Chapter 4 (Pure Mathematics): Graphing Techniques

Objectives

At the end of the chapter, you should be able to:

(a) understand the characteristics of graphs with the help of a graphic calculator, locate the turning points, and determine the asymptotes (horizontal and vertical), axes of symmetry, and restrictions on the possible values of x and y;

(b) select the appropriate "window" of a graphic calculator that would display the critical features of the functions when sketching graphs;

(c) use a graphic calculator to draw and compare the graphs of a variety of functions;

(d) understand the relationship between a graph and its associated algebraic equation.

Content

4.1 Functions

4.2 Basic Graphs

4.2 Characteristics of a Graph
- 4.2.1 Axes Intercepts
- 4.2.2 Asymptotes
- 4.2.3 Stationary Points
- 4.2.4 Restrictions on values of x and y
- 4.2.5 Symmetry

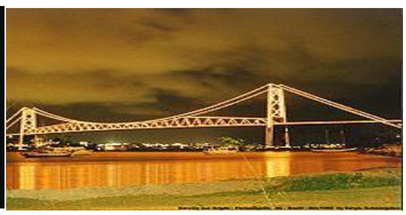
4.3 Conics Section
- 4.3.1 Circles
- 4.3.2 Ellipse
- 4.3.3 Using GC
- 4.3.4 General Forms of Conic

Relevant websites

1. Applet on circles and ellipses: http://mathinsite.bmth.ac.uk/applet/ellipses/ellipses.html
2. Applet on graphs of various functions: http://www.analyzemath.com/

Introduction

For concept learning systems, graphical data representation is crucial. A good representation might make possible the learning of a concept that was not learnable using other representations. For example, in the case of the earthquake domain (earthquake data such as its location, epicenter, intensity, depth, type, etc.), we need important information such as the distance between the earthquakes' epicentres and the difference in time between the earthquakes. This information could make possible the learning of an important concept. This is known as a graph-based concept learning system. Below are some examples of "graphs" you see in physical world.

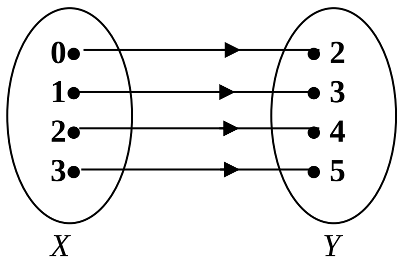
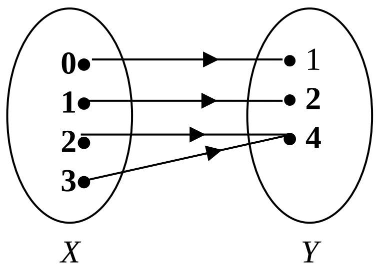
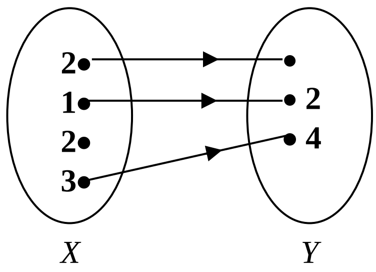
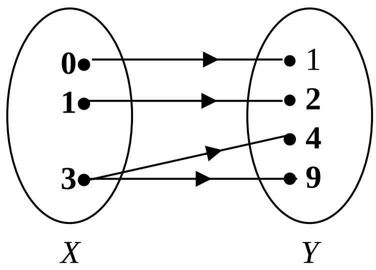
4.1 Definition of Function

A function is a **rule or relationship** where for every input there is **only one output**.

An example of a function is $\mathrm{f}(x) = x^2 + 5$, where x is the input to the function $\mathrm{f}(x)$ and $x^2 + 5$ is the output of the function $\mathrm{f}(x)$.

For every input value of x, there is only one output value. In short, a function is a process.

Example 1
For each of the mappings below, state, with a reason, whether it is a function.

(i)		Yes, since every input to function f has only one output. i.e. $\mathrm{f}(x) = x + 2$
(ii)		Yes, since every input to function f has only one output.
(iii)		No, since 2 has no output.
(iv)		No, since 3 has two outputs.

4.2 Basic Graphs

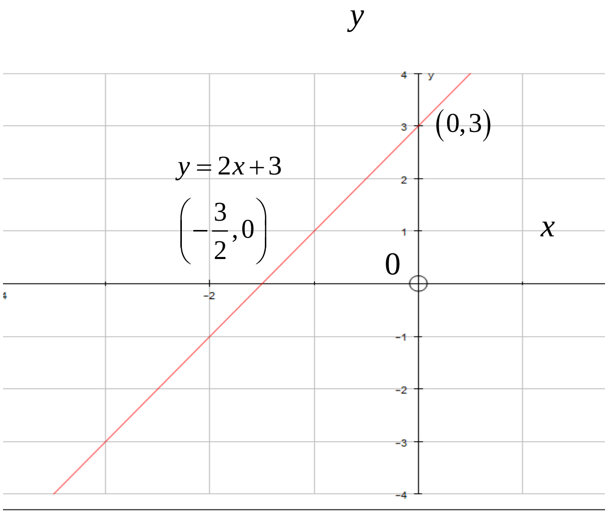
(A) Polynomial Graphs

Graphs with equation in the form $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$ and n is a positive integer.

(i) $y = ax + b$ (Linear Functions)

Linear functions have equations in the form $y = ax + b$ where a is the gradient and b is the y-intercept of the line.

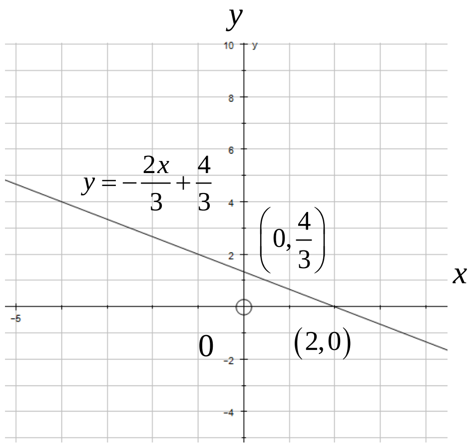
Example: $y = 2x + 3$ $(a > 0)$



$x = 0 \Rightarrow y = 3$

$y = 0 \Rightarrow x = -\frac{3}{2}$

Example: $y = -\frac{2x}{3} + \frac{4}{3}$ $(a < 0)$



$y = 0 \Rightarrow \frac{2x}{3} = \frac{4}{3}$

$x = 2$

(ii) $y = ax^2 + bx + c$ where $a \neq 0$ (Quadratic Functions)

Quadratic functions have equations in the form $y = ax^2 + bx + c$, where $a \neq 0$.

The graph of a quadratic function is a parabola symmetrical about a vertical axis.

It is easier to sketch a quadratic function in completed square form,

i.e. $y = a(x-p)^2 + q$.

Note:

(i) For $a > 0$, the curve has a minimum point ("U - shape").

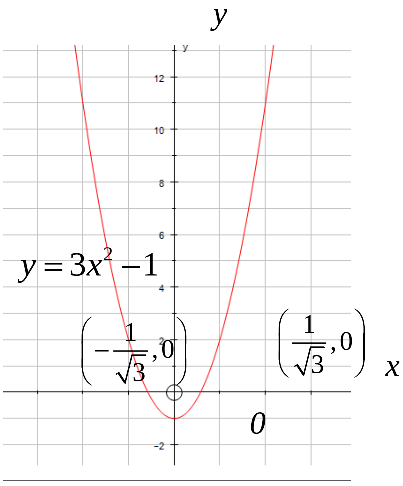
For $a < 0$, the curve has a maximum point ("Inverted U - shape").

(ii) The maximum/minimum point has coordinates (p, q).

(iii) The x-intercepts (if any) and y-intercept can by found by fixing $y = 0$ and $x = 0$ respectively.

(iv) The x-intercepts (if any) are given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example: $y=3x^2-1$ $(a>0)$

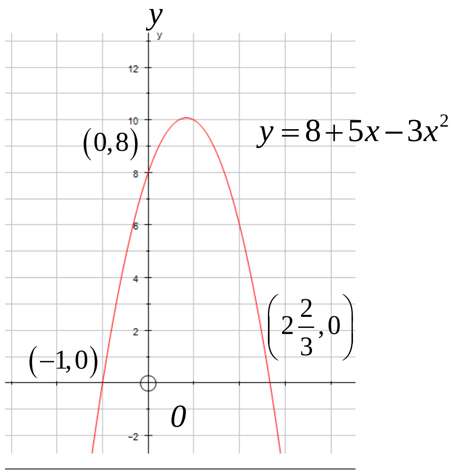


min point : $(0,-1)$

Example: $y=8+5x-3x^2$ $(a<0)$

$$y=8+5x-3x^2=-3\left(x^2-\frac{5}{3}x\right)+8$$

$$=-3\left[\left(x-\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2\right]+8=-3\left(x-\frac{5}{6}\right)^2+10\frac{1}{12}$$



max point : $\left(\frac{5}{6},5\frac{11}{12}\right)$

$x=0\Rightarrow y=8$

$y=0\Rightarrow 3x^2-5x-8=0$

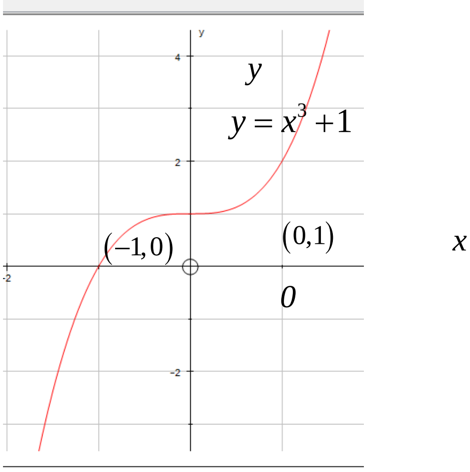
$$x=\frac{-5\pm\sqrt{25-4(3)(-8)}}{2(3)}=\frac{-5\pm\sqrt{121}}{6}=\frac{16}{6}\text{ or }-1$$

$$=\frac{8}{3}\text{ or }-1$$

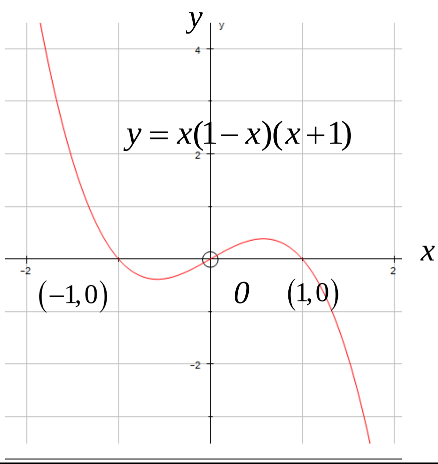
(iii) $y=ax^3+bx^2+cx+d$ (Cubic Functions)

Cubic functions have equations in the form $y=ax^3+bx^2+cx+d$, $x\in\mathbb{R}$, where $a\neq 0$.

Example: $y=x^3+1$ $(a>0)$



Example: $y=x(1-x)(x+1)$ $(a<0)$

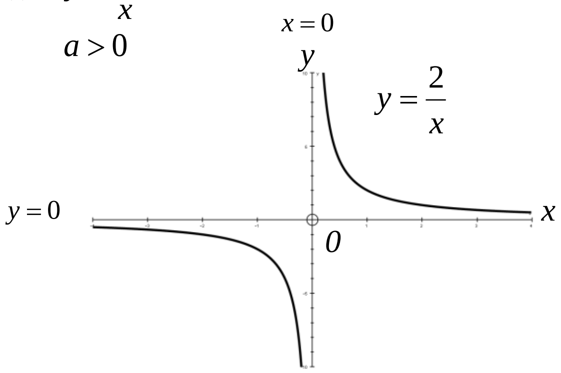


(B) Reciprocal Functions $y=\frac{a}{x^n}$ where *n* is a positive integer

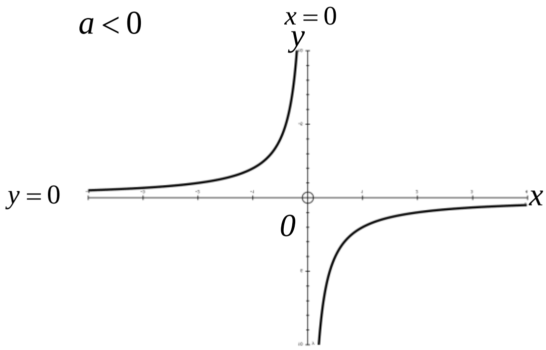
Reciprocal functions have equations in the form $y=\frac{1}{x}$, $x\in\mathbb{R}$, $x\neq 0$. The graph tends towards the asymptotes $x=0$ and $y=0$.

(i) $y=\frac{2}{x}$

$a>0$

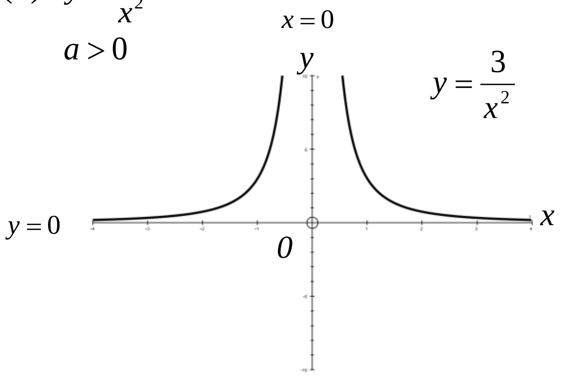


$a<0$

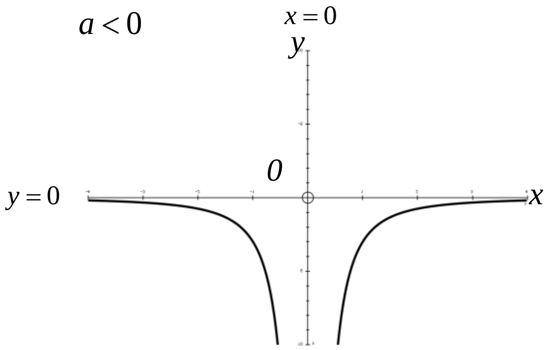


(ii) $y=\frac{a}{x^2}$

$a>0$



$a<0$



Note:

1. The graphic calculator has to be in function mode to graph $y=\mathrm{f}(x)$.

2. GC will not show you the asymptotes, axes intercepts and max and min points. Therefore, DO NOT copy the graph from the GC blindly.

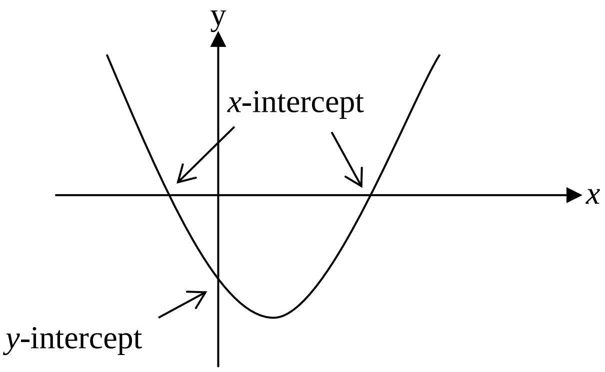
4.2 Characteristics of a Graph

The following are important features of a graph to take note:
(A) Axial Intercepts
(B) Asymptotes
(C) Stationary Points
(D) Symmetries

4.2.1 Axial Intercepts

An *x*-intercept of a graph is a point where the graph intersects the *x*-axis.
To find the *x*-intercepts of a graph, let $y=0$ and solve the equation for *x*.

A *y*-intercept of a graph is a point where the graph intersects the *y*-axis.
To find the *y*-intercepts of a graph, let $x=0$ and solve the equation for *y*.



