RVHS H2 Mathematics Remedial Programme

Topic: Vectors I, II

Basic Mastery Questions

1. **ACJC Promo 9758/2021/Q10(i)**
Referred to the origin *O*, the points *A*, *B* and *C* have position vectors $4\mathbf{i}-2\mathbf{j}$, $\alpha\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $-\mathbf{i}-7\mathbf{j}+\beta\mathbf{k}$ respectively, where α and β are constants.

Given that *A*, *B* and *C* are collinear, show that α =5, and find the value of β. [3]

Answer: $\beta=-10$

$$\overrightarrow{OA}=\begin{pmatrix}4\\-2\\0\end{pmatrix}\quad \overrightarrow{OB}=\begin{pmatrix}\alpha\\-1\\2\end{pmatrix}\quad \overrightarrow{OC}=\begin{pmatrix}-1\\-7\\\beta\end{pmatrix}$$

A, B and C are collinear

$\therefore \overrightarrow{AB}=k\overrightarrow{AC}$

$$\begin{pmatrix}\alpha-4\\1\\2\end{pmatrix}=k\begin{pmatrix}-5\\-5\\\beta\end{pmatrix}\Rightarrow\begin{cases}\alpha-4=-5k\\1=-5k\\2=k\beta\end{cases}\Rightarrow\begin{cases}k=-\frac{1}{5}\\\alpha=5\\\beta=-10\end{cases}$$

2. **JPJC Prelim 9758/2021/01/Q2**
Referred to the origin *O*, the points *A* and *B* have position vectors **a** and **b** such that

$$\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k} \quad\text{and}\quad \mathbf{b}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}\,.$$

(i) Find the size of angle *OAB*. [2]

The point *C* has position vector **c** given by $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$, where λ and μ are positive constants. Given that the area of triangle *OAC* is twice that of triangle *OBC*,

(ii) find μ in terms of λ, [3]

(iii) hence, if *OC* = $\sqrt{118}$, find the position vector **c**. [4]

Answer: (i) $144.7°$ **(ii)** $\mu = 2\lambda$ **(iii)** $\underset{\sim}{c} = \begin{pmatrix} 3\sqrt{2} \\ 5\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}$

(i)

$$\overrightarrow{BA} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$\cos \measuredangle OAB = \frac{\overrightarrow{OA} \bullet \overrightarrow{BA}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{BA}\right|} = \frac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}}{\sqrt{3}\sqrt{2}} = \frac{-2}{\sqrt{6}}$$

$$\measuredangle OAB = 144.7°$$

(ii)

Area of $OAC = \frac{1}{2}\left|\underset{\sim}{a} \times \left(\lambda \underset{\sim}{a} + \mu \underset{\sim}{b}\right)\right| = \frac{\mu}{2}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right|$ since $\mu > 0$

Area of $OBC = \frac{1}{2}\left|\underset{\sim}{b} \times \left(\lambda \underset{\sim}{a} + \mu \underset{\sim}{b}\right)\right| = \frac{\lambda}{2}\left|\underset{\sim}{b} \times \underset{\sim}{a}\right|$ since $\lambda > 0$

Given area of triangle *OAC* is twice that of triangle *OBC*,

$$\frac{\mu}{2}\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| = 2\frac{\lambda}{2}\left|\underset{\sim}{b} \times \underset{\sim}{a}\right|$$

since $\left|\underset{\sim}{a} \times \underset{\sim}{b}\right| = \left|\underset{\sim}{b} \times \underset{\sim}{a}\right|$

$\therefore \mu = 2\lambda$

(iii)

$OC=\sqrt{118}$

$\left|\lambda \underset{\sim}{a}+\mu \underset{\sim}{b}\right|=\sqrt{118}$

$\left|\lambda \underset{\sim}{a}+2\lambda \underset{\sim}{b}\right|=\sqrt{118}$ since $\mu=2\lambda$

