H

TEMASEK JUNIOR COLLEGE

2023 JC1 PROMOTIONAL EXAMINATION

Higher 2

FURTHER MATHEMATICS 9649

25 Sep 2023

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of both booklets. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise. Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 Determine whether the following sets are subspaces of $\mathbb{R}^3$. Justify your answer.

(a) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 2x^2 - y^2 = 0 \right\}$,

(b) $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 2y, y - z = 0 \right\}$. [5]

2 $\{u_n\}$ is a geometric progression with first term 2, common ratio r and sum to infinity S.

Given that u_1 and u_3 are the k^{th} and $(k+1)^{\text{th}}$ terms of an arithmetic progression with common difference d, show that d is negative. [3]

Given further that u_7 is the $(k+2)^{\text{th}}$ term of the arithmetic progression, find the exact value of the common difference. [5]

3 **(a)** An ellipse E is given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b \in \mathbb{R}^+$ and $a > b$.

F_1 and F_2 are the foci of E, where F_1 has a positive x-coordinate and P is a point on E.

(i) Explain why the perimeter of triangle F_1PF_2 is always a constant, and state its value in terms of a and b. [2]

(ii) If F_1PF_2 can form an equilateral triangle, find the equations of the directrices of E in terms of a and/or b. [3]

(b) The polar equation of a conic C is $r = \frac{3}{2 + 2\sin\theta}$ where $0 \le \theta \le 2\pi$.

(i) Find the eccentricity of C and identify the conic. [2]

(ii) State the cartesian equation of the directrix of C. (You may assume that the pole is located at the origin of the cartesian plane.) [1]

4 A model for the population size of a particular species of bird in a particular forest is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$$

where P is the population size (in thousands), t is the elapsed time (in number of months), and k and N are constants.

(a) State the contextual significance of k and of N in this model. [2]

(b) Given that $N = 4$, and that there are initially 2 000 birds in the forest, find P in terms of k and t. [5]

(c) Explain what happens to the bird population in the long run if $k = 0.6$. [2]

5 Mary intends to save a total of \$520 000 for her overseas tertiary education. She saves \$40 in the first month and \$50 in the second month. For each subsequent month, she intends to save a total of the amount she saved in the previous month and three-quarters of the amount she saved two months ago.

Let u_n be the amount of money in dollars that she saves in the n^{th} month.

(a) Write down a recurrence relation for u_n in terms of u_{n-1} and u_{n-2}. [1]

(b) Solve the recurrence relation in **(a).** [6]

(c) Determine the number of months needed to save a total of \$520 000. [3]

6 ~~It is given that -2 is an eigenvalue with corresponding eigenvector $\begin{pmatrix}1\\1\\0\end{pmatrix}$ of a matrix **A**~~

~~where $\mathbf{A} = \begin{pmatrix}0 & -2 & 2\\ k & 1 & k\\ -1 & 1 & k\end{pmatrix}$.~~

~~**(a)** Show that $k = 3$.~~ ~~[1]~~

~~**(b)** Find, without using a graphing calculator and showing your working clearly, the other two eigenvalues and their corresponding eigenvectors.~~ ~~[5]~~

~~Hence write down a matrix **Q** and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{QDQ}^{-1}$.~~ ~~[2]~~

~~**(c)** Find a matrix **P** and a diagonal matrix **C** such that $\mathbf{B} = \mathbf{PCP}^{-1}$ where $\mathbf{B} = (\mathbf{A} - 3\mathbf{I})^4$.~~ ~~[3]~~

[Turn over

7 The polar curve C is represented by the equation $r = \tan\theta + \sec\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

It is given that as $\theta \to -\frac{\pi}{2}, r \to 0$.

(a) Show that C has a vertical asymptote and write down its equation. [2]

(b) Sketch the graph of C. [2]

(c) Find the exact area of the finite region bounded by C and the initial line. [5]

(d) Find the length of the segment of C where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. [3]

8 Let $\mathrm{T}: \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation $\mathrm{T}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \mathbf{A}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$.

It is given that the matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -3 & 2 & \alpha - 6 \\ 1 & 5\alpha - 2 & 1 & -1 \\ 0 & 1 & 0 & \alpha - 4 \end{pmatrix}$ where α is a real constant.

(a) Find the values of α for which the dimension of the null space of T is 1. [4]

For the rest of the question, take $\alpha = 0$.

(b) Find a basis for the range space of T. [1]

(c) Given that the vector $\begin{pmatrix} p \\ -7 \\ q \\ 3 \end{pmatrix}$ lies in the range space of T, find the values of p and q. [3]

(d) Find a basis for the null space of T. [3]

(e) Hence, find the set of solutions for $\mathbf{Ax} = \begin{pmatrix} -2 \\ -3 \\ -2 \\ 1 \end{pmatrix}$. [2]

9 **(a)** Given that $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, show that $I_n = \frac{1}{n-1} - I_{n-2}$ for all integers $n \geq 3$. [3]

Hence find the exact value of I_3. [3]

(b) The region bounded by the curve $y = x(\sin x + 1)$ where $0 \leq x \leq \frac{5\pi}{2}$, the y-axis and the line $y = 5\pi$ is rotated 2π radians about the y-axis. Find the exact volume of the solid generated. [8]

10 ~~During a carnival, ice cream popsicles are given away to children for free. Ten popsicles are left to be given away before the end of the carnival. It is given that four of the popsicles are durian flavour, two are chocolate flavour and the remaining flavours are apple, orange, lime and berry. Popsicles that are of the same flavour are indistinguishable from one another. Suppose that four popsicles are given to 4 children, such that each child receives exactly one popsicle. Find the number of ways this can be done if~~

~~**(a)** all four popsicles are of different flavour, [2]~~

~~**(b)** there are no restrictions on the flavours of the popsicles. [5]~~

~~The four children together with their father and mother went to play at a Merry-Go-Round at the carnival. The Merry-Go-Round has ten identical chairs placed in a circle. The family of six and four other people are seated such that each person occupies one seat.~~

~~**(c)** Find the number of ways the ten of them can be seated if the family of six are seated together such that all the four children are seated between their parents. [3]~~