Example 2

Find the axes intercepts of the graph $y=\dfrac{x+1}{x+2}$.

Solution:

When $x=0,\ y=\dfrac{1}{2}$.

When $y=0,\ x=-1$.

Therefore the *x*-intercept is $(-1,0)$ and *y*-intercept is $\left(0,\dfrac{1}{2}\right)$.

<u>Example 3 (Using GC)</u>

Find the axial intercepts of $y = x^2 e^x - 1$.

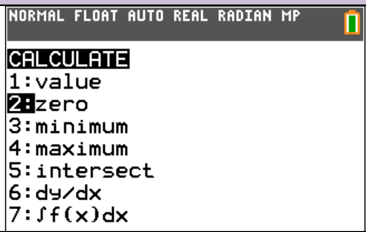
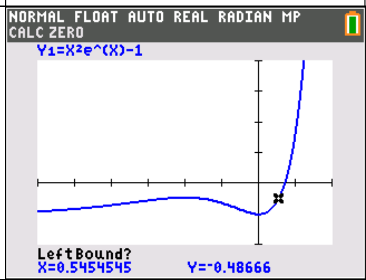
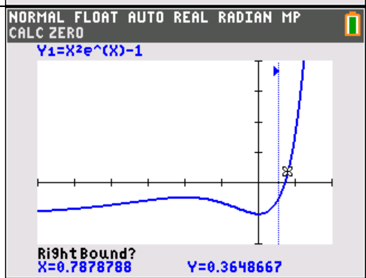
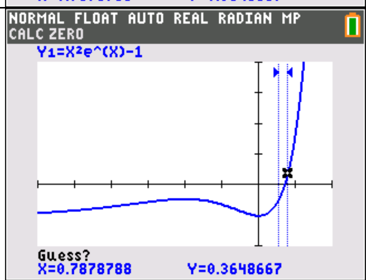
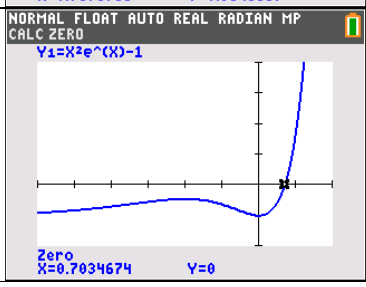
Solution:

When $x = 0$, $y = -1$.

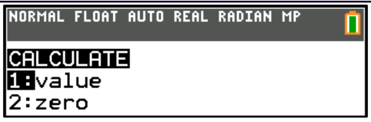
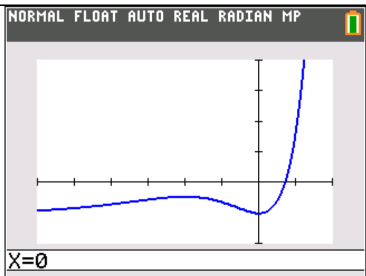
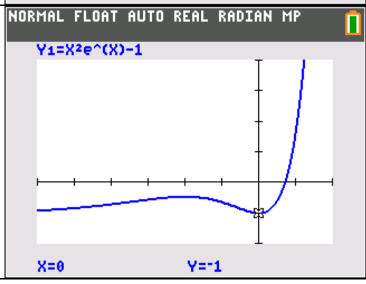
When $y = 0$, $x^2 e^x - 1 = 0$ (Using GC)

This equation is difficult to solve by using algebraic method. We use GC to determine x-intercept.

<u>Using GC to find x-intercept</u>

Steps	Screenshot	Notes
1. Press [2nd][CALC] Select 2: zero or simply press [2]		
2. Respond to “Left bound?”, move the cursor to the left of the x-intercept and press [ENTER]		
3. Respond to “Right bound?”, move the cursor to the right of the x-intercept and press [ENTER]		
4. When prompted “Guess?”, press [ENTER]		
5. The x-intercept is $x = 0.7034674$.		

Using GC to find *y*-intercept

Steps	Screenshot	Notes
1. Press [2nd][CALC] Select 1: value Press [ENTER]		
2. Enter the value $x = 0$. Press [ENTER]		
3. The *y*-intercept is (0, – 1)		

4.2.2 Asymptotes

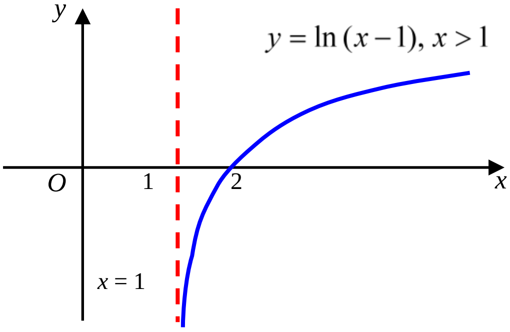
(a) Vertical Asymptotes

In the graph of $y = \mathrm{f}(x)$, if there exist a constant a such that $x \to a,\ y \to +\infty$ or as $x \to a,\ y \to -\infty$, then the line $x = a$ is a vertical asymptote of the graph.

Method: Look for values of x for which $\mathrm{f}(x)$ is undefined.
If $\mathrm{f}(x)$ is undefined when $x = a$, then $x = a$ is a vertical asymptotes.

In our syllabus, we will learn two types of asymptotes, namely, vertical asymptote and horizontal asymptote. Asymptotes are usually drawn as **dotted lines.**

For example, in the graph of $y = \ln(x-1),\ x > 1$, the value of y tends to $-\infty$ when x gets closer and closer to 1. (In fact, x approaches to 1 from the right side, i.e. $x \to 1^+$). Then the line $x = 1$ is a vertical asymptote.

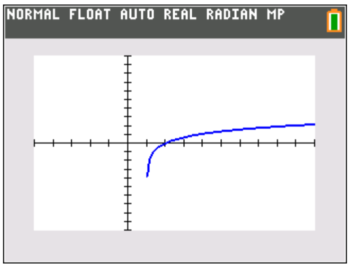


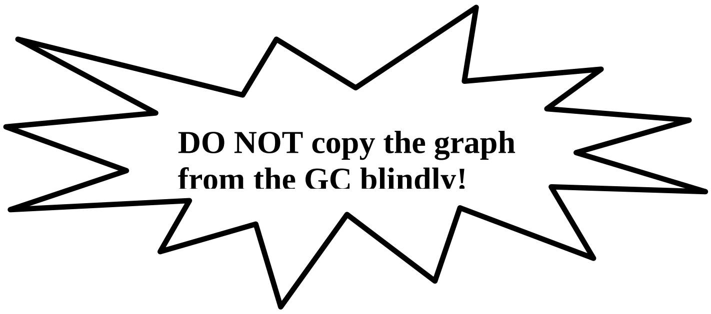
Note:

1. A graph does not touch its vertical asymptote(s) at all times.
2. GC does not indicate the presence of asymptotes.

   For example, the graph $y = \ln(x - 1)$ obtained from GC is shown as if appears as if the graph discontinues at the point (1, – 3)!

   This is due to the limitation of GC.





3. A **rational** function is of the form $y = \dfrac{C(x)}{D(x)}$, where $C(x)$ and $D(x)$ are polynomials of x. To find vertical asymptotes, we let $D(x) = 0$ and solve for x. See the next example (**Example 4a**) for illustration. We will study rational functions in more detail in Section 4.2.2.1

Example 4
Find the equation of vertical asymptotes (if any) of the following graphs with equation **(a)** $y = \dfrac{x}{(x+1)(x-3)}$ **(b)** $y = \dfrac{x^3}{x^2+5}$ **(c)** $y = \ln(1-2x)$

Solution:

(a) Let $(x + 1)(x – 3) = 0$, $\Rightarrow$ $x = -1$ and $x = 3$
The vertical asymptotes are $x = -1$ and $x = 3$.

(b) Since $x^2 + 5 \neq 0$, there is no vertical asymptote.

(c) $y = \ln(1-2x)$ is undefined when $1 – 2x = 0$, the vertical asymptote is $x = \dfrac{1}{2}$.

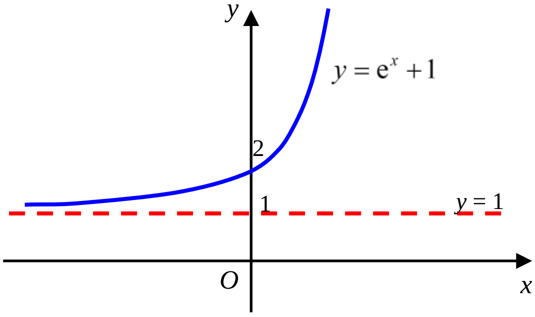
(b) Horizontal Asymptotes

In the graph of $y = \mathrm{f}(x)$, if there exist a constant k such that $x \to +\infty, y \to k$ **or** as $x \to -\infty, y \to k$, then the line $x = k$ is a horizontal asymptote of the graph.

Method: Examine the values of y as $x \to \infty$ and as $x \to -\infty$

For example, for the graph of $y = \mathrm{e}^x + 1$,
As $x \to \infty, y \to \infty$,
but $x \to -\infty, y \to 1$.

The line $y = 1$ is a horizontal asymptote.



Note:

1. $f(x) \to \infty$ is read as $f(x)$ tends to or approaches to infinity.

2. A graph cannot cut a vertical asymptote but it is possible for a graph to cut a horizontal asymptote.

Example 5

Find the equation of horizontal asymptotes (if any) of the following graphs with equation **(a)** $y = 1 - \frac{1}{x}$ **(b)** $y = e^{2x} - 3$ **(c)** $y = \ln(2x+1)$

Solution:

(a) The horizontal asymptote is $y = 1$.

(b) The horizontal asymptote is $y = -3$.

(c) As $x \to \infty$, $y \to \infty$ and $x \to -\infty$, $y \to -\infty$ there is no horizontal asymptote.

4.2.2.1 Asymptotes of Rational Functions

A rational function is of the form $f(x) = \frac{P(x)}{Q(x)}$ where P(x) and Q(x) are **polynomials**[1] with Q(x) ≠ 0.

Expressions like $\frac{x-1}{x+2}$, $\frac{x-1}{x^2+4}$ and $\frac{x^2+1}{2x^3-2x+3}$ are rational functions.

Vertical Asymptotes

Vertical asymptotes of a rational function are given by the values of x where the function is undefined i.e. $Q(x) = 0$.

Therefore to find vertical asymptotes of a rational function, let $Q(x) = 0$ and solve for x.

For example, $y = \frac{2}{1+x}$.

Horizontal asymptotes

If $y = \frac{P(x)}{Q(x)}$ is a proper fraction, then the equation of the horizontal asymptote is $y = 0$.

[1] A **polynomial function** is a **function** of the form: $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$
Example: linear, quadratic, cubic, quartic functions are polynomial functions.

For example, $y=\frac{2}{7x}$.

If $y=\frac{P(x)}{Q(x)}$ is an improper fraction, do long division to express $y=\frac{P(x)}{Q(x)}$ as $y=A(x)+\frac{R(x)}{Q(x)}$.

As $x\to\pm\infty$, $y\to A(x)$. Thus equation of horizontal asymptote is $y=A(x)$.

For example, $y=\frac{x}{1+x}=1-\frac{1}{1+x}$.

Note:
Refer to Annex 1 for the method to do long division.

Example 6 (If $y=\frac{P(x)}{Q(x)}$ is a proper fraction)

Sketch $y=-\frac{2}{1+x}$, giving the equations of asymptotes and coordinates of any points of intersection with the axes.

Solution:
Asymptotes:

Vertical asymptote : $x=-1$

Horizontal asymptote : $x\to\pm\infty$, $y\to 0$. Therefore, **horizontal** asymptote is $y=0$.

Stationary (turning) points:

$$\frac{dy}{dx}=\frac{2}{(1+x)^2}\neq 0$$

There are no turning points.

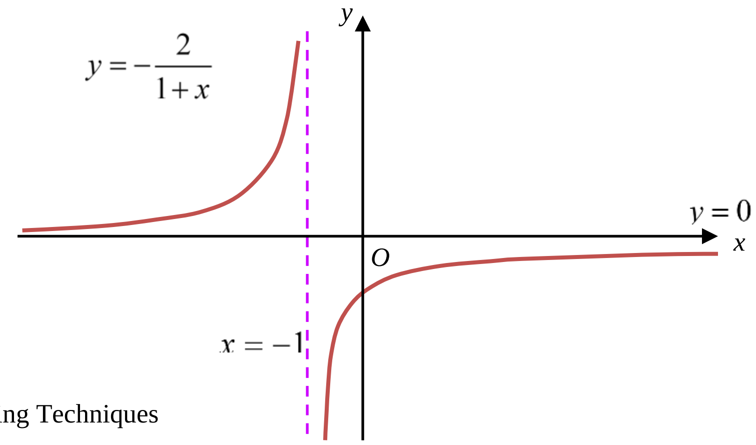
Using GC to obtain the sketch:

Intercepts:

When x = 0, y = - 2,

When y = 0, (no real solution)

In this case there is no x intercept.



Example 7 (If $y=\dfrac{P(x)}{Q(x)}$ is not a proper fraction)

Sketch $y=\dfrac{x}{1+x}$, giving the equations of asymptotes and coordinates of any points of intersection with the axes.

Solution:

$y=\dfrac{x}{1+x}=1-\dfrac{1}{1+x}$ (by long division. Why?)

Asymptotes:

Vertical asymptote : $x=-1$

Horizontal asymptote: $x\to\pm\infty,\ \dfrac{1}{1+x}\to 0,\ 1-\dfrac{1}{1+x}\to 1$. Therefore, **horizontal** asymptote is $y=1$.

Stationary (turning) points:

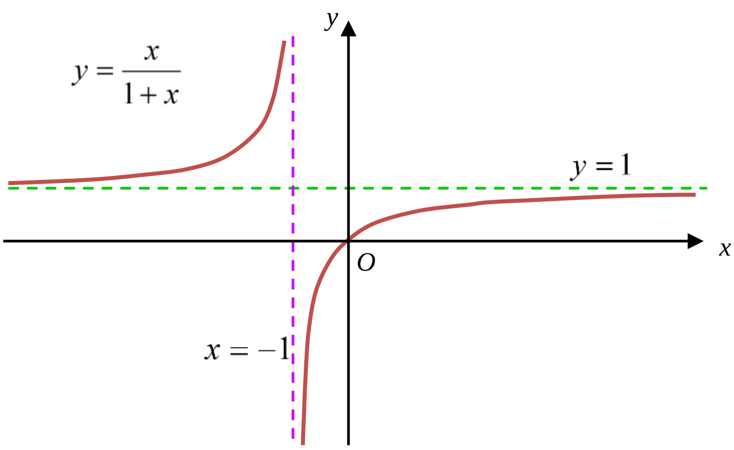
$$\frac{dy}{dx}=\frac{1}{(1+x)^2}\neq 0$$

There are no turning points.

Axes Intercepts:

When $x = 0$, $y = 0$, $\Rightarrow$ The curve passes through the origin.

Using GC to obtain the sketch:



Note:

This curve in Example 6 is called a **rectangular hyperbola**.

A function of the form $y=\dfrac{ax+b}{cx+d}$, $c\neq 0$ is always a rectangular hyperbola.

Exercise 1

Find the asymptotes of the following graphs

(a) $y = \dfrac{-3}{2x}$

(b) $y = \dfrac{2x-5}{x-3}$

(c) $y = \dfrac{5x^2-15x+13}{x^2-3x+2}$.

