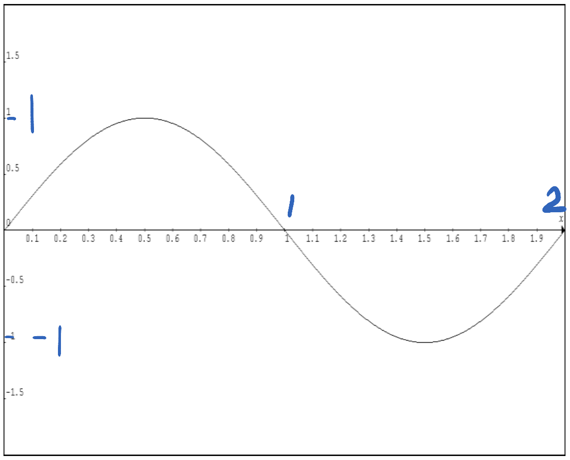
HCI H2 Math Prelim Paper 2 Solution

<table>
<tr><th>Qn.</th><th>Solutions</th></tr>
<tr><td>1
(i)</td><td>


Since y = 0.5 cuts the graph twice, f is not a one-one function. Therefore f^{-1} does not exist.

Restrict the domain to $\frac{1}{2} \le x \le \frac{3}{2}$ so that the range of h is equal to the range of f.

$\therefore \quad m = \frac{1}{2} \quad , \; n = \frac{3}{2}$

Since $h(\frac{5}{6}) = \frac{1}{2}$, therefore $h^{-1}(\frac{1}{2}) = \frac{5}{6}$.
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<tr><td>(ii)</td><td>
With domain $\frac{1}{2} \le x \le \frac{3}{2}$,

$R_h = [-1, \; 1]$, $D_g = [-1, \; \infty)$,

Since $R_h \subseteq D_g$, therefore gh exists.

Substituting R_h as D_g into $y = g(x)$ graph,
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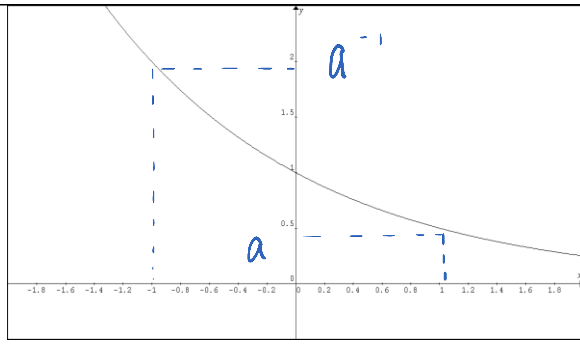
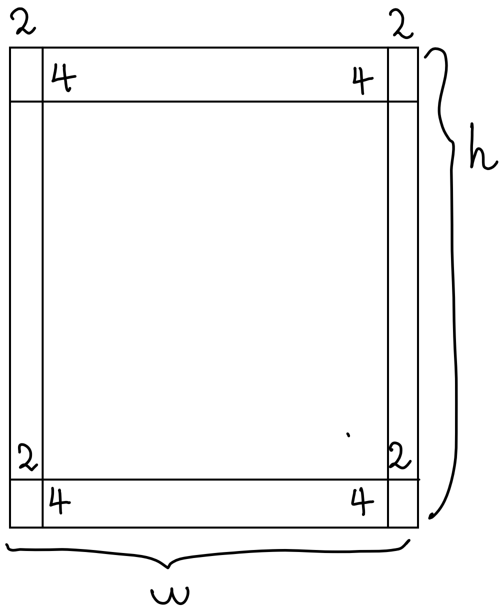


$R_{gh} = [a,\ a^{-1}]$

2



Let w be the width of the paper used and h be the length of the paper used.

