)$, express $\mathrm{f}(x)$ in partial fractions. Hence find u_n in terms of n, giving your answer in exact form. [8]

5 The curve C has an equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = 2x - \sin x - \frac{1}{3}$.

(a) Show algebraically that C cuts the x-axis exactly once in the interval $0 < x < 1$. [2]

Let $\mathrm{f}(\alpha) = 0$.

(b) Use Newton-Raphson method with initial approximation $x_0 = 0$ to find an estimate for α, correct to 4 decimal places. [3]

The region R_1 is bounded by C and both axes. The region R_2 is bounded by C, the x-axis and the line $x = 1$. Let A be the total area of R_1 and R_2.

(c) Explain why Simpson's Rule with four strips might not give a good estimation of the value of A. [1]

(d) Use Simpson's Rule with six strips to estimate the value of A, giving your answer correct to 5 decimal places. [3]

6 **(a)** The points $O(0,0)$, $P(k_1,0)$, $Q(0,k_2)$ and $R(k_1,k_2)$ have images O', P', Q' and R' respectively under the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) Find the coordinates of O', P', Q' and R' in terms of a, b, c, d, k_1 and k_2. [2]

(ii) Show that the area of the geometrical shape $O'P'R'Q'$ is equivalent to

$$\left|\det(\mathbf{A})\right| \times \text{area of rectangle } OPRQ.$$ [2]

(b) An $n \times n$ square matrix $\mathbf{M}$ is said to be an orthogonal matrix if $\mathbf{MM}^T = \mathbf{M}^T\mathbf{M} = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix.

(i) Explain why the product of two orthogonal matrices is still orthogonal. [2]

(ii) Find the possible value(s) of the determinant of an orthogonal matrix. [2]

(iii) Show that there exists a 2×2 non-orthogonal matrix with determinant of a value found in part **(ii)**. [2]

(iv) Let $\mathbf{B} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$. Verify that $\mathbf{B}$ is orthogonal and hence write down $\mathbf{B}^{-1}$, showing clearly how you obtain your answer. [3]

[Turn Over

7 **(a)** It is given that $\begin{pmatrix} 12 \\ -3 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{M} = \begin{pmatrix} 3 & 1 & 3 \\ -1 & -1 & k \\ 1 & 3 & k \end{pmatrix}$, where k is a real constant to be determined. Find the eigenvalues and corresponding eigenvectors of **M**. Show your working clearly. [7]

(b) The transformation T is the linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the matrix **M**. The plane P is invariant under T, such that if any point in P undergoes the transformation T, the image of the point will still lie in P. Given that P intersects the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\mu \in \mathbb{R}$, at a unique point, find a cartesian equation of P. [4]

8 **(a)** The line L has cartesian equation $(\cos\theta)x + (\sin\theta)y = r$, where $r > 0$. Find, in terms of r, the shortest distance from the origin to the line L and explain the significance of θ. [4]

The curve C has cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$.

(b) Determine, in terms of a and b, the equation of the tangent to C at the point $P(x_0, y_0)$, in the form $\mathrm{f}(x, y) = 1$, where f is a function of x and y. [3]

The line l passes through the origin and is perpendicular to the tangent to C at P. The line l and the tangent line intersect at point X.

(c) The locus of X as P varies is known as the pedal curve of C. Use the results of parts **(a)** and **(b)** to show that the polar equation of the pedal curve of C is $a^2\cos^2\theta + b^2\sin^2\theta = r^2$. [3]

(d) Given than $a = 3$ and $b = 2$, find the arc length of the pedal curve of C. [3]

9 For an object falling through the atmosphere, the 'terminal velocity' is the value approached by the velocity after a long time.

A skydiver of mass m kg falls vertically after jumping off the aircraft at a high altitude. He opens his parachute when he reaches terminal velocity v_1 ms^{-1}, which is when his velocity is constant. At time t seconds after the skydiver opens his parachute, his vertical displacement is s meters from the initial point where he opened his parachute. It is given that $s = 0$ and $\frac{\mathrm{d}s}{\mathrm{d}t} = v_1$ at $t = 0$. The net force acting on the skydiver is given by the following differential equation

$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = mg - K\frac{\mathrm{d}s}{\mathrm{d}t},$$

where g ms^{-2} is the gravitational acceleration constant and K kgs^{-1} is the air resistance proportionality constant associated with an open parachute.