(d) $y = \dfrac{4+3x}{x^2-1}$

Solution:

(a) $y = \dfrac{-3}{2x}$

Horizontal asymptote: $y = 0$

To find vertical asymptote: Let $2x = 0$

$x = 0$

(b) $y = \dfrac{2x-5}{x-3} = 2 + \dfrac{1}{x-3}$

Horizontal asymptote: $y = 2$

To find vertical asymptote: Let $x - 3 = 0$

$x = 3$

(c) $y = \dfrac{5x^2-15x+13}{x^2-3x+2} = 5 + \dfrac{3}{x^2-3x+2}$

Horizontal asymptote: $y = 5$

To find vertical asymptote: Let $x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2$ or $x = 1$

(d) $y = \dfrac{4+3x}{x^2-1}$

Horizontal asymptote: $y = 0$

To find vertical asymptote: Let $x^2 - 1 = 0$

$(x+1)(x-1) = 0$

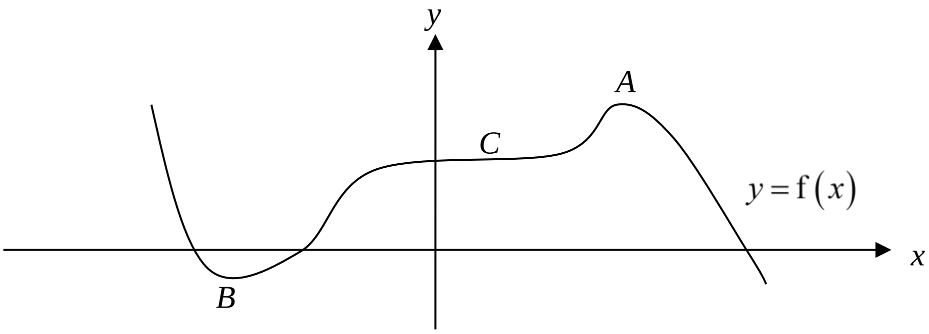
$x = -1$ or $x = 1$

4.2.3 Stationary Points

A point on a curve $y=\mathrm{f}(x)$ is called a stationary point if the gradient at the point is zero.
At a stationary point, the curve is flat and the tangent drawn is parallel to the x-axis.

There are three types of stationary points:
(a) maximum point,
(b) minimum point and
(c) point of inflexion.

Maximum and minimum points are also called turning points.



In the diagram above, A is a maximum point, B is a minimum point and C is a stationary point of inflexion. A, B and C are all stationary points.

To locate all stationary points, let $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.

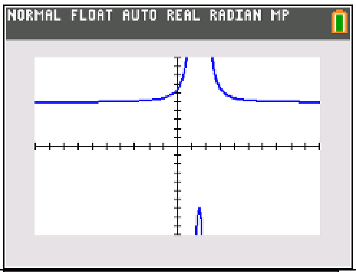
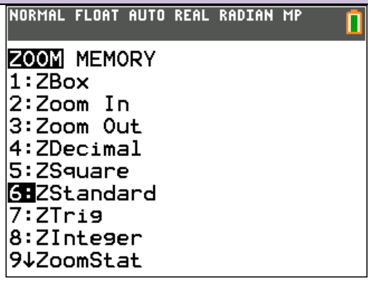
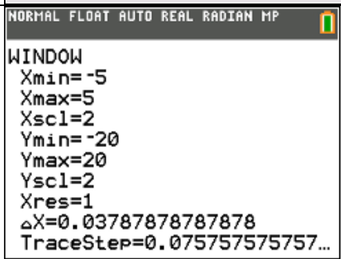
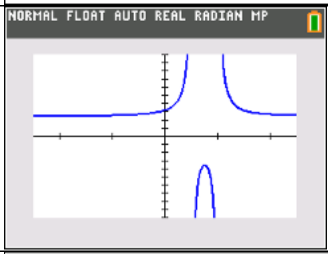
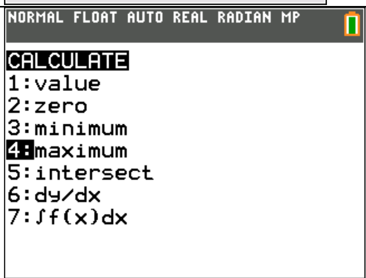
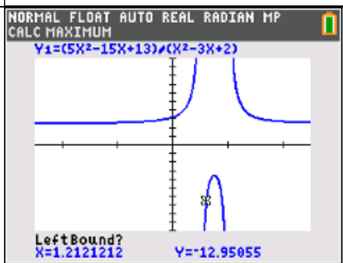
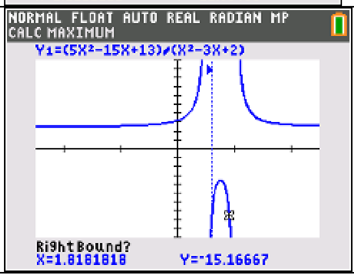
If the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ has no real roots, then the curve has no stationary points.
Note: You will learn how to find stationary points and the nature of the stationary points using analytical method in Chapter 6.

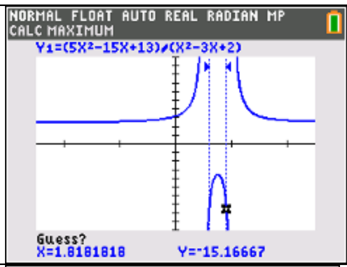
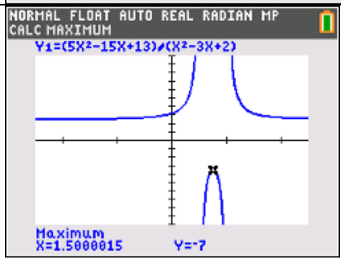
Example 8

Find the stationary point(s) of the graph $y=\frac{5x^2-15x+13}{x^2-3x+2}$.

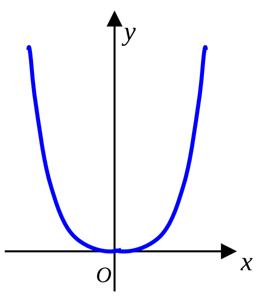
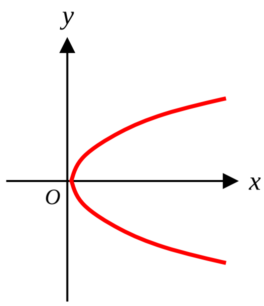
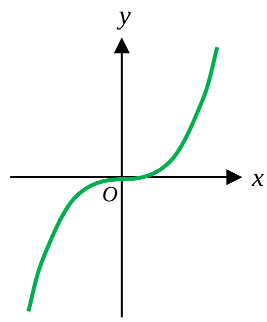
Solution:
Using GC, the maximum point is (1.50, -7).

Using GC to find the maximum point

Steps	Screenshot	Notes
1. Press $\boxed{\text{Y=}}$ and enter the expression $y=\dfrac{5x^2-15x+13}{x^2-3x+2}$, and press $\boxed{\text{GRAPH}}$ or $\boxed{\text{ZOOM}}\boxed{6}$	 	 The standard window settings shows the graph for $-10\le x\le 10$ and $-10\le y\le 10$.
2. Press $\boxed{\text{WINDOW}}$ to change window setting Press $\boxed{\text{GRAPH}}$		
3. Press $\boxed{\text{GRAPH}}$		
4. Press $\boxed{\text{2nd}}\boxed{\text{CALC}}$ Select 4: maximum or simply press $\boxed{4}$ Press $\boxed{\text{ENTER}}$		
5. Respond to "Left bound?", move the cursor to the left of the maximum point and press $\boxed{\text{ENTER}}$		
6. Respond to "Right bound?", move the cursor to the right of the maximum point and press $\boxed{\text{ENTER}}$		

7. When prompted “Guess?”, press $\boxed{\text{ENTER}}$		
8. The x-value and the y-value of the maximum point is shown on the screen.		The maximum point that we found from the analytical method is $(1.5, -7)$.

4.2.4 Symmetry

Axis/Point of Symmetry	***y*-axis**	***x*-axis**	**origin**
Algebraically	Equation of the curve remains unchanged when x is replaced by $-x$. If (x, y) is a point on the graph, then $(-x, y)$ is also a point on the same graph.	Equation of the curve remains unchanged when y is replaced by $-y$. If (x, y) is a point on the graph, then $(x, -y)$ is also a point on the same graph.	Equation of the curve remains unchanged when x is replaced by $-x$ and y by $-y$. If (x, y) is a point on the graph of $y = \mathrm{f}(x)$ then $(-x, -y)$ is also a point on the same graph.
Graph	eg. $y = x^4$ 	eg. $y^2 = x$ 	eg. $y = x^3$ 

Example 9

Sketch $y = \dfrac{x+1}{x+2}$.

Solution:

$y = \dfrac{x+1}{x+2}$

$y = 1 - \dfrac{1}{x+2}$

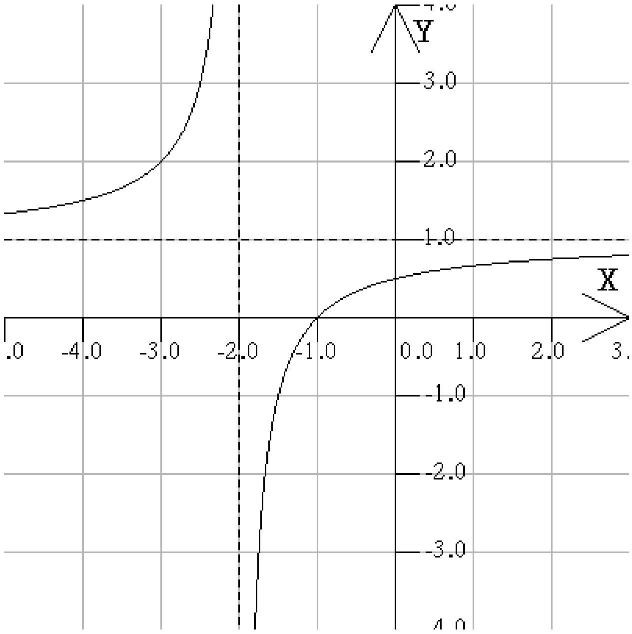
As $x \to \infty, y \to 1$

$y = 1$ is the asymptotes.

$x = -2$ is the vertical asymptotes

When $x = 0$, $y = \dfrac{1}{2}$.

The graph is a rectangular hyperbola. It cuts the *y*-axis at $y = \dfrac{1}{2}$



When $y = 0, x = -1$

It cuts the *x*-axis at $x = -1$

Example 10 (Specimen Paper 2006)

Consider the curve $y = \dfrac{3x-6}{x(x+6)}$.

(i) State the coordinates of any points of intersections with the axes.

(ii) State the equations of the asymptotes.

(iii) Sketch the curve $y = \dfrac{3x-6}{x(x+6)}$.

Solution:

(i) When $y = 0 \Rightarrow 3x - 6 = 0 \Rightarrow x = 2$

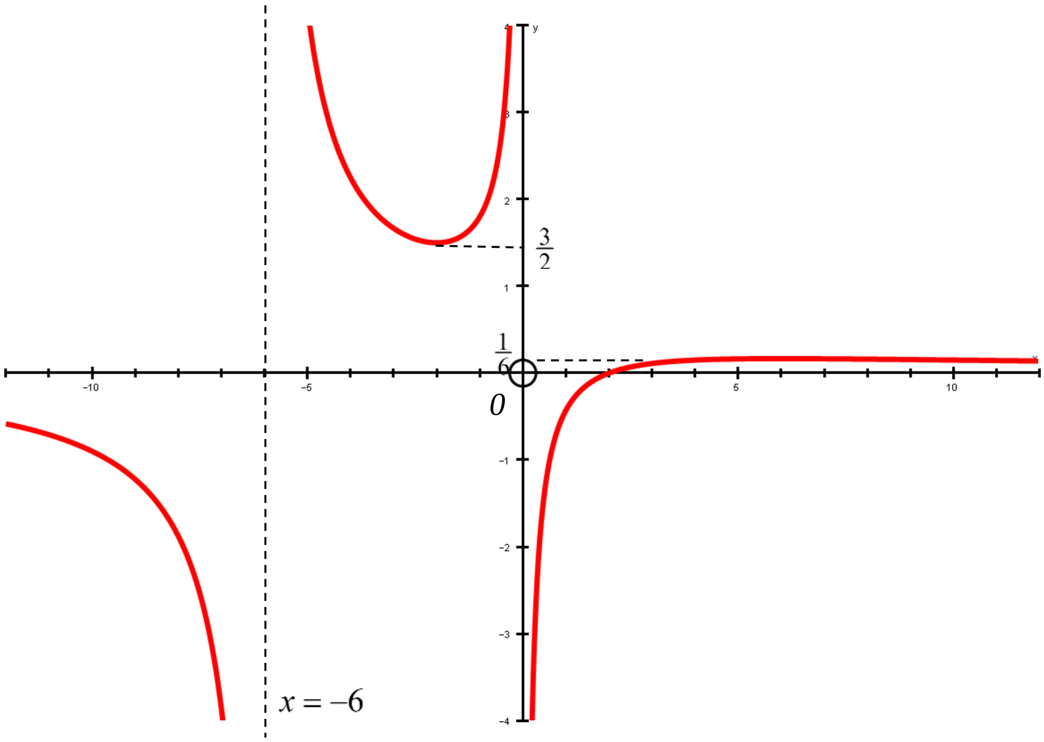
Point of intersection with the x-axis is $(2,0)$.

(ii) Vertical asymptotes: $x = 0$ or $x = -6$

Horizontal asymptote: $y = 0$

(Since $\dfrac{3x-6}{x(x+6)}$ is a proper fraction, $y = 0$ is a horizontal asymptote.)

(iii)



Exercise 2

1. Sketch the curves represented by the following equations.

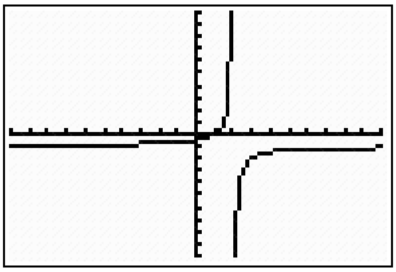
(a) $y=\dfrac{1-x}{x-2}$ (b) $y=\ln(x+2)$

(c) $y=e^{-x-1}$

Solution:

(a) $y=\dfrac{1-x}{x-2}$

$$y=\frac{1-x}{x-2}=\frac{-(x-2)-1}{x-2}=-1-\frac{1}{x-2}$$



As $x\to\pm\infty, y\to-1$

$y=-1$ is the asymptotes.

$x=2$ is the vertical asymptotes

When $x=0$, $y=-\dfrac{1}{2}$.

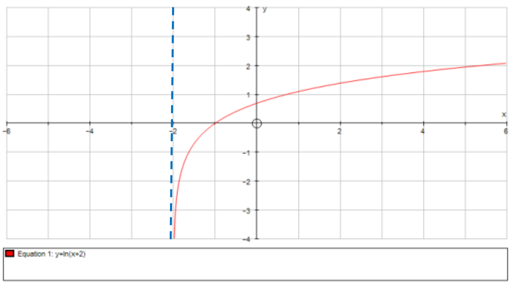
The graph is a rectangular hyperbola. It cuts the y-axis at $y=-\dfrac{1}{2}$

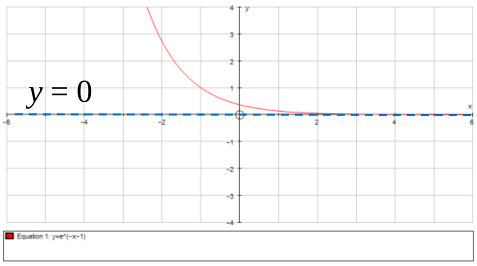
When $y=0, x=1$

It cuts the x-axis at $x=1$

(b) $y=\ln(x+2)$ (c) $y=e^{-x-1}$

$x=-2$





2. The equation of a curve C is $y=1+\dfrac{6}{x-2}-\dfrac{24}{x+3}$.

(i) Write down the equations of the asymptotes.