Area of paper used, $A = wh$

We have $(w-4)(h-8) = 1352$

$$\Rightarrow w = \frac{1352}{(h-8)} + 4$$

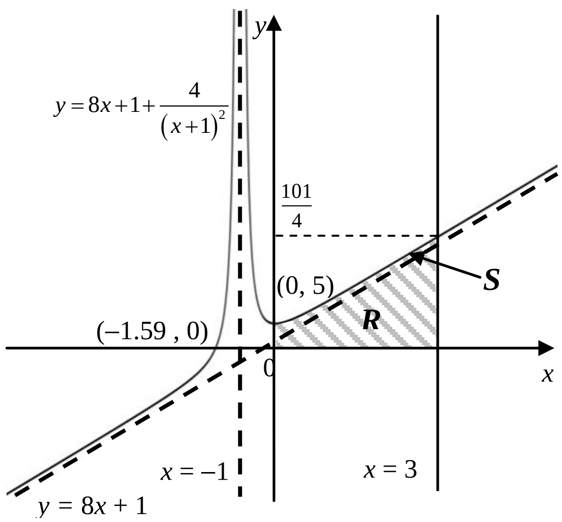
Therefore $A = wh = \left(\frac{1352}{(h-8)} + 4\right)h$

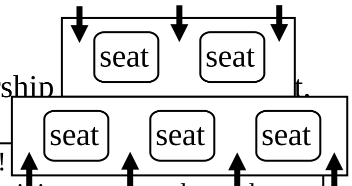
$$= \frac{1352h}{(h-8)} + 4h$$

	$\frac{\mathrm{d}A}{\mathrm{d}h}=\frac{(h-8)(1352)-(1352h)(1)}{(h-8)^2}+4$ $=\frac{-10816}{(h-8)^2}+4$ To minimise the paper used, $\frac{\mathrm{d}A}{\mathrm{d}h}=0$ $\Rightarrow \frac{10816}{(h-8)^2}=4$ $\Rightarrow (h-8)^2=2704$ $\Rightarrow h=\sqrt{2704}+8=60$ First derivative test: <table><tr><td>h</td><td>$(60)^-$</td><td>60</td><td>$(60)^+$</td></tr><tr><td>$\frac{\mathrm{d}A}{\mathrm{d}h}$</td><td>-ve</td><td>0</td><td>+ve</td></tr></table> OR Second derivative test: $\frac{\mathrm{d}^2A}{\mathrm{d}h^2}=\frac{21632}{(h-8)^3}>0$ Hence at $h=60$, the amount of paper used is minimised. At $h=60$, $\Rightarrow w=\frac{1352}{52}+4$ $w=30$ Hence the dimension of the paper is 30 cm by 60 cm.
3 (i)	$z^n+\frac{1}{z^n}=\mathrm{e}^{\mathrm{i}n\theta}+\frac{1}{\mathrm{e}^{\mathrm{i}n\theta}}=\mathrm{e}^{\mathrm{i}n\theta}+\mathrm{e}^{-\mathrm{i}n\theta}$ $=\cos n\theta+\mathrm{i}\sin n\theta+\cos n\theta-\mathrm{i}\sin n\theta$ $=2\cos n\theta$ (shown)

(ii)	By taking $n=1$, $z+\frac{1}{z}=2\cos\theta$ $\Rightarrow \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$ $\cos^3\theta=\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^3$ $=\frac{1}{8}\left(z+\frac{1}{z}\right)^3$ $=\frac{1}{8}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)$ (shown) $\cos^3\theta=\frac{1}{8}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)$ $=\frac{1}{8}\left(z^3+\frac{1}{z^3}+3\left(z+\frac{1}{z}\right)\right)$ $=\frac{1}{8}\left(2\cos3\theta+3\left(2\cos\theta\right)\right)$ $=\frac{1}{4}\cos3\theta+\frac{3}{4}\cos\theta$ (shown)
(iii)	<u>Method 1</u> $\int\cos^3 3\theta\ \mathrm{d}\theta=\int\frac{1}{4}\cos9\theta+\frac{3}{4}\cos3\theta\ \mathrm{d}\theta$ $=\frac{1}{36}\sin9\theta+\frac{1}{4}\sin3\theta+C$ <u>Method 2</u> $\int\cos^3 3\theta\ \mathrm{d}\theta=\int\cos3\theta\cos^2 3\theta\ \mathrm{d}\theta$ $=\int\cos3\theta\left(1-\sin^2 3\theta\right)\ \mathrm{d}\theta$ $=\int\cos3\theta-\cos3\theta\sin^2 3\theta\ \mathrm{d}\theta$ $=\frac{1}{3}\sin3\theta-\frac{1}{9}\sin^3 3\theta+C$