$\left|\begin{pmatrix}\lambda+2\lambda\\ \lambda+4\lambda\\ \lambda+4\lambda\end{pmatrix}\right|=\sqrt{118}$

$(3\lambda)^2+(5\lambda)^2+(5\lambda)^2=118$

$9\lambda^2+25\lambda^2+25\lambda^2=118$

$59\lambda^2=118$

$\lambda=\pm\sqrt{2}$

Since $\lambda>0$

$\lambda=\sqrt{2}$, $\underset{\sim}{c}=\begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$

Or

$\left|\begin{pmatrix}3\lambda\\ 5\lambda\\ 5\lambda\end{pmatrix}\right|=\sqrt{118}$

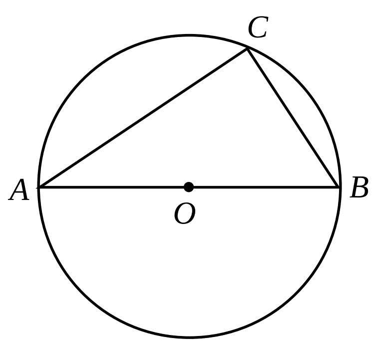
$\lambda\left|\begin{pmatrix}3\\ 5\\ 5\end{pmatrix}\right|=\sqrt{118}$ since $\lambda>0$

$\lambda\sqrt{59}=\sqrt{118}$

$\lambda=\sqrt{2}$

$\underset{\sim}{c}=\begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$

3. **RI Prelim 9758/2021/02/Q4(a)(i)**



Referred to the origin *O*, points *A*, *B* and *C* have position vectors **a**, **b** and **c** respectively. The three points lie on a circle with centre *O* and diameter *AB* (see diagram).
Using a suitable scalar product, show that the angle *ACB* is $90°$. [4]

$$\overrightarrow{AC}\cdot\overrightarrow{BC} = (\overrightarrow{OC}-\overrightarrow{OA})\cdot(\overrightarrow{OC}-\overrightarrow{OB})$$

$$= (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{b})$$

$$= (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}+\mathbf{a}) \qquad \text{(since } \mathbf{b} = -\mathbf{a}\text{)}$$

$$= \mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{a}$$

$$= \mathbf{c}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{a}$$

$$= |\mathbf{c}|^2-|\mathbf{a}|^2 \qquad \text{(since } \mathbf{c}\cdot\mathbf{c}=|\mathbf{c}|^2, \mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2\text{)}$$

$$= 0 \qquad \left(\text{since } |\mathbf{c}| = |\mathbf{a}| = \text{radius}\right)$$

Standard Questions

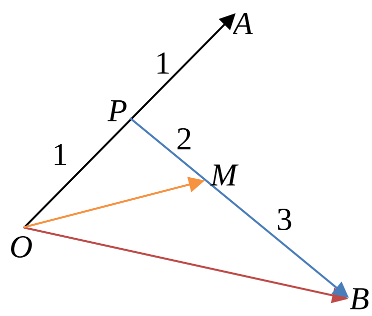
1. **MI Promo 9758/2021/PU2/02/Q4**

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The mid-point of OA is P and the point M on PB is such that $PM : MB = 2:3$.

By finding $\overrightarrow{OM}$, show that the area of triangle OMP can be written as $k|\mathbf{a}\times\mathbf{b}|$ where k is a constant to be found. [5]

Given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = \sqrt{2}$ and the angle AOB is $\frac{\pi}{4}$ radians, show that PM is perpendicular to OA. [4]