(ii) Find the coordinates of the points where C meets the axes.

(iii) Sketch C.

Solution:

(i) Vertical Asymptotes: x = 2 and x = -3
Horizontal asymptotes: y = 1

(ii) When x = 0, y = 1+(-3)-8 = -10 (y axis intercept)

When y = 0,

$$0 = 1 + \frac{6}{x-2} - \frac{24}{x+3}$$

$$0 = \frac{(x-2)(x+3)+6(x+3)-24(x-2)}{(x-2)(x+3)}$$

$$x^2 + x - 6 + 6x + 18 - 24x + 48 = 0$$

$$x^2 - 17x + 60 = 0$$

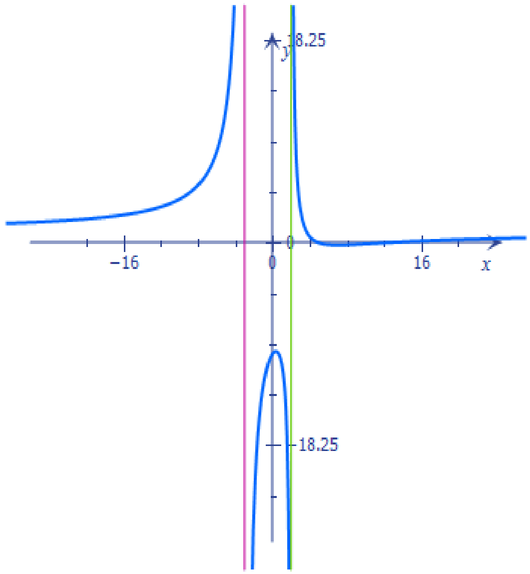
$$(x-12)(x-5) = 0$$

$$x = 12 \text{ or } x = 5$$

Coordinates of the axial intercepts are (0, -10), (5, 0), (12, 0).

(iii) $y = 1 + \frac{6}{x-2} - \frac{24}{x+3}$

Asymptotes:
$x = 2$
$x = -3$



3. Find the equations of the asymptotes of the graph $y = \dfrac{3-2x}{x-2}$ and sketch the graph. On the same diagram, sketch the graph of $y = 1 - e^{-2x}$. Hence find the number of real roots of $(3x-5)e^{2x} = x-2$.

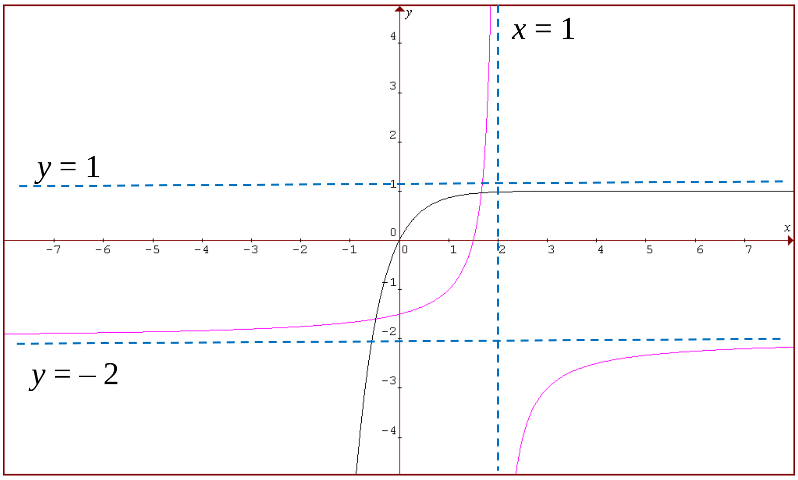
Solution:

The vertical asymptote is $x = 2$.

$y = \dfrac{3-2x}{x-2} = -2 - \dfrac{1}{x-2} \Rightarrow$ The horizontal asymptote is $y = -2$.

When $x = 0$, $y = -3/2$.

When $y = 0$, $x = 3/2$.



$$(3x-5)e^{2x} = x-2$$

$$\frac{3x-5}{x-2} = e^{-2x}$$

$$3 + \frac{1}{x-2} = e^{-2x}$$

$$-3 - \frac{1}{x-2} = -e^{-2x}$$

$$1 - 3 - \frac{1}{x-2} = 1 - e^{-2x}$$

$$-2 - \frac{1}{x-2} = 1 - e^{-2x}$$

$$\frac{3-2x}{x-2} = 1 - e^{-2x}$$

Since there are two points of intersection, number of roots = 2.

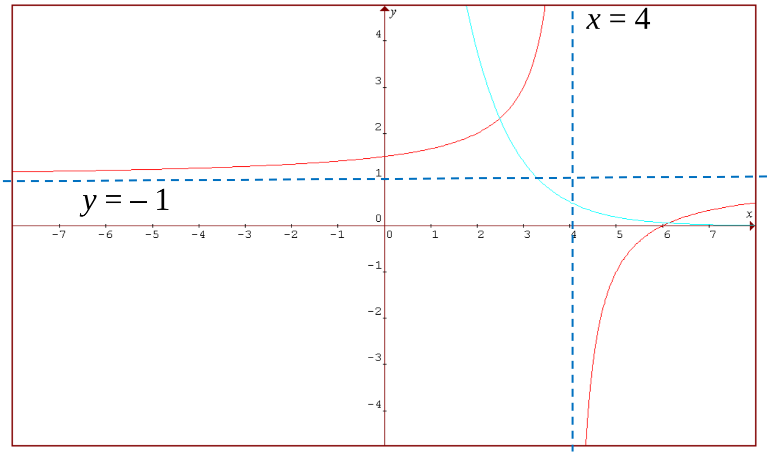
4. Sketch the graph of $y = \dfrac{6-x}{4-x}$, stating the equation of the asymptotes. By first sketching the graph of a suitable function on your diagram, find an interval of the form [n, n+1], where n is an integer, containing the larger root, α, of the equation $\dfrac{6-x}{4-x} = 28e^{-x}$.

Solution:

$y = \dfrac{6-x}{4-x} = 1 + \dfrac{2}{4-x}$

Asymptotes: $x = 4$ and $y = 1$.

When $x = 0$, $y = 3/2$.
When $y = 0$, $x = 6$.



From the graph, the larger root α lies in the interval [6,7] and so $n = 6$.

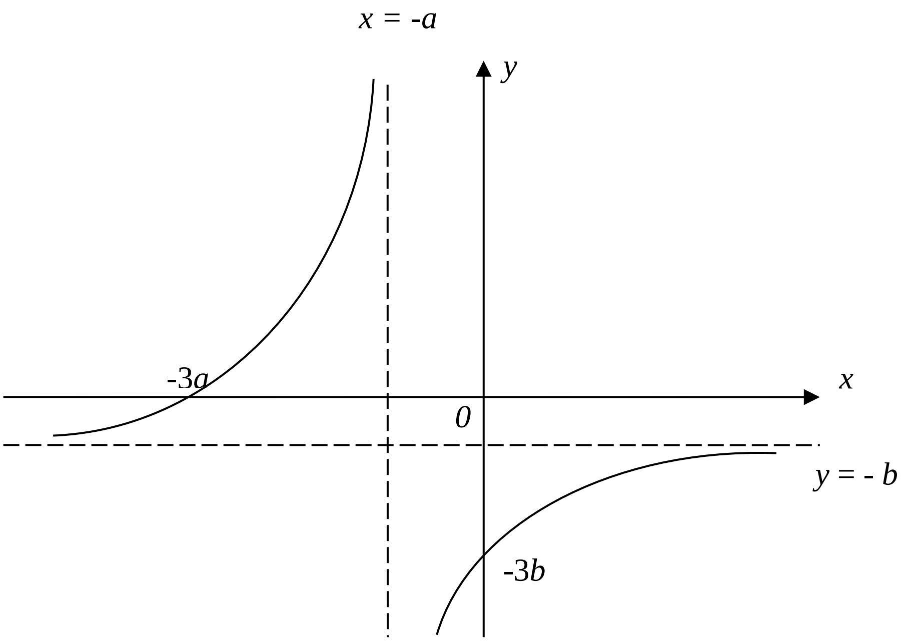
5. The curve C has equation $y = -\dfrac{b(x+3a)}{x+a}$ where *a* and *b* are positive constants. State, in terms of *a* and *b*, the coordinates of intersection of C with the axes and the equations of asymptotes.

Solution:

$$y = -\frac{b(x+3a)}{x+a} = -b - \frac{2ab}{x+a}$$

When $y = 0$, $x = -3a$.
When $x = 0$, $y = -3b$.
Coordinates of intersection: $(0, -3b)$ and $(-3a, 0)$

Equations of asymptotes: $y = -b$ and $x = -a$.

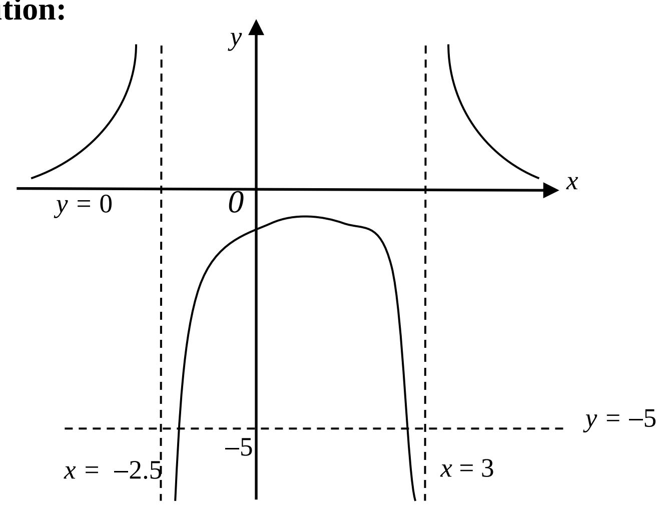


6. A graph has equation $y = \dfrac{6}{(x-3)(2x+5)}$.

(i) Sketch the graph, indicating all the asymptotes.

(ii) Hence solve the inequality $0 < \dfrac{1}{(3-x)(2x+5)} < \dfrac{5}{6}$.

Solution:



$$0 < \frac{1}{(3-x)(2x+5)} < \frac{5}{6}$$

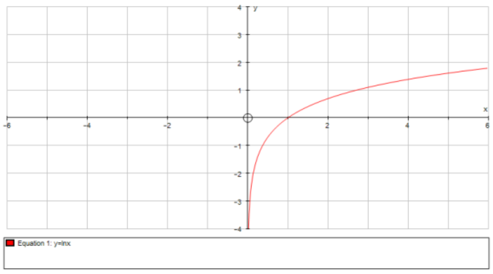
$$-5 < \frac{6}{(x-3)(2x+5)} < 0$$

$$\therefore\ -2.39 < x < 2.89$$

7. Sketch the graph of $y = \ln x$ for $x > 0$.

Express $xe^{x} = 7.39$ in the form $\ln x = ax + b$ and state the value of a and of b.
Insert on your sketch the additional graph required to illustrate how a graphical solution of the equation $xe^{x} = 7.39$ may be obtained. Hence, find $xe^{x} > 7.39$.

Solution:



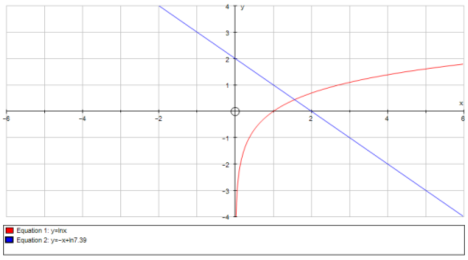
$$xe^{x} = 7.39$$

$$x + \ln x = \ln 7.39$$

$$\ln x = -x + \ln 7.39$$

$$a = -1\ ,\quad b = \ln 7.39$$

Sketch the graph of $y = -x + \ln 7.39$



The intersection point is (1.56 , 7.39)

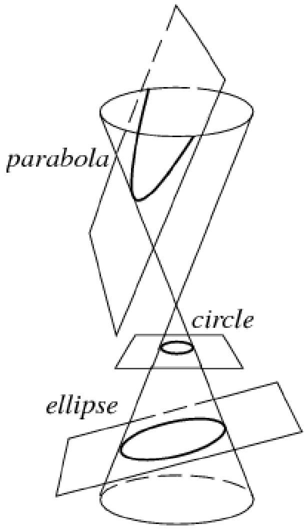
For $xe^x > 7.39$, $x > 1.56$

Answer

2 (i) x = 2 and x = -3, y = 1 (ii) (0, -10), (5, 0), (12, 0)	3. y = -2, 2
4. x = 4 and y =1, n = 6	5. (0, -3b) and (-3a, 0) y = -b and x = -a
6. $-2.39 < x < 2.89$	7. $a = -1$, $b = \ln 7.39$, $x > 1.56$

4.3 Conic Section

A conic section is the intersection of a plane and a cone as shown in the diagram below. This creates the shapes of parabola, circle, ellipse and hyperbola. These shapes are known as conics.



Applet on conics: https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG

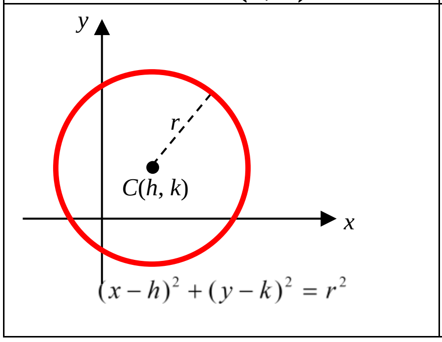
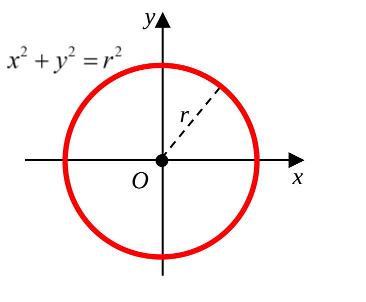
4.3.1 Circle

The standard form of a circle with centre (h, k) and radius r is

$$\boxed{(x-h)^2+(y-k)^2=r^2}$$

where $h,k,r\in\mathbb{R}$ and $r>0$.

Recall from O levels Mathematics, the equation of the circle is $(x-h)^2+(y-k)^2=r^2$, centered at (h, k) with radius r. In particular, the equation of the circle is $x^2+y^2=r^2$ centered at (0, 0) with radius r.

Centre (*h*, *k*)	**Centre (0, 0)**
	

Example 11

Sketch the graphs of $x^2+y^2-2x+4y=4$.

Solution

$$x^2+y^2-2x+4y=4$$
$$(x^2-2x)+(y^2+4y)=4$$

(In this example, we need to complete the square for x^2 and y^2 to express in the form $(x-h)^2+(y-k)^2=r^2$)

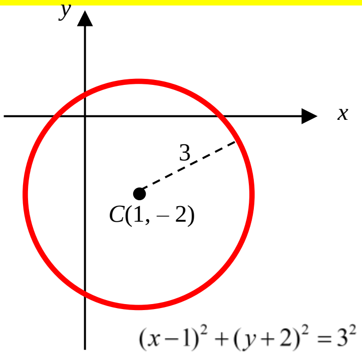
Completing the squares, we have

$$(x-1)^2-1+(y+2)^2-4=4$$

Converting to standard form:

$$(x-1)^2+(y+2)^2=3^2$$

This is a circle with centre at $(1,-2)$ and radius 3

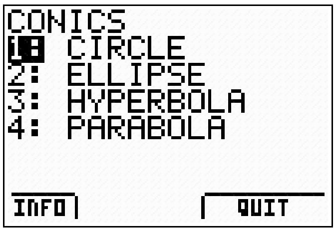


4.3.2 Using GC

There are two ways of using the GC to sketch the graphs of circles and ellipses.