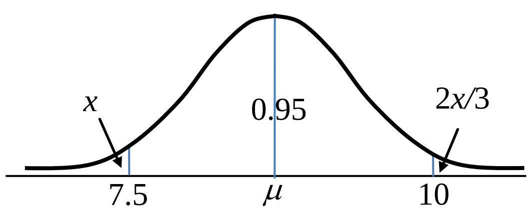
4 (a)	$AB \perp OP \Rightarrow (\mathbf{b}-\mathbf{a})\cdot(\mathbf{a}+5\mathbf{b})=0$ $\Rightarrow -4\mathbf{a}\cdot\mathbf{b}-\lvert\mathbf{a}\rvert^2+5\lvert\mathbf{b}\rvert^2=0$ Since we have $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ=\mathbf{a}\cdot\mathbf{b}\Rightarrow \mathbf{a}\cdot\mathbf{b}=\frac{\lvert\mathbf{a}\rvert}{2}$ $\therefore -2\lvert\mathbf{a}\rvert-\lvert\mathbf{a}\rvert^2+5=0$ $\lvert\mathbf{a}\rvert^2+2\lvert\mathbf{a}\rvert-5=0$ $\lvert\mathbf{a}\rvert=\frac{-2\pm\sqrt{24}}{2}=\sqrt{6}-1$ or $-\sqrt{6}-1(\text{rej})$ $\therefore \lvert\mathbf{a}\rvert=\sqrt{6}-1$
(b)	$\overrightarrow{OC}=\frac{\mathbf{a}}{2}\Rightarrow\overrightarrow{OE}=\frac{3\overrightarrow{OB}+4\overrightarrow{OC}}{7}=\frac{3\mathbf{b}+2\mathbf{a}}{7}$ Let $AD:AB=\lambda$, then $\overrightarrow{OD}=\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OA}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$ Since O, E, D are collinear, $\overrightarrow{OE}=k\overrightarrow{OD}$ for some constant k $\quad\Rightarrow k\left[\frac{3\mathbf{b}+2\mathbf{a}}{7}\right]=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$ $\lambda=\frac{3}{5}$ $AD:AB=3:5$
5 (i)	For $x>0$, $(x+1)^2>0$ Since a, b and c are positive, for $x>0$, $\frac{c}{(x+1)^2}>0$ and $ax+b>0$ $\therefore\ y=ax+b+\frac{c}{(x+1)^2}>0$ for $x>0$ (shown)
(ii)	Given area of region R $= 42$ $\int_0^3 ax+b+\frac{c}{(x+1)^2}\,\mathrm{d}x = 42$ $\left[\frac{1}{2}ax^2+bx-\frac{c}{x+1}\right]_0^3 = 42$ $\left[\frac{9}{2}a+3b-\frac{c}{4}\right]-[-c] = 42$ $\frac{9}{2}a+3b+\frac{3}{4}c=42$ (shown) —— (1)
(iii)	$y=ax+b+\frac{c}{(x+1)^2}$

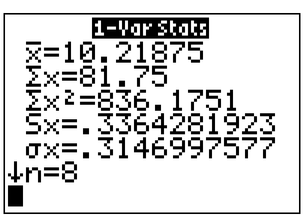
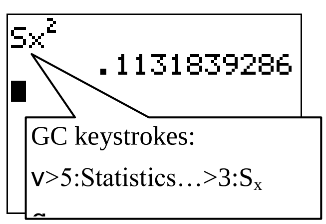
	At (0, 5), $b + c = 5$ ——— (2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a - \dfrac{2c}{(x+1)^3}$ At (0, 5), $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $a - 2c = 0$ ——— (3) From GC, $a = 8,\ b = 1,\ c = 4$ $\therefore$ equation of G is $y = 8x + 1 + \dfrac{4}{(x+1)^2}$.
(iv)	 [Note: x-intercept (−1.585171 , 0)]
(v)	Required volume $= \int_0^3 \pi \left[8x + 1 + \dfrac{4}{(x+1)^2}\right]^2 \mathrm{d}x - \int_0^3 \pi (8x+1)^2 \mathrm{d}x$ $= 163.2774895$ $= 163$ units^3
6(i)	Let X be the time spent by a customer at a supermarket. If $X \sim \mathrm{N}(35, 30^2)$, $P(X < 0) = 0.122$. i.e. the probability that the time spent being less than zero is significantly big. However, X is a non-negative quantity. Therefore a normal distribution will not provide an adequate model.
(ii)	Let X be the time spent by a customer at a supermarket. $X \sim \mathrm{N}(35, 10^2)$