(a) Find s in terms of t, v_1, m, g and K. [7]

(b) Write down, in terms of m, g and K, the second terminal velocity, v_2, reached by the skydiver after opening the parachute. [1]

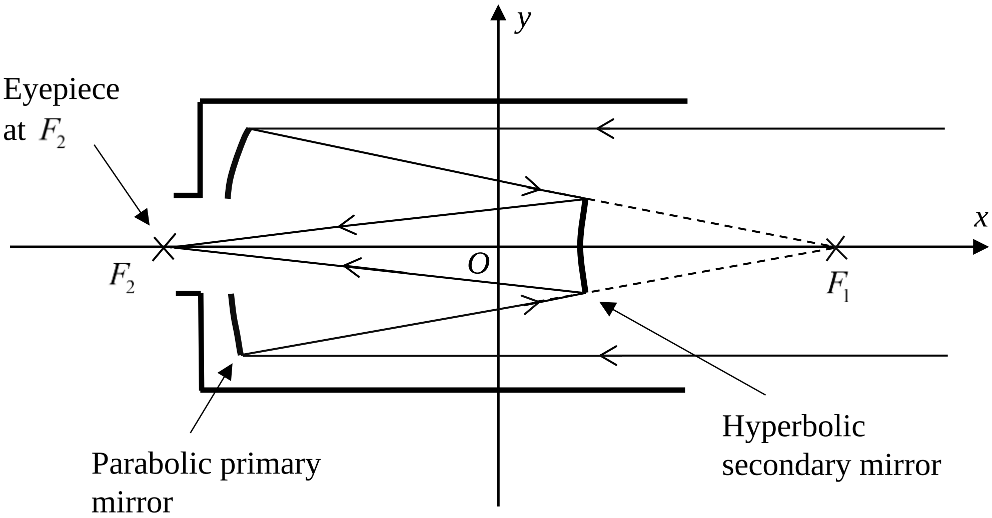
It is given that $m = 75$, $g = 9.8$, $K = 110$ and $v_1 = 54$.

(c) If the skydiver opens the parachute at an altitude of 1000 metres from the ground, find the altitude of the skydiver when he is within 2% of v_2. [4]

10 **(a)** A solid ceramic display is to be built at the centre of an observatory deck, as part of an art exhibition. The display can be modelled by rotating the region bounded by the curve $y = \mathrm{e}^{-x^2}$, the x-axis and the lines $x = r$ and $x = -r$ through π radians about the y-axis, where r is a positive constant. Determine the theoretical maximum volume of the solid ceramic display. [3]

[Turn Over

(b) A tourist uses a Cassegrain telescope on the observatory deck to observe stars at a distance. The telescope has a parabolic primary mirror and a hyperbolic secondary mirror. Incoming light rays are first reflected by the parabolic primary mirror with focus F_1, to the hyperbolic secondary mirror with focus F_2, before passing through a hole in the parabolic primary mirror to be observed through an eyepiece placed at F_2.



Cross-Section of a Cassegrain telescope

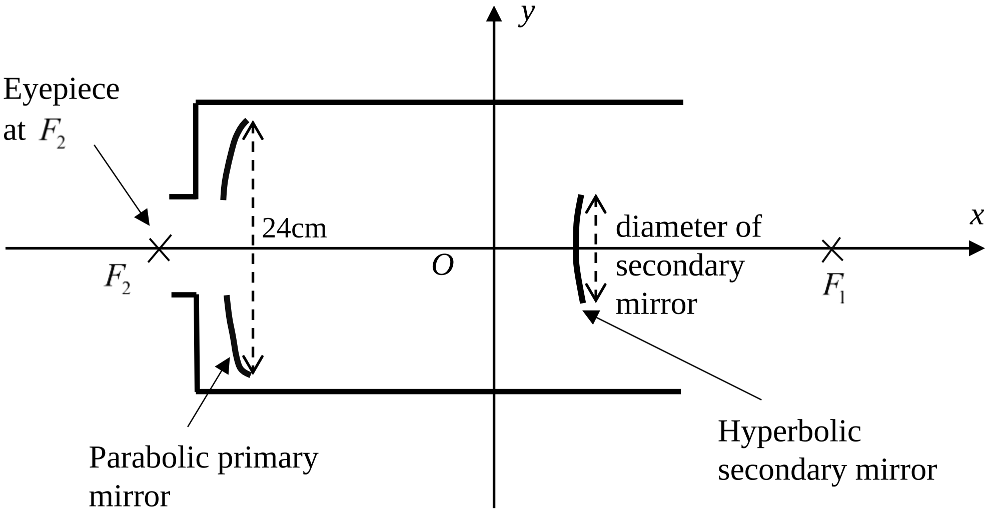
The diagram above shows the cross-section of the Cassegrain telescope (not drawn to scale). The foci F_1 and F_2 are equidistant from O. It is given further that the distance between F_1 and F_2 is fixed at 100cm and the focal length of the parabolic primary mirror is 80cm.

You may assume that F_1 and F_2 lie on the x-axis, and the hyperbolic secondary and parabolic primary mirrors are symmetrical about the x-axis.

(i) Find the cartesian equation of the parabola representing the cross-section of the parabolic primary mirror. [2]

The hyperbolic secondary mirror is positioned such that its vertex is 30cm away from F_1.

(ii) Determine the eccentricity of the hyperbola representing the cross-section of the hyperbolic secondary mirror. [2]



Cross-Section of Cassegrain Telescope with Diameter of Mirrors

(iii) Given that the diameter of the parabolic primary mirror is 24cm, find the minimum diameter of the hyperbolic secondary mirror, such that it captures all the light rays reflected by the parabolic primary mirror. [6]

End of Paper