Method 1
Use 'Conics' application.

- Press the [APPS] button, scroll down to find Conics and press [enter]



However this method can only be used to sketch the graphs, it cannot be used to find intersection points between a conic and another graph. The good old way of using [Y=] works.

Method 2
Use 'Equation Editor' to key the equations of the graph.

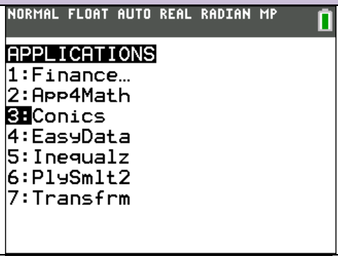
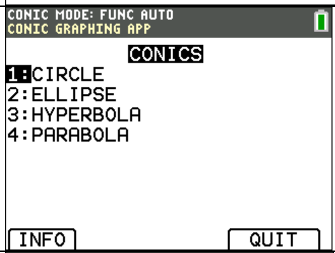
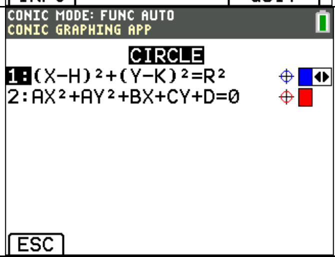
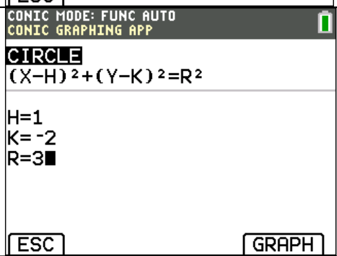
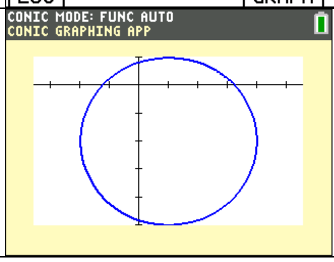
- Press the [Y=] button, key in the equations and press [GRAPH]

<u>Example 12</u>

Sketch the graph of $(x-1)^2+(y+2)^2=9$.

Solution:

Write the equation in standard form : $(x-1)^2+(y-(-2))^2=3^2$

Steps	**Screenshot**	**Remarks**
Press [APPS] Select the Conics mode press [ENTER]		
Press [1] to select CIRCLE		
Press [ENTER]		
Key in the appropriate values. In this case, H = 1, K = -2, R = 3		
Press [GRAPH]		This is the correct graph of the circle with Centre = $(1,-2)$ Radius = 3

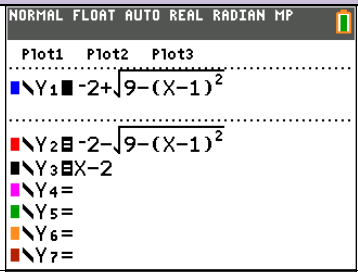
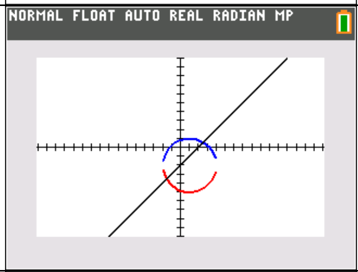
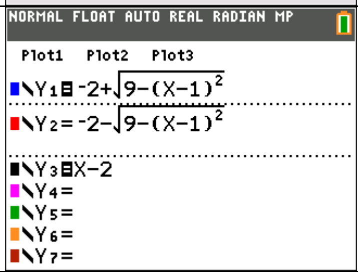
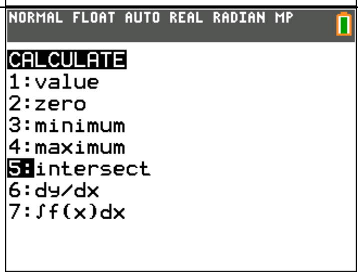
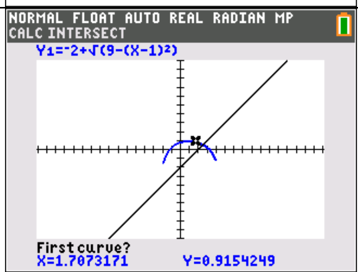
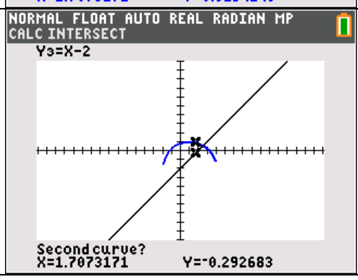
<u>Example 13</u>

Find the points of intersection of $(x-1)^2+(y+2)^2=9$ and the line $y=x-2$.

Solution:

We cannot use the Conics App here to sketch the circle, this is because in the Conics App, we cannot find the point of intersection.

First, we make y the subject of the equation: $y=-2\pm\sqrt{9-(x-1)^2}$.

<table>
<tr><th>Steps</th><th>Screenshot</th><th>Notes</th></tr>
<tr><td>Press [Y=] and enter the expression
$-2+\sqrt{9-(x-1)^2}$ into Y_1
$-2-\sqrt{9-(x-1)^2}$ into Y_2,
and $x-2$ into Y_3.</td><td></td><td></td></tr>
<tr><td>Press [ZOOM][5]</td><td></td><td>Notice that there is a gap between two parts of the circle, this is due to limitation of GC.</td></tr>
<tr><td>To find the point of intersection between the upper circle and the line, De-select Y_2 by simply move the cursor on Y_2 then press [ENTER].</td><td></td><td></td></tr>
<tr><td>Press [2nd][CALC]
select [5]</td><td></td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
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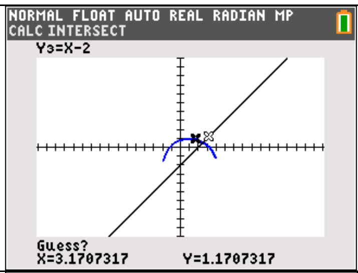
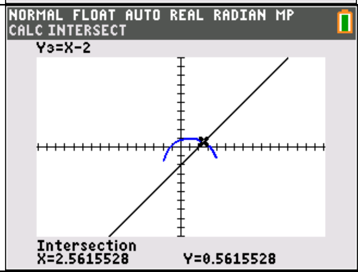
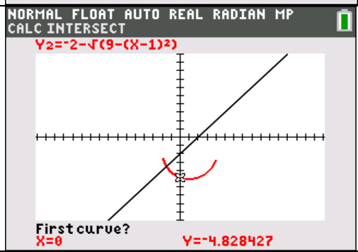
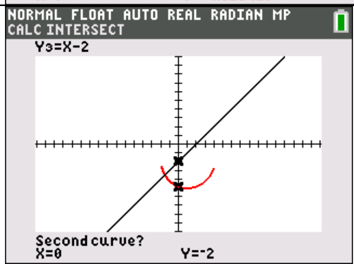
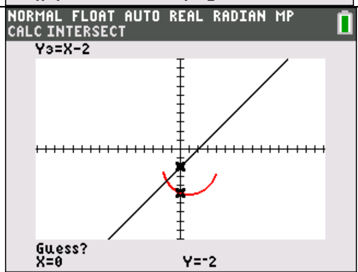
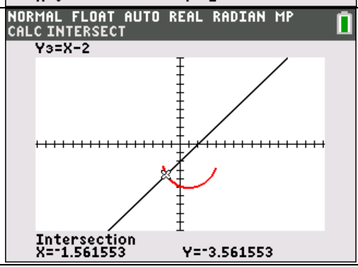
		
		Repeat the steps to find the intersection between Y_2 and Y_3
		
		
		
		The points of intersection are $(2.56, 0.562)$ and $(-1.56, -3.56)$.

The points of intersection are $(2.56, 0.561)$ and $(-1.56, -3.56)$.

4.3.3 Parabola

The standard form of the equation of a parabola with vertex (h, k) is as follows.

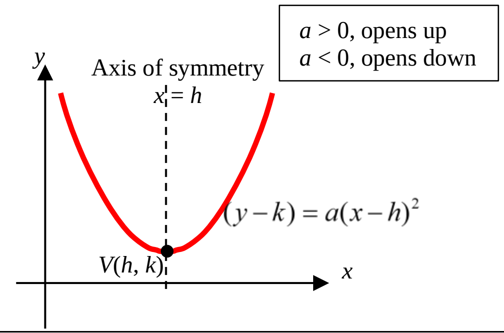
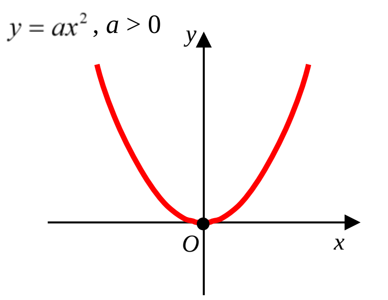
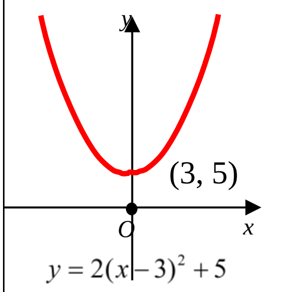
$$(y-k)=a(x-h)^2, \quad a\neq 0, \text{ axis of symmetry: } x=h$$

$$(y-k)^2=a(x-h), \quad a\neq 0, \text{ axis of symmetry: } y=k$$

When the vertex is the origin (0, 0), the equation of the parabolas takes the form

$$y=ax^2 \qquad \text{axis of symmetry: } y-\text{axis}$$

$$y^2=ax \qquad \text{axis of symmetry: } x-\text{axis}$$

<table>
<tr><th></th><th>Vertex (h, k)</th><th>Vertex (0, 0)</th><th>Example</th></tr>
<tr><td>Vertical Axis</td><td>
</td><td>
</td><td>
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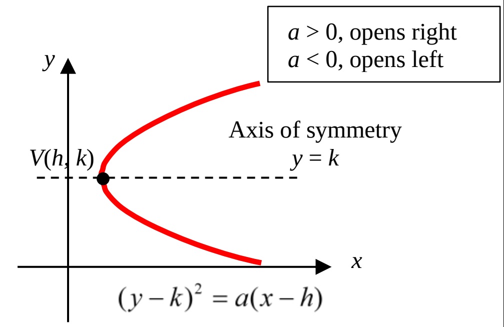
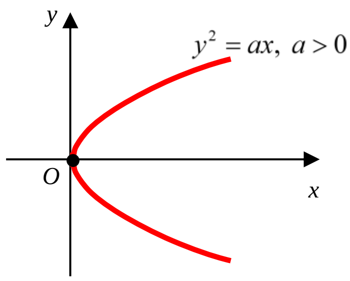
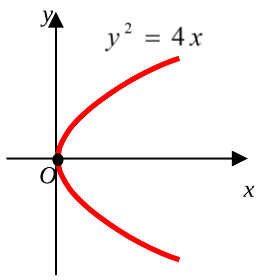
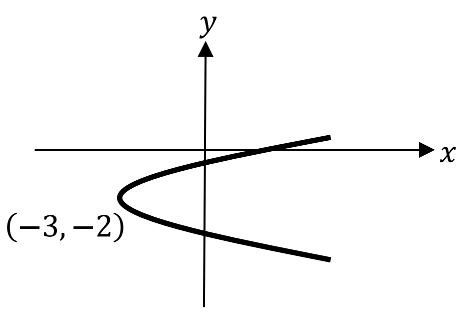
<tr><td>Horizontal Axis</td><td>
</td><td>
</td><td>
</td></tr>
</table>

Example 14

Sketch $y^2-2x+4y=2$.

Solution:

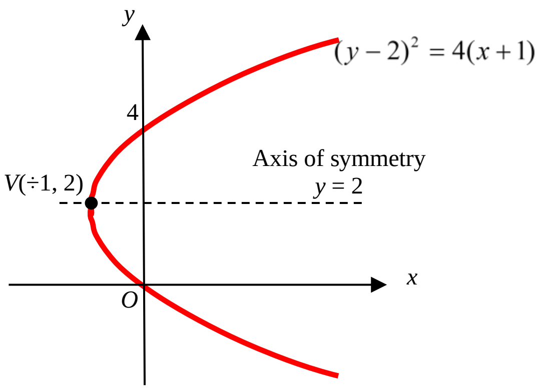
$$y^2-2x+4y=2$$
$$(y+2)^2-4-2x=2$$
$$(y+2)^2-6=2x$$
$$x=\frac{1}{2}(y+2)^2-3$$



Example 15

Sketch the graph of $(y-2)^2 = 4(x+1)$ and state the axis of symmetry

Solution:



4.3.4 General Forms of Conic

The general form of the equation of a circle, ellipse and parabola can be written as $Ax^2 + By^2 + Cx + Dy + E = 0$ as follows:

Conic	General form: $Ax^2 + By^2 + Cx + Dy + E = 0$
Circle	$A = B$
Parabola	$A = 0$ or $B = 0$

Note:
If we complete the square and re-arrange the equations in the general form, we will obtain the standard forms.

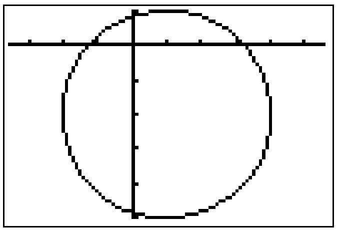
Example 16
Sketch the following graphs on separate diagrams, making clear the main relevant features of the graphs.
(a) $x^2 + y^2 - 2x + 4y = 4$
(b) $y^2 - 2x + 4y = 4$

Solution:
(a) $x^2 + y^2 - 2x + 4y = 4$

$(x-1)^2 - 1 + (y+2)^2 - 4 = 4$

$(x-1)^2 + (y+2)^2 = 3^2$

This is a circle with centre at $(1, -2)$ and radius 3.

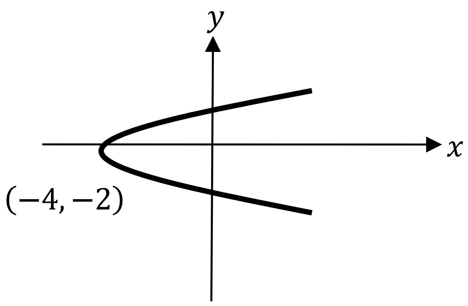


(b) $y^2 - 2x + 4y = 4$

$(y+2)^2 - 4 - 2x = 4$

$(y+2)^2 - 8 = 2x$

$x = \frac{1}{2}(y+2)^2 - 4$



Exercise 3

1 Identify and sketch the curves represented by the following equations.

(a) $x^2 - 7x + y^2 + 6 = 0$ (b) $x^2 - 7x + 5y - 9 = 0$

Solution:

(a) $x^2 - 7x + y^2 + 6 = 0$

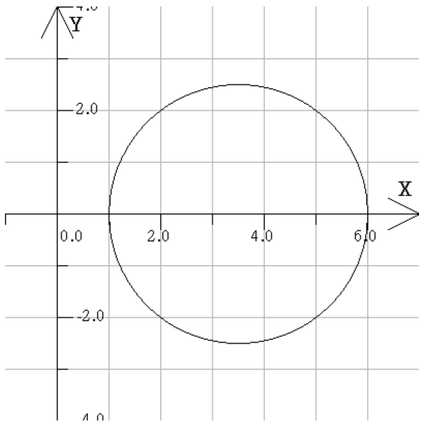
$$x^2 - 2\left(\frac{7}{2}\right)x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 6 + y^2 = 0$$

$$\left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + 6 + y^2 = 0$$

$$\left(x - \frac{7}{2}\right)^2 - \frac{25}{4} + y^2 = 0$$

$$\left(x - \frac{7}{2}\right)^2 + y^2 = \frac{25}{4}$$

$$\left(x - \frac{7}{2}\right)^2 + y^2 = \left(\frac{5}{2}\right)^2$$

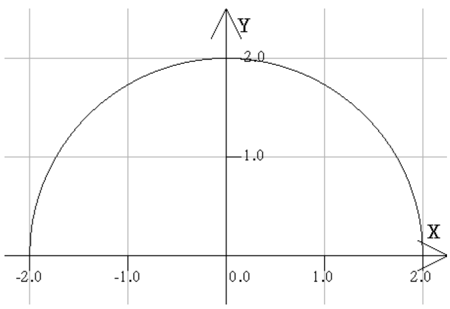


This is the equation of a circle with radius $\frac{5}{2}$, centre at $\left(\frac{7}{2}, 0\right)$.