Answer: $k = \frac{1}{10}$

(i)	$\overrightarrow{OP} = \frac{1}{2}\mathbf{a}$ $\overrightarrow{OM} = \frac{2}{5}\overrightarrow{OB} + \frac{3}{5}\overrightarrow{OP}$ $= \frac{2}{5}\mathbf{b} + \frac{3}{5}\left(\frac{1}{2}\mathbf{a}\right)$ $= \frac{2}{5}\mathbf{b} + \frac{3}{10}\mathbf{a}$  Area of triangle OMP $= \frac{1}{2}\left|\overrightarrow{OM}\times\overrightarrow{OP}\right|$ $= \frac{1}{2}\left|\left(\frac{2}{5}\mathbf{b} + \frac{3}{10}\mathbf{a}\right)\times\frac{1}{2}\mathbf{a}\right|$ $= \frac{1}{2}\left|\frac{2}{5}\mathbf{b}\times\frac{1}{2}\mathbf{a} + \frac{3}{10}\mathbf{a}\times\frac{1}{2}\mathbf{a}\right|$ $= \frac{1}{2}\left|\frac{1}{5}\mathbf{b}\times\mathbf{a} + \frac{3}{20}\mathbf{a}\times\mathbf{a}\right|$ $= \frac{1}{10}\left|\mathbf{b}\times\mathbf{a}\right|$ since $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ $= \frac{1}{10}\left|\mathbf{a}\times\mathbf{b}\right|$ where $k = \frac{1}{10}$.

(ii)	$\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$ $= \frac{2}{5}\mathbf{b} + \frac{3}{10}\mathbf{a} - \frac{1}{2}\mathbf{a}$ $= \frac{2}{5}\mathbf{b} - \frac{1}{5}\mathbf{a}$ $\overrightarrow{PM} \bullet \overrightarrow{OA} = \left(\frac{2}{5}\mathbf{b} - \frac{1}{5}\mathbf{a}\right) \bullet \mathbf{a}$ $= \frac{2}{5}\mathbf{b} \bullet \mathbf{a} - \frac{1}{5}\mathbf{a} \bullet \mathbf{a}$ $= \frac{2}{5}\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\measuredangle AOB - \frac{1}{5}\lvert\mathbf{a}\rvert^2$ $= \frac{2}{5}\left(\sqrt{2}\right)(2)\cos\left(\frac{\pi}{4}\right) - \frac{1}{5}(2)^2$ $= \frac{4\sqrt{2}}{5}\left(\frac{\sqrt{2}}{2}\right) - \frac{4}{5}$ $= 0$ Since $\overrightarrow{PM} \bullet \overrightarrow{OA} = 0$, PM is perpendicular to OA. **<u>Alternative method:</u>** $\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ $\overrightarrow{PB} \bullet \overrightarrow{OA} = \left(\mathbf{b} - \frac{1}{2}\mathbf{a}\right) \bullet \mathbf{a}$ $= \mathbf{b} \bullet \mathbf{a} - \frac{1}{2}\mathbf{a} \bullet \mathbf{a}$ $= \lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\cos\measuredangle AOB - \frac{1}{2}\lvert\mathbf{a}\rvert^2$ $= \left(\sqrt{2}\right)(2)\cos\left(\frac{\pi}{4}\right) - \frac{1}{2}(2)^2$ $= 2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) - 2$ $= 0$ Since $\overrightarrow{PB} \bullet \overrightarrow{OA} = 0$, PB is perpendicular to OA. Since $\overrightarrow{PM} = \frac{2}{5}\overrightarrow{PB}$, PM is perpendicular to OA. Hence PM is perpendicular to OA.

2. **MI Promo 9758/2020/PU2/P1/Q7**

In the parallelogram *OABC*, $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OC}=\mathbf{c}$. The point *M* on *OA* is such that *OM* : *MA* = 2 : 1 and the point *N* on *AB* is such that *AN* : *NB* = 1 : 2. It is given that the lines *CM* and *ON* intersect at point *R*.