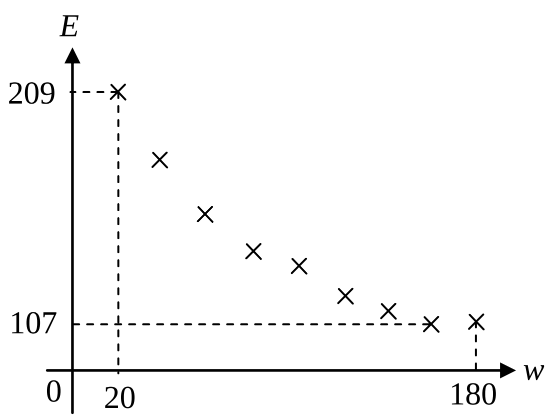
	$X_1 + X_2 - 3X \sim \mathrm{N}(-35, 1100)$ or $3X - X_1 + X_2 \sim \mathrm{N}(35, 1100)$ $\mathrm{P}\left(X_1 + X_2 > 3X\right)$ $= \mathrm{P}\left(X_1 + X_2 - 3X > 0\right)$ $= 0.146$
7(i)	No. of ways $= 7315 \times 4! = 175560$
(ii)	$n(n-1)(n-2)(n-3) = 175560$ or $\ {}^{n}\mathrm{P}_4 = 175560$ or $\ {}^{n}\mathrm{C}_4 = 7315$ Using GC or guess and check, $n = 22$. <u>Method 1</u>:  Consider the 4 scholarship t. 7 ways to slot the 4 scholarship recipients. 4! recipients among themselves. 9! ways to arrange remaining applicants. No. of ways $= 7 \times 4! \times 9!$ $= 60963840$ <u>Method 2</u>: Consider the 4 scholarship recipients as 1 unit. Case 1: The 4 scholarship recipients are in the row with 6 seats. Case 2: The 4 scholarship recipients are in the row with 7 seats. No. of ways 7! ways to arrange remaining applicants in row with 7 seats. $= \left(4! \times {}^{9}C_2 \times 3! \times 7!\right) + \left(4! \times {}^{9}C_3 \times 4! \times 6!\right)$ 3! ways to arrange 1 unit of scholarship recipients and 2 non-scholarship recipients. 4! ways to arrange scholarship recipients among themselves. ${}^{9}C_2$ ways to choose 2 non-scholarship recipients in row with 6 seats. $= 60963840$
8(i)	$\mathrm{P}(A \mid B)$ $= \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ $= \dfrac{\mathrm{P}(2\,\text{Curry}, 1\,\text{Spicy}, 1\,\text{others}) + \mathrm{P}(3\,\text{Curry}, 1\,\text{Spicy})}{\mathrm{P}(\text{exactly } 1 \text{ Spicy})}$