(b) $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \Rightarrow x^2 + y^2 = 2^2$

This is the equation of a semi-circle with radius 2, centre at $(0,0)$. It is a semi-circle and not a circle because $y = \sqrt{4 - x^2}$ and not $y = \pm\sqrt{4 - x^2}$

It cuts the x-axis at $x = -2$ *and* $x = 2$



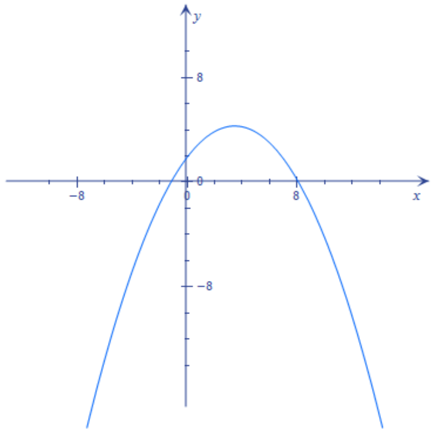
(b) $x^2 - 7x + 5y - 9 = 0$

$$\left(x-\frac{7}{2}\right)^2-\left(\frac{7}{2}\right)^2+5y-9=0$$

$$5y=-\left(x-\frac{7}{2}\right)^2+\frac{85}{4}$$

$$y=-\frac{1}{5}\left(x-\frac{7}{2}\right)^2+\frac{17}{4}$$

There is a maximum point at $(\frac{7}{2},\frac{17}{4})$



Exercise 4 (Revision)

1. Sketch:

(a) $y=e^x-x^2$

(b) $y=1+6x-3x^2-4x^3$

(c) $y=0.5^x-\ln(1+x)$

(d) $y=1-e^{1-2x}$

(e) $y=2^x-x^2$

(f) $y=\ln(2x-3)$

(g) $y=6-4x^3-3x^4$

(h) $y=2x^3-5x^2-4x+3$

2. Solve:

(a) $2e^{2x}\geq 9-3e^x$ using $u=e^x$

(b) $3e^{2x}\leq 4\left(e^{-2x}-1\right)$ using $u=e^{2x}$

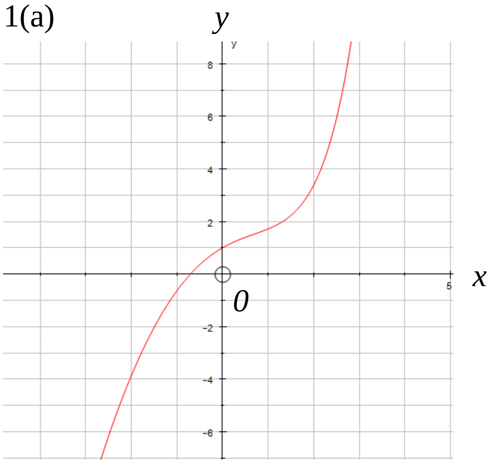
3. **<u>2011 A Level Paper Q2 (modified)</u>**

(i) On a single diagram, sketch the graphs of $y=2-(0.6)^x$ and $y=x^2-1$, stating clearly the coordinates of any points of intersection with the y-axis.

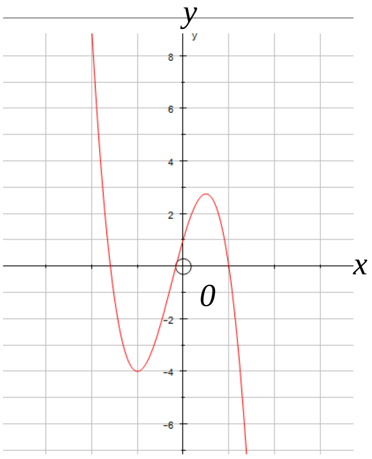
(ii) Find the x-coordinates of the points of intersection of $y=2-(0.6)^x$ and $y=x^2-1$, giving your answers correct to 4 decimal places.

Solution:

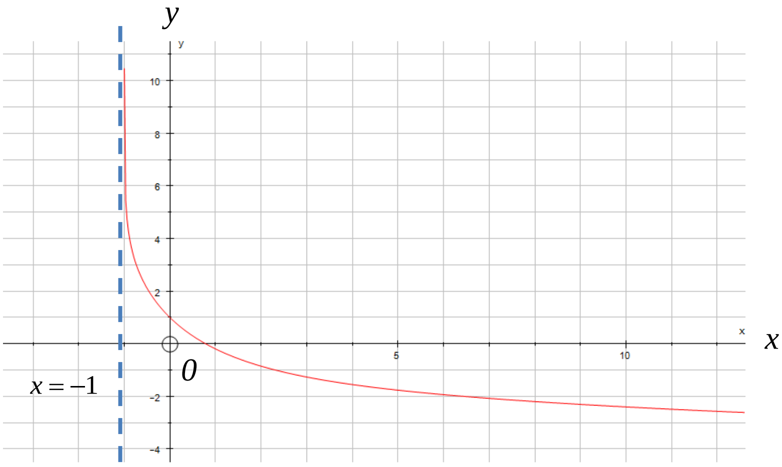
1(a)



(b)

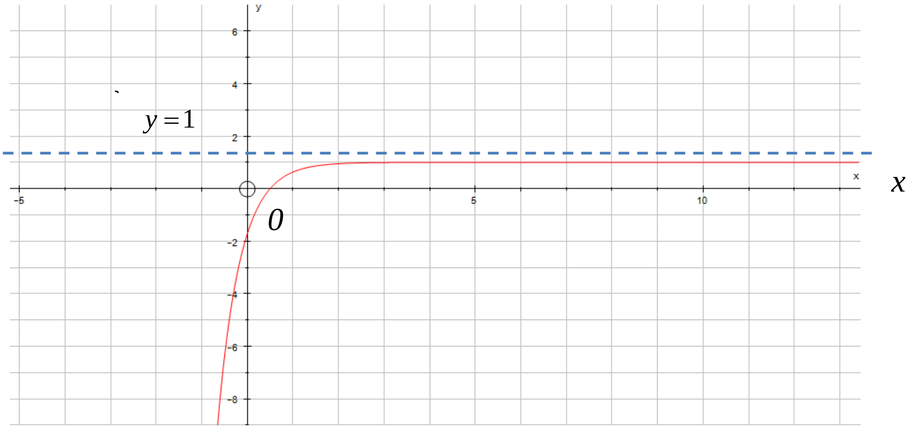


(c)



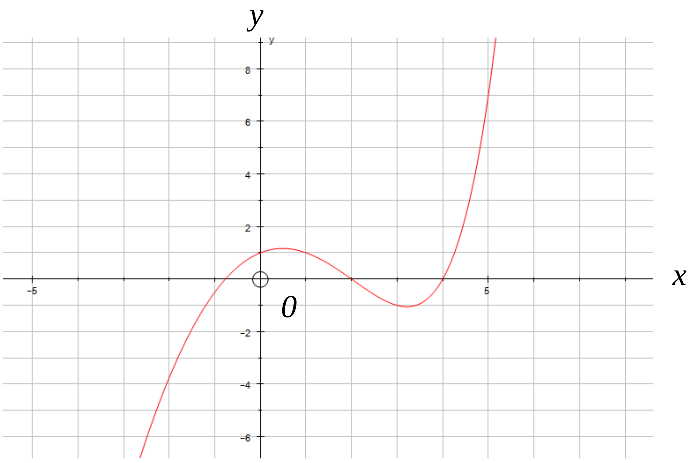
Vertical asymptote: x = -1.

(d)

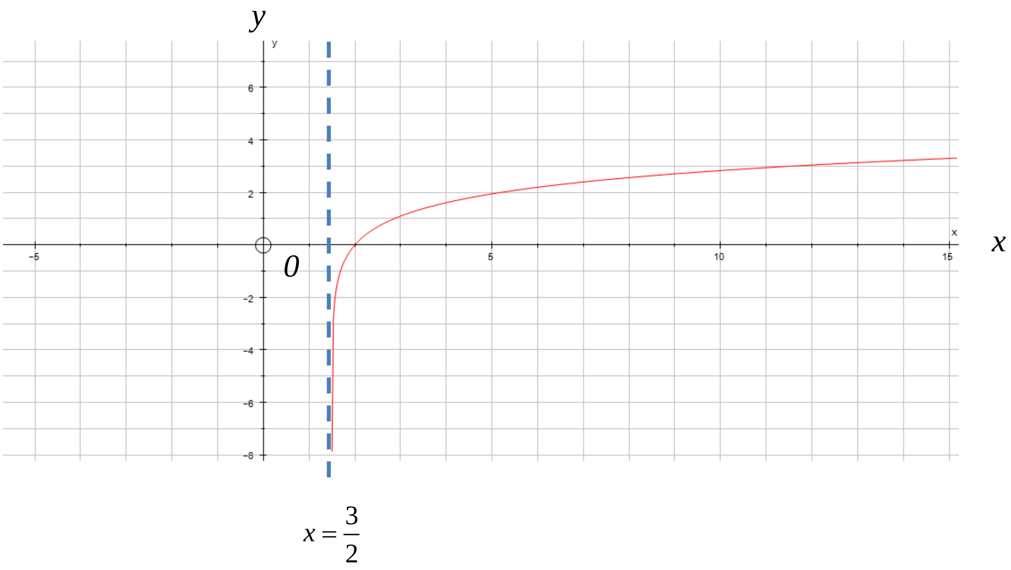


Horizontal asymptote: $y = 1$.

(e)

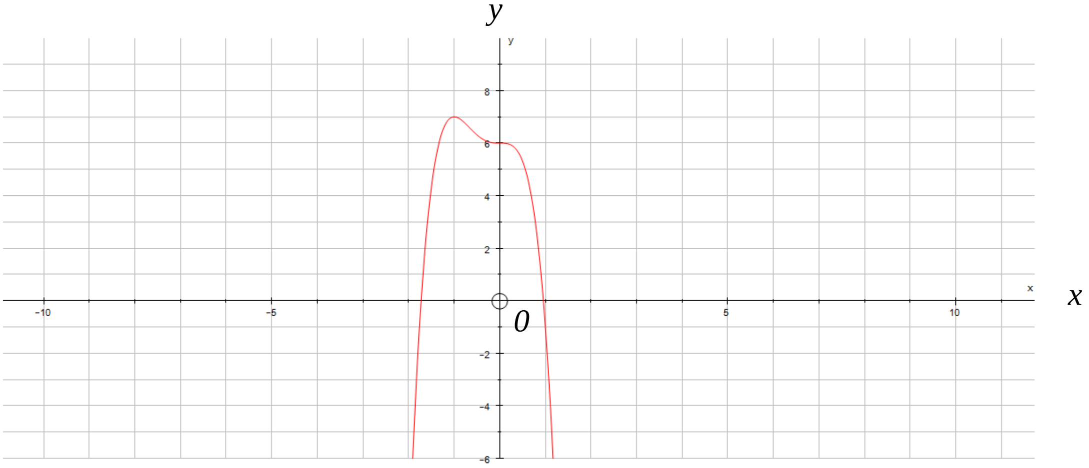


(f)

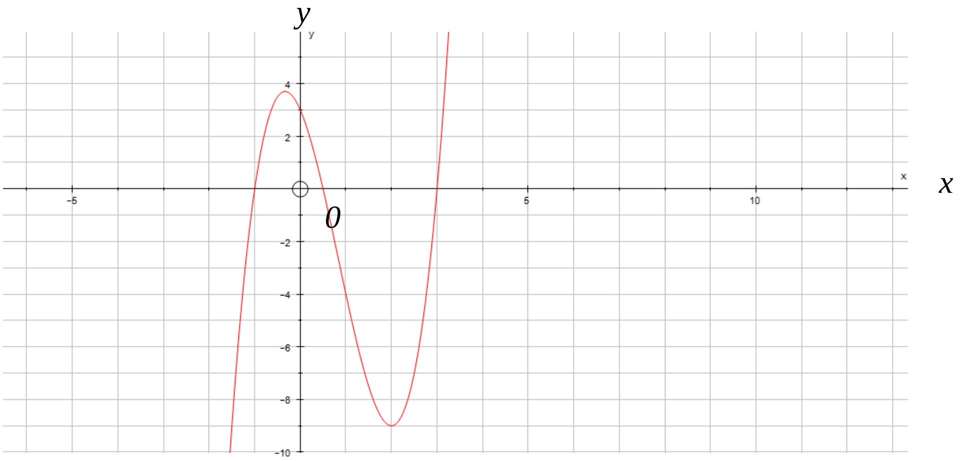


Vertical asymptote: $x = \dfrac{3}{2}$.

(g)



(h)



2(a)

$2e^{2x} \geq 9 - 3e^{x}$

$2u^{2} \geq 9 - 3u$

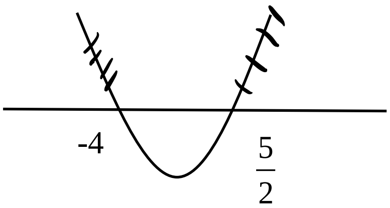
$2u^{2} + 3u - 9 \geq 0$

$(2u-5)(u+4) \geq 0$

$u \leq -4$ or $u \geq \frac{5}{2}$

(reject as $e^{x} > 0$) $\quad e^{x} \geq \frac{5}{2}$

$x \geq \ln \frac{5}{2}$



(b) $3e^{2x} \leq 4\left(e^{-2x} - 1\right)$

$3u \le 4\left(u^{-1} - 1\right)$

$3u \le 4\left(\frac{1}{u} - 1\right)$

$3u^2 \le 4(1-u)$

$3u^2 + 4u - 4 \le 0$

$(3u-2)(u+2) \le 0$

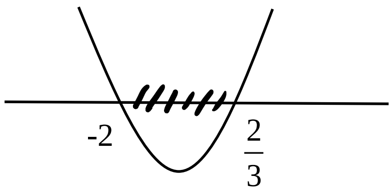
$-2 \le u \le \frac{2}{3}$

since $u = \mathrm{e}^{2x}$ is positive,

$e^{2x} \le \frac{2}{3}$

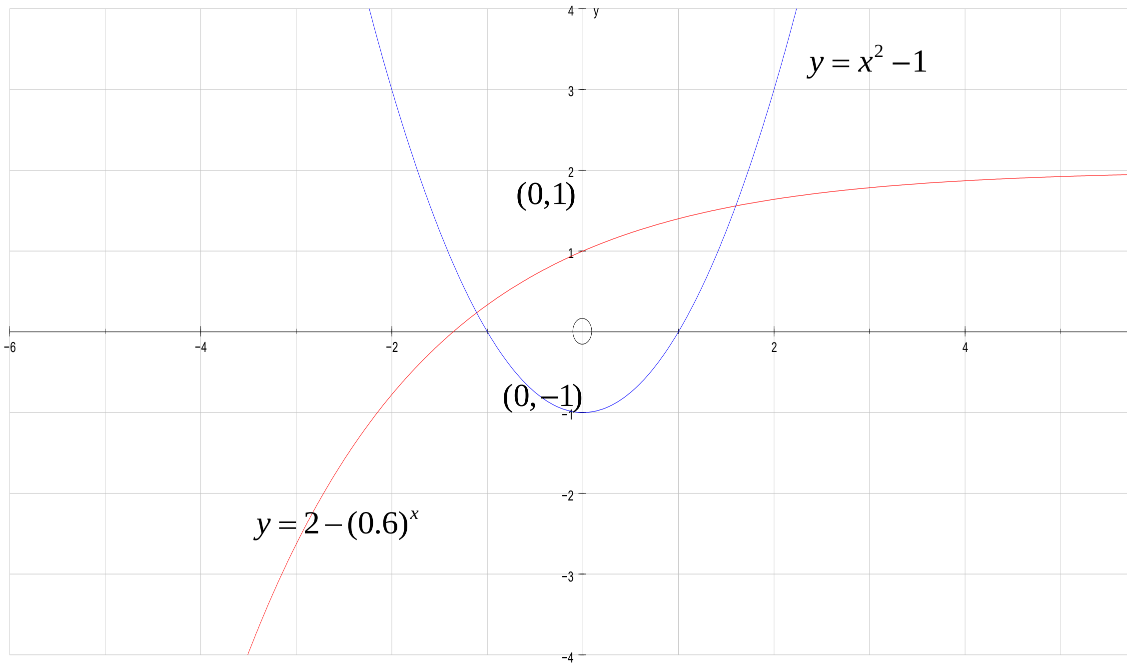
$2x \le \ln\left(\frac{2}{3}\right)$

$x \le \frac{1}{2}\ln\left(\frac{2}{3}\right)$



Q3.