(i) Find $\overrightarrow{OM}$ and $\overrightarrow{ON}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{c}$. [2]

(ii) Show that $\overrightarrow{OR}=\frac{6}{11}\mathbf{a}+\frac{2}{11}\mathbf{c}$. [4]

(iii) Hence find the ratio $CR:RM$. [1]

(iv) State, with a reason, whether the points *O*, *B* and *R* are collinear. [2]

(i)	$\overrightarrow{OM}=\frac{2}{3}\mathbf{a}$ $\overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=\mathbf{a}+\frac{1}{3}\mathbf{c}$
(ii)	<u>Method 1</u> Let $CR:RM=\mu:1-\mu$ and $OR:RN=\lambda:1-\lambda$. Using $\overrightarrow{CM}$, by ratio theorem, $\overrightarrow{OR}=\dfrac{(1-\mu)\mathbf{c}+\mu\dfrac{2\mathbf{a}}{3}}{1}=\dfrac{2\mu}{3}\mathbf{a}+(1-\mu)\mathbf{c}$ Using $\overrightarrow{ON}$, $\overrightarrow{OR}=\lambda\overrightarrow{ON}=\lambda\left(\mathbf{a}+\dfrac{\mathbf{c}}{3}\right)=\lambda\mathbf{a}+\dfrac{\lambda}{3}\mathbf{c}$ $\therefore\dfrac{2\mu}{3}\mathbf{a}+(1-\mu)\mathbf{c}=\lambda\mathbf{a}+\dfrac{\lambda}{3}\mathbf{c}$ Comparing coefficient of $\mathbf{a}$: $\dfrac{2\mu}{3}=\lambda$ Comparing coefficient of $\mathbf{c}$: $1-\mu=\dfrac{\lambda}{3}$ Solving simultaneously, $\lambda=\dfrac{6}{11},\ \mu=\dfrac{9}{11}$. $\therefore\overrightarrow{OR}=\dfrac{2}{3}\left(\dfrac{9}{11}\right)\mathbf{a}+\left(1-\dfrac{9}{11}\right)\mathbf{c}=\dfrac{6}{11}\mathbf{a}+\dfrac{2}{11}\mathbf{c}$. == <u>Method 2</u>

	$l_{ON}: \mathbf{r} = \lambda\left(\mathbf{a} + \frac{1}{3}\mathbf{c}\right),\ \lambda \in \mathbb{R}$ $\overrightarrow{CM} = \overrightarrow{OM} - \overrightarrow{OC} = \frac{2}{3}\mathbf{a} - \mathbf{c}$ $l_{CM}: \mathbf{r} = \mathbf{c} + \mu\left(\frac{2}{3}\mathbf{a} - \mathbf{c}\right),\ \mu \in \mathbb{R}$ Since the lines intersect, $\lambda\left(\mathbf{a} + \frac{1}{3}\mathbf{c}\right) = \mathbf{c} + \mu\left(\frac{2}{3}\mathbf{a} - \mathbf{c}\right)$. $\lambda\mathbf{a} + \frac{\lambda}{3}\mathbf{c} = \frac{2\mu}{3}\mathbf{a} + (1-\mu)\mathbf{c}$ Comparing coefficient of $\mathbf{a}$: $\lambda = \frac{2\mu}{3}$ Comparing coefficient of $\mathbf{c}$: $\frac{\lambda}{3} = 1 - \mu$ Solving simultaneously, $\lambda = \frac{6}{11},\ \mu = \frac{9}{11}$. $\therefore \overrightarrow{OR} = \frac{2}{3}\left(\frac{9}{11}\right)\mathbf{a} + \left(1 - \frac{9}{11}\right)\mathbf{c} = \frac{6}{11}\mathbf{a} + \frac{2}{11}\mathbf{c}$.
(iii)	From above, $\overrightarrow{CR} = \frac{9}{11}\overrightarrow{CM}$. $\therefore CR : RM = 9 : 2$.
(iv)	$\overrightarrow{OR} = \frac{6}{11}\mathbf{a} + \frac{2}{11}\mathbf{c}$ $\overrightarrow{OB} = \mathbf{a} + \mathbf{c}$ Since $\overrightarrow{OR} \neq k\overrightarrow{OB}$ for any real constant *k*, *O*, *B* and *R* are not collinear.