	$= \dfrac{\left(\frac{10}{80}\times\frac{9}{79}\times\frac{10}{78}\times\frac{60}{77}\times\frac{4!}{2!}\right)+\left(\frac{10}{80}\times\frac{9}{79}\times\frac{8}{78}\times\frac{10}{77}\times\frac{4!}{3!}\right)}{\frac{10}{80}\times\frac{70}{79}\times\frac{69}{78}\times\frac{68}{77}\times 4} = \dfrac{141}{2737}$ or $0.0515162587 = 0.0515$ (3 s.f.) OR $\mathrm{P}(A\mid B) = \dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ $= \dfrac{\dfrac{({}^{10}C_2\times{}^{10}C_1\times{}^{60}C_1)+({}^{10}C_3\times{}^{10}C_1)}{{}^{80}C_4}}{\dfrac{{}^{10}C_1\times{}^{70}C_3}{{}^{80}C_4}}$ $= \dfrac{141}{2737}$ or $0.0515162587 = 0.0515$ (3 s.f.)
(ii)	$\mathrm{P}(A)$ $= 1-\mathrm{P}(0\text{ curry})-\mathrm{P}(1\text{ curry})$ $= 1-\left(\frac{70}{80}\times\frac{69}{79}\times\frac{68}{78}\times\frac{67}{77}\right)-\left(\frac{10}{80}\times\frac{70}{79}\times\frac{69}{78}\times\frac{68}{77}\times 4\right)$ $\left[\text{or } 1-\dfrac{\binom{70}{4}}{\binom{80}{4}}-\dfrac{\binom{10}{1}\binom{70}{3}}{\binom{80}{4}}\right]$ $= 0.0741568558 = 0.0742$ (3 s.f.) Or $\mathrm{P}(A)$ $= \mathrm{P}(2\text{ curry})+\mathrm{P}(3\text{ curry})+\mathrm{P}(4\text{ curry})$ $= \left(\frac{10}{80}\times\frac{9}{79}\times\frac{70}{78}\times\frac{69}{77}\times\frac{4!}{2!2!}\right)+\left(\frac{10}{80}\times\frac{9}{79}\times\frac{8}{78}\times\frac{70}{77}\times\frac{4!}{3!}\right)$ $+\left(\frac{10}{80}\times\frac{9}{79}\times\frac{8}{78}\times\frac{7}{77}\right)$ $\left[\text{or } \dfrac{\binom{10}{2}\binom{70}{2}}{\binom{80}{4}}+\dfrac{\binom{10}{3}\binom{70}{1}}{\binom{80}{4}}+\dfrac{\binom{10}{4}}{\binom{80}{4}}\right]$ $= 0.0741568558 = 0.0742$ (3 s.f.) <u>Method 1</u>: Show $\mathrm{P}(A\mid B)\neq\mathrm{P}(A)$. Hence A and B are not independent, and $\therefore$ A' and B'

	are not independent. From (i), $\mathrm{P}(A \mid B) = 0.0515$ Since $0.0515 = \mathrm{P}(A \mid B) \neq \mathrm{P}(A) = 0.0742$, A and B are not independent. Hence A' and B' are not independent. <u>Method 2</u>: Show $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \cdot \mathrm{P}(B)$. Hence A and B are not independent, and $\therefore$ A' and B' are not independent. From (i), $\mathrm{P}(A \cap B)$ $= \left(\frac{10}{80} \times \frac{9}{79} \times \frac{10}{78} \times \frac{60}{77} \times \frac{4!}{2!}\right) + \left(\frac{10}{80} \times \frac{9}{79} \times \frac{8}{78} \times \frac{10}{77} \times \frac{4!}{3!}\right)$ $\left[\text{or } \frac{({}^{10}C_2 \times {}^{10}C_1 \times {}^{60}C_1) + ({}^{10}C_3 \times {}^{10}C_1)}{{}^{80}C_4}\right]$ $= 0.017830271 = 0.0178$ (3 s.f.) $\mathrm{P}(B) = \frac{10}{80} \times \frac{70}{79} \times \frac{69}{78} \times \frac{68}{77} \times 4$ $\left[\text{or } \frac{{}^{10}C_1 \times {}^{70}C_3}{{}^{80}C_4}\right]$ $= 0.3461095866 = 0.346$ (3 s.f.) $\mathrm{P}(A) \cdot \mathrm{P}(B) = 0.0742 \times 0.346 = 0.0256663987 = 0.0257$ Since $0.0178 = \mathrm{P}(A \cap B) \neq \mathrm{P}(A) \cdot \mathrm{P}(B) = 0.0257$, A and B are not independent. Hence A' and B' are not independent. Note: Students can choose to show $\mathrm{P}(A' \cap B') \neq \mathrm{P}(A') \cdot \mathrm{P}(B')$ or $\mathrm{P}(A' \mid B') \neq \mathrm{P}(A')$ or $\mathrm{P}(B' \mid A') \neq \mathrm{P}(B')$
9(i)	Let X be the number of defective SIM cards, out of 10. $X \sim \mathrm{B}(10, 0.15)$ We assume that the probability of a SIM card being defective remains constant at 0.15. OR We assume that the defective SIM cards produced are independent of one another.