i)



ii) Using GC, $x = -1.1116$
$x = 1.5995$

Practice Questions

1. MI Promo 8865/2018/Q8

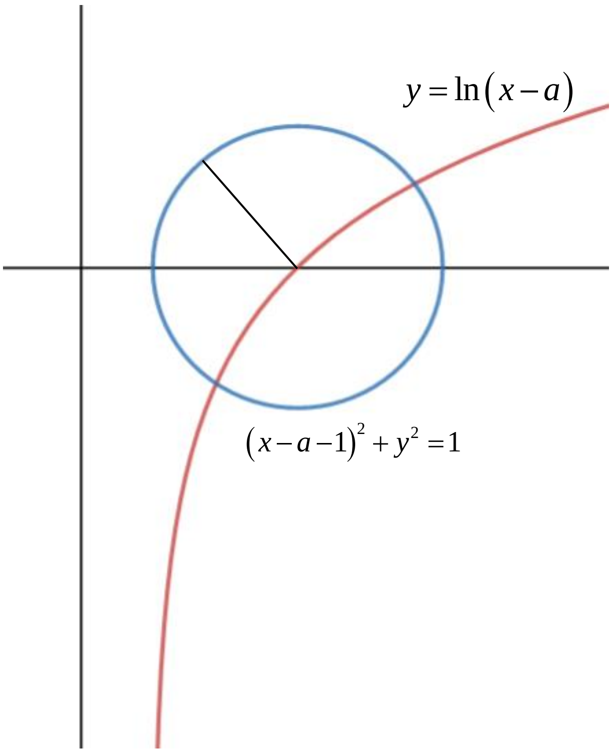
(a) On the same diagram, sketch the graphs of $y=\ln(x-a)$, where $0<a<1$ and $(x-a-1)^2+y^2=1$, stating the coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

Given that $a=\frac{1}{2}$, find the x-coordinates of the points of intersection between the two curves, giving your answer correct to 2 decimal places. [1]

(b) Given that $\ln x=m$ and $\ln y=n$, where m and n are positive numbers, express $\ln xy^3$ and $\ln x^{\ln\frac{y}{x}}$ in terms of m and n. [5]

Solution

1(a)



1(a) Intersection of $y=\ln\left(x-\frac{1}{2}\right)$ and $\left(x-\frac{3}{2}\right)^2+y^2=1$

From GC:
$x=2.31$ (2 d.p) or $x=0.94$ (2 d.p)

1(b)
$$\ln xy^3=\ln x+3\ln y$$
$$=m+3n$$

$$\ln x^{\ln\frac{y}{x}} = \left(\ln\frac{y}{x}\right)\ln x$$
$$= (\ln y - \ln x)\ln x$$
$$= (n-m)m$$

2. RI Promo 8865/2018/Q6

The curve C has equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = 6x - 3x^2 - 4x^3$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the coordinates of the stationary points on the curve. [4]

(ii) Use a non-calculator method to determine the nature of each of the stationary points. [2]

(iii) Sketch the graph of C, stating the coordinates of any points where the curve crosses the x-axis. [2]

(iv) Find the integer solution of the equation $\mathrm{f}(x) = 8$, and prove algebraically that there are no other real solutions. [3]

Solution:

2(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = 6 - 6x - 12x^2$

For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\Rightarrow 6 - 6x - 12x^2 = 0 \Rightarrow x = -1$ or $\frac{1}{2}$

When $x = -1$, $y = -5$

When $x = \frac{1}{2}$, $y = \frac{7}{4}$

$\therefore$ Coordinates of the stationary points are $(-1, -5)$ and $\left(\frac{1}{2}, \frac{7}{4}\right)$.

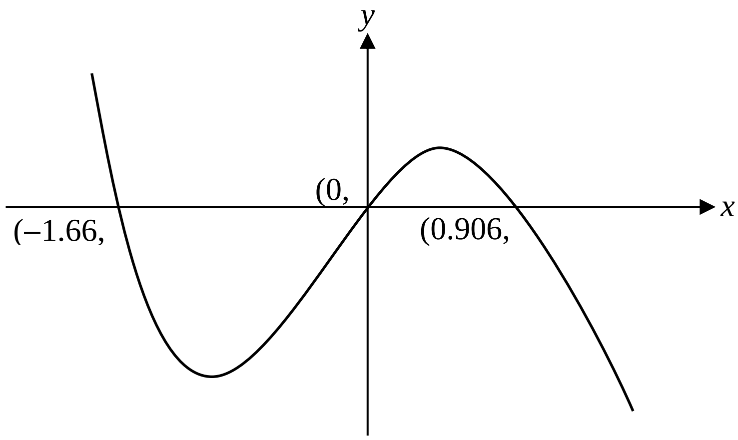
(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6 - 24x$

When $x = -1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 18 > 0$ (minimum point)

When $x = \frac{1}{2}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -18 < 0$ (maximum point)

$\therefore$ $(-1, -5)$ is a minimum point and $\left(\frac{1}{2}, \frac{7}{4}\right)$ is a maximum point.

2(iii)



2(iv) Note that f $(-2) = 6(-2) - 3(-2)^2 - 4(-2)^3 = 8$

$\therefore$ $x = -2$ is an integer solution of f $(x) = 8$.

$6x - 3x^2 - 4x^3 - 8 = -(x + 2)(4x^2 - 5x + 4)$

Consider $4x^2 - 5x + 4 = 0$

Discriminant $D = (-5)^2 - 4(4)(4) = -39 < 0$

Hence there are no other real roots.

Alternatively:

$4x^2 - 5x + 4 = 4\left(x - \frac{5}{8}\right)^2 + \frac{39}{16} > 0$ for all real values of x

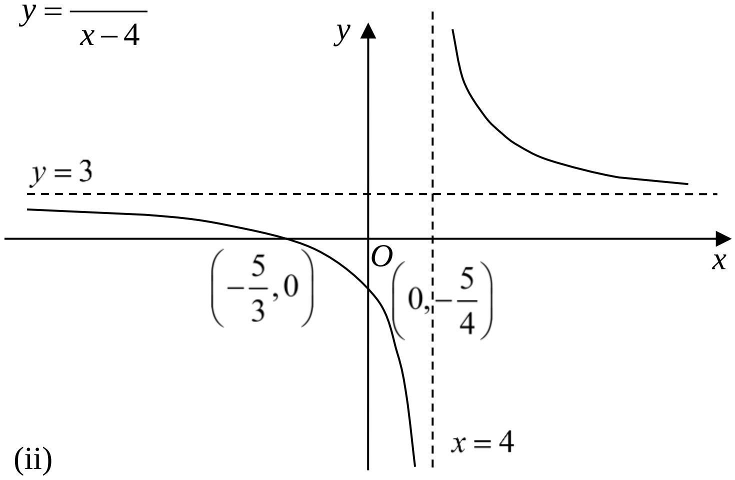
3. RVHS Promo 8865/2018/Q4

(i) Sketch the graph of $y = \frac{3x+5}{x-4}$, labelling clearly the equations of any asymptotes and the exact coordinates of any points where the curve crosses the axes. [3]

(ii) Hence, find the exact solution(s) of the inequality $\frac{3x^2+5}{x^2-4} > 0$. [4]

(iii) By adding a suitable curve to the sketch in part **(i)**, solve $e^{\frac{3x+5}{x-4}} \le x+3$. [4]

3 (i)



(ii)

Solving $\dfrac{3x+5}{x-4} > 0$ from the graph in (i), we have

$x < -\dfrac{5}{3}$ or $x > 4$

Replacing x with x^2, we have

$x^2 < -\dfrac{5}{3}$ (rej) or $x^2 > 4$

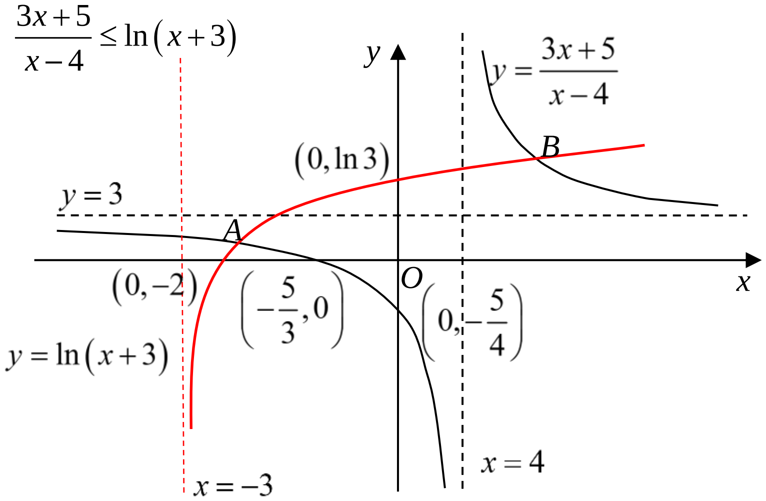
$x < -2$ or $x > 2$

(iii)

$$e^{\frac{3x+5}{x-4}} \le x+3$$

$$\ln e^{\frac{3x+5}{x-4}} \le \ln(x+3)$$

$$\frac{3x+5}{x-4} \le \ln(x+3)$$



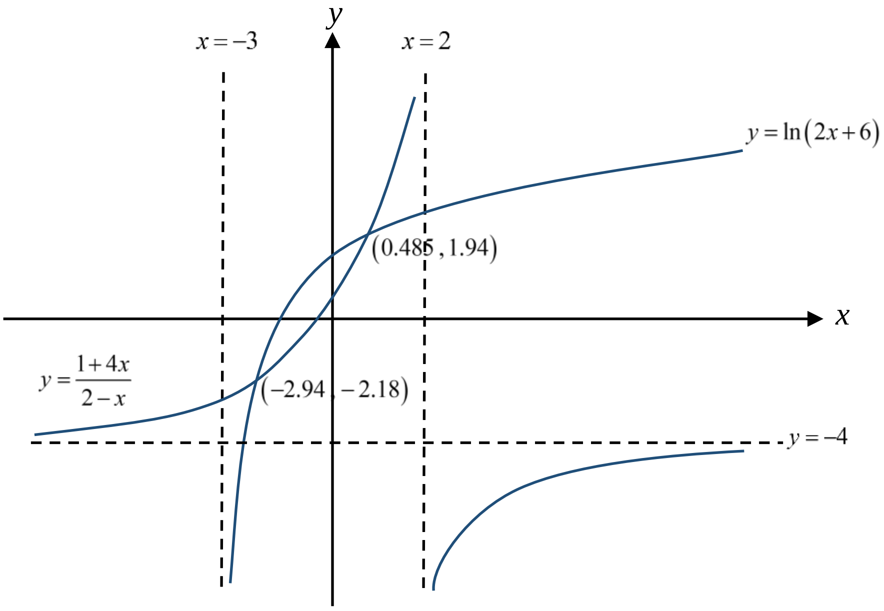
Intersections : A(-1.88, 0.110) and B(33.1, 3.59)

$-1.88 \le x < 4 \quad \text{or} \quad x \ge 33.1$

4. TJC Promo 8865/2018/Q5

Sketch the curves $y = \dfrac{1+4x}{2-x}$ and $y = \ln(2x+6)$ on the same diagram, stating clearly the equation of any asymptotes and coordinates of any intersection points.

Hence solve the inequality $\dfrac{1+4x}{2-x} \ge \ln(2x+6)$. [6]



$\dfrac{1+4x}{2-x} \ge \ln(2x+6)$

$-3 < x \le -2.94 \quad \text{or} \quad 0.485 \le x < 2$

5. VJC Promo 8865/2018/Q8

Tom and Jerry cycle with speed v km/h and u km/h respectively, such that t hours after passing a fixed point O, v and u are given by

$$v = e^{-2t} + \frac{100}{t+5} \quad \text{and} \quad u = -t^2 + 25.$$

Both of them decide to stop when any one of them first stops.

(i) Show that Tom's speed is always decreasing. [3]

(ii) On the same diagram, sketch both speed-time graphs, indicating clearly the coordinates of the point of intersection between the two graphs. [3]

(iii) Find the area bounded by the graph of

(a) $v = e^{-2t} + \dfrac{100}{t+5}$, the axes and the line $t = 5$, [2]

(b) $u = -t^2 + 25$ and the axes, giving the exact value. [3]

(iv) Determine the value of the difference of your answers for parts **(a)** and **(b)**, and explain in context what this value means. [2]

Solution:

(i) $v = \mathrm{e}^{-2t} + \dfrac{100}{(t+5)}$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2\mathrm{e}^{-2t} - \frac{100}{(t+5)^2}$$

$\therefore \quad p = -2, \qquad q = -100$

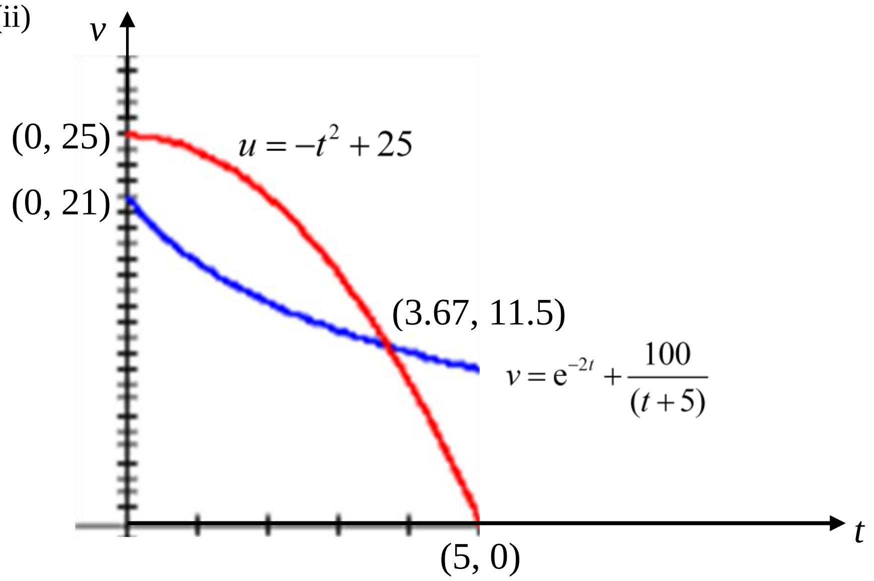
$$\frac{\mathrm{d}v}{\mathrm{d}t} = a = -2\mathrm{e}^{-2t} - \frac{100}{(t+5)^2}$$

$$= -2\left[\mathrm{e}^{-2t} + \frac{50}{(t+5)^2}\right] < 0$$

($\because$ e^{-2t} and $(t+5)^2 > 0$ for all $x \in \mathbb{R}$)

Therefore, velocity is always decreasing.

(ii)



(iii) (a) Required area = $\displaystyle\int_0^5 \mathrm{e}^{-2t} + \frac{100}{(t+5)}\,\mathrm{d}t = 69.8$

(a)

$$\text{Required area} = \int_0^5 -t^2 + 25\,\mathrm{d}t$$

$$= \left[-\frac{t^3}{3} + 25t\right]_0^5$$

$$= \left(-\frac{125}{3} + 125\right) - 0$$

$$= \frac{250}{3}$$

(iv) Difference = $\frac{250}{3} - 69.8 = 13.5$
Jerry travelled 13.5 m more than Tom when they both stopped cycling 5 hours later.

6. AJC Promo 8865/2018/Q4

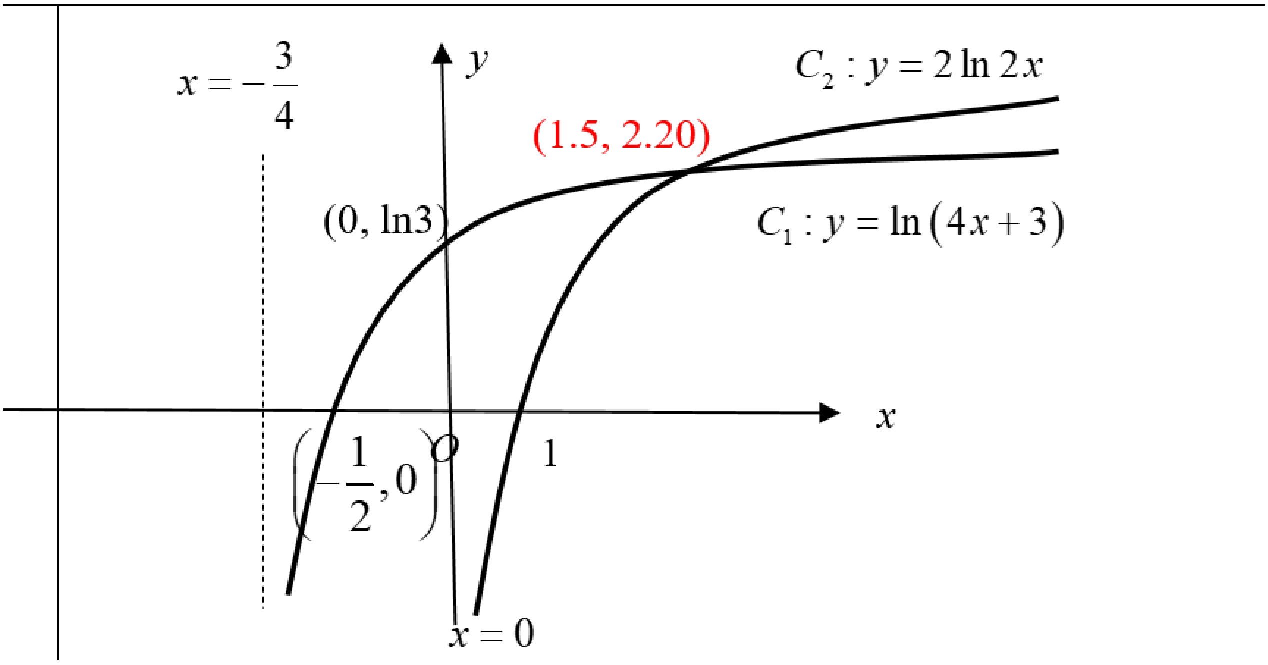
The curves C_1 and C_2 have equations $y = \ln(4x+3)$ and $y = 2\ln 2x$ respectively.

(i) Sketch the graph of C_1, indicating clearly the coordinates of the points where the curve crosses the axes, and the equation of any asymptote(s). [3]

(ii) Insert the graph of C_2 in your diagram, indicating clearly the coordinates of the point where the curve intersects C_1, and the equation of any asymptote(s). [2]

(iii) Deduce the range of values of x that satisfy the inequality $\ln(4x+3) \le 2\ln 2x$. [1]

Solution:



To find intercepts
At y-axis: x=0, y=ln3

At x-axis: y=0 $\ln(4x+3) = 0 \Rightarrow 4x+3 = 1 \Rightarrow x = -\frac{1}{2}$

C_1 crosses axes at $\left(-\frac{1}{2}, 0\right)$ and (0,ln3)

(0, ln3) and $\left(-\frac{1}{2}, 0\right)$ to be indicated on graph

(ii)For $\ln(4x+3) \le 2\ln x$ (C_1 above C_2)

$x \ge \frac{3}{2}$

7. AJC Promo 8865/2018/Q6

The curve C has the equation $y = 5e^{-2x} - 1$.

(i) Find the exact coordinates of the points where C crosses the axes, showing your working clearly. [2]

(ii) Sketch the graph of C, indicating clearly the equation(s) of any asymptote(s). [2]

(iii) Without using a calculator, find the equation of the tangent to C at the point where the curve crosses the y-axis, giving your answers in form $y = mx + c$, where m and c are exact constants to be found.
Hence, find the exact x co-ordinates of the point where this tangent cuts the x-axis. [4]

Solution:

7(i) When $y = 0$,

$$0 = 5e^{-2x} - 1 \qquad \text{or} \qquad 0 = 5e^{-2x} - 1$$

$$e^{-2x} = \frac{1}{5} \qquad\qquad 5e^{-2x} = 1$$

$$\qquad\qquad \ln\left(5e^{-2x}\right) = \ln 1$$

$$-2x = \ln\left(\frac{1}{5}\right) \qquad \ln 5 + (-2x)\ln e = 0$$

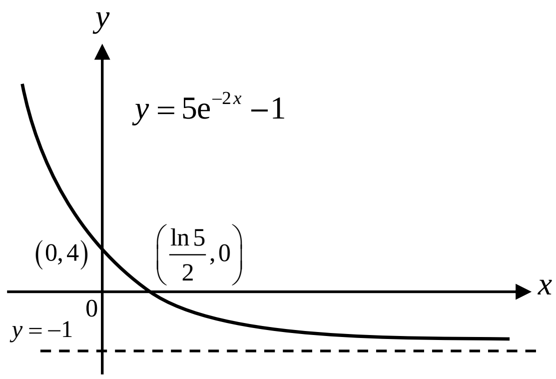
$$x = -\frac{1}{2}\ln\frac{1}{5} = \frac{1}{2}\ln 5 \qquad x = \frac{1}{2}\ln 5$$

When $x = 0$,

$y = 5e^{0} - 1 = 5 - 1 = 4$

x-intercept: $\left(\frac{\ln 5}{2}, 0\right)$ and y-intercept: $(0, 4)$

(ii)



(iii) $\frac{dy}{dx} = -10e^{-2x}$

When $x = 0$, gradient of tangent $= -10$

$\therefore$ Equation of tangent at $x = 0$ is $y - 4 = -10(x - 0)$

$$y = -10x + 4$$

Tangent cuts x-axis : $x = \frac{2}{5}$. Point is $\left(\frac{2}{5}, 0\right)$

Q8. **DHS Promo 8865/2018/Q5**

The curve C_1 is a semicircle with centre (2,0) and radius 3 units. It has equation $(x-2)^2+y^2=9$, for $y\geq 0$. The curve C_2 has equation $y=2e^{2-x}+1$.

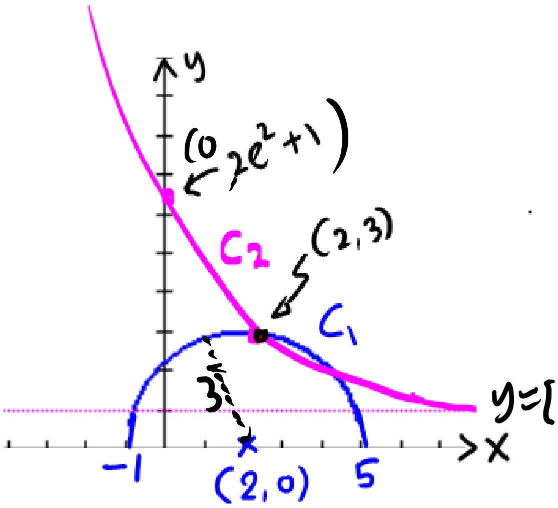
(i) Sketch C_1. [2]

(ii) On the same diagram, sketch C_2, giving the equation of the asymptote and the coordinates of the point of intersection with the y-axis. [2]

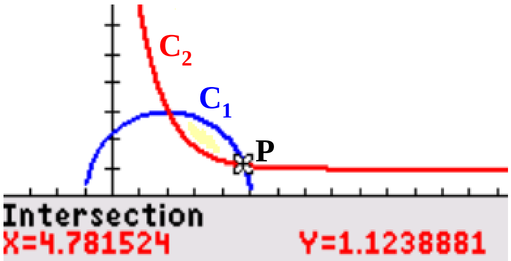
(iii) Hence find the values of x for which $\sqrt{(9-(x-2)^2)}\geq 2e^{2-x}+1$. [2]

Solution:

8(i),(ii)



Section view



8(iii) $\sqrt{\left(9-\left(x-2\right)^2\right)}\geq 2e^{2-x}+1$ i.e. interval of x where C_1 is above C_2

Using GC at P, $2\leq x\leq 4.78$ (3s.f.)

Answers:

Q1	$x = 2.31$ (2 d.p) or $x = 0.94$ (2 d.p) $m+3n$ $(n-m)m$
Q2	(i) $6 - 6x - 12x^2$, $(-1, -5)$, $\left(\frac{1}{2}, \frac{7}{4}\right)$ (ii) $(-1, -5)$ min, $\left(\frac{1}{2}, \frac{7}{4}\right)$ max (iv) $x = -2$
Q3	$x < -2$ or $x > 2$; $-1.88 \le x < 4$ or $x \ge 33.1$
Q4	$-3 < x \le -2.94$ or $0.485 \le x < 2$
Q5	(iiia) 69.8 (iiib) $\frac{250}{3}$ (iv) 13.5
Q6	(iii) $x \ge \frac{3}{2}$
Q7	(i)x-intercept: $\left(\frac{\ln 5}{2}, 0\right)$ and y-intercept: $(0,4)$ (iii) $y = -10x + 4$; $x = \frac{2}{5}$
Q8	(iii) $2 \le x \le 4.78$ (3 s.f.)

Summary

1. Circle
   - $x^2 + y^2 = r^2$, circle with centre (0, 0) and radius = r
   - $(x-a)^2 + (y-b)^2 = r^2$, circle with centre (a, b) and radius = r

2. Rational functions

   To sketch the graph of a rational function $y = \dfrac{P(x)}{Q(x)}$, do the following (if necessary):

   - Find axes intercepts: when x = 0 and when y = 0.

   - Find all asymptotes:
     - Vertical asymptotes: Let denominator be 0, find x.
     - Horizontal asymptotes:
       - Case1: $\dfrac{P(x)}{Q(x)}$ is a proper fraction

         Hence horizontal asymptote is y = 0.
       - Case2: $\dfrac{P(x)}{Q(x)}$ is an improper fraction e.g. $y = \dfrac{ax+b}{cx+d}$
         - Use long division or other methods to obtain the form $y = \alpha + \dfrac{\beta}{cx+d}$, where α and β are constants to be determined.

           Horizontal asymptote is $y = \alpha$.

3. Find Stationary Points ($\dfrac{dy}{dx} = 0$) and determine their nature.

4. For other graphs, you may also consider the following when necessary:
   - Behaviour of y when $x \to \pm\infty$
   - Restrictions on values of x and/or y
   - Symmetry

     If y is an even function, i.e. f(–x) = f(x), the graph is symmetrical about the y-axis.

     If y is an odd function, i.e. f(–x) = –f(x), the graph is symmetrical about the origin.

Graphing Techniques – A Checklist of Concepts

- ☐ I understand the characteristics of graphs such as Stationary points (**S**), intersections with the axes (**I**) and asymptotes (**A**).
- ☐ I am able to determine equations of asymptote (horizontal and vertical), axes of symmetry and restrictions of possible values of *x*/or *y* of a graph;
- ☐ I know how to use a graphic calculator to draw a given function, locating the turning points, axial intercepts, and adjusting the 'window setting' that would display the essential features of the functions
- ☐ I am able to sketch the graphs of rational functions such as

$$y = \frac{a}{bx+c} \quad \text{and} \quad y = \frac{dx+e}{fx+g}$$

- ☐ I can recognize different types of conics such as circle, parabola and sketch their graphs with or without the use of a graphing calculator
- ☐ I am able to convert the conics equation to a form such that I can identify its characteristics such as the center and radius for a circle;

Relevant websites

1) Applet on conics: https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG
2) Applet on rational function:

   https://www.geogebra.org/m/Fsnt4mRk

   https://www.geogebra.org/m/naKtmybb

   https://www.geogebra.org/m/dFP5VqR7
3) Applet on graphs of parametric equations: https://www.geogebra.org/m/z3sh6xSE
4) Applet on graphs of various functions: http://www.analyzemath.com/

Annex 1

Polynomial

A polynomial in x is an expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0, \quad a_n \neq 0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_{n-1}, a_n$ are real constants called the real coefficients of the polynomial.

The integer n is called the degree of the polynomial.

Expressions like $2x^2+3x+5$ and $-3x^3+\frac{1}{2}x^2-4x+2$ are polynomials in x while $x^2+\frac{1}{x}$ and $x^2+2\sin x+1$ are not.

Long Division of Polynomials

Firstly, let us take a look at an example of division in arithmetic. To divide 32 by 5, the steps taken can be shown as follows:

$$\begin{array}{r}
6 \\
5\overline{)32} \\
\underline{30} \\
2
\end{array}$$

We write $\frac{32}{5} = 6 + \frac{2}{5}$.

An example on division of polynomials:

$$\begin{array}{r}
x+2 \\
x+1\overline{)x^2+3x+3} \\
\underline{x^2+x} \\
2x+3 \\
\underline{2x+2} \\
1
\end{array}$$

We have the following relationship:

$$\frac{x^2+3x+3}{\boxed{x+1}} = \boxed{x+2} + \frac{\boxed{1}}{x+1}$$

Remainder

Divisor

Quotient

Example 13

Find the quotient and remainder when x^4+x^3-2x-2 is divided by $x-3$.

$$
\begin{array}{r@{}l}
 & x^3+4x^2+12x+34 \\
x-3 \big) & x^4+x^3-2x-2 \\
 & \underline{x^4-3x^3} \\
 & \quad 4x^3 \\
 & \quad \underline{4x^3-12x^2} \\
 & \qquad 12x^2-2x \\
 & \qquad \underline{12x^2-36x} \\
 & \qquad\qquad 34x-2 \\
 & \qquad\qquad \underline{34x-102} \\
 & \qquad\qquad\quad 100
\end{array}
$$

The quotient is $x^3+4x^2+12x+34$ and the remainder is 100.
We may then write:

$$\frac{x^4+x^3-2x-2}{x-3}=x^3+4x^2+12x+34+\frac{100}{x-3}.$$