(ii)	$\mathrm{P}(X<2)=\mathrm{P}(X\le 1)=0.5442998$ Let Y be number of 'good' batches, out of 33. $Y \sim \mathrm{B}(33, 0.5442998)$ Since n = 33 is large, np = 17.962 > 5, n(1–p) = 15.038 > 5, $Y \sim \mathrm{N}(17.962, 8.18524)$ approx $\mathrm{P}(Y>33\times 0.8)=\mathrm{P}(Y>26.4)$ $=\mathrm{P}(Y\ge 27)\xrightarrow{cc}\mathrm{P}(Y>26.5)=0.00142$
10 (a) (i)	The customers are ordered as they queue up at the payment counter. $\frac{120}{30}=4$ The first customer is chosen from the 1st four customers using simple random sampling. Every 4th customer is chosen thereafter.
(ii)	No. As we cannot gather the profile of the customers and break down into strata proportional to that of the 120 customers as the survey is done before the customers leave.
(b)	Let X be the waiting time. Since n = 100 is large, by CLT, $\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{100}\right)$ approximately. $\mathrm{P}(7.5<\overline{X}<10)=0.95$ and $\mathrm{P}(\overline{X}>10)=\frac{2}{3}\mathrm{P}(\overline{X}<7.5)$  $x+\frac{2x}{3}=0.05$ $x=0.03$

	$\mathrm{P}\left(\overline{X}<7.5\right)=0.03 \;\Rightarrow\; \mathrm{P}\left(Z<\dfrac{7.5-\mu}{\frac{\sigma}{10}}\right)=0.03$ $\mathrm{P}\left(\overline{X}<10\right)=0.98 \;\Rightarrow\; \mathrm{P}\left(Z<\dfrac{10-\mu}{\frac{\sigma}{10}}\right)=0.98$ $\dfrac{7.5-\mu}{\frac{\sigma}{10}}=-1.8808$ $10\left(7.5-\mu\right)=-1.8808\sigma$ $10\mu-1.8808\sigma=75 \;---(1)$ $\dfrac{10-\mu}{\frac{\sigma}{10}}=2.0537$ $10\left(10-\mu\right)=2.0537\sigma$ $10\mu+2.0537\sigma=100 \;---(2)$ Solving (1) and (2), $\mu=8.70$
11 (i)	Let T be an estimator of the population variance σ^2. Then T is an unbiased estimator of σ^2 if $\mathrm{E}(T)=\sigma^2$.
(ii)	Using GC, An unbiased estimate of mean of X, $\bar{x}=\dfrac{327}{32}=10.21875$ (exact) An unbiased estimate of variance of X, $s^2=0.1131839286=0.113$ (3 s.f.)  
(iii)	Assumption: X follows a normal distribution. Let μ mm be the population mean diameter of a miniature wheel. $\mathrm{H}_0: \mu=10.00$ $\mathrm{H}_1: \mu\neq10.00$

	Test statistic: $T = \dfrac{\bar{X} - 10.00}{\sqrt{\dfrac{0.1131839286}{8}}} \sim t(7)$ Level of Significance: 10% Critical region: Reject H_0 if p-value ≤ 0.10 Assuming H_0 is true, using GC, p–value = 0.10849 Since p-value = 0.10849 $\geq$ 0.10, we do not reject H_0 at the 10% level of significance and conclude that there is insufficient evidence that the mean diameter of the miniature wheel is different from expected.
(iv)	$H_0: \mu = 10.00$ $H_1: \mu > 10.00$ Test statistic: $Z = \dfrac{\bar{X} - 10.00}{\sqrt{\dfrac{0.45^2}{8}}} \sim N(0,1)$ Level of Significance: 5% Critical region: Reject H_0 if p-value ≤ 0.05 i.e. Reject H_0 if $z \geq 1.644853626$ Hence $z = \dfrac{\bar{x} - 10.00}{\sqrt{\dfrac{0.45^2}{8}}} \geq 1.644853626$ $\bar{x} \geq 10.26169461$ $\therefore \{\bar{x} \in \mathbb{R} : \bar{x} \geq 10.3\}$
12 (i)	
(ii)	The scatter diagram seems to indicate a curvi-linear relationship between the points. Hence the linear model $E = a + bw$ is probably not appropriate.

(iii)	For w and $\ln E$, $r = -0.9785715974 = -0.979$ For w and $\frac{1}{E}$, $r = 0.9902693221 = 0.990$ Since $\lvert r \rvert$ for w and $\frac{1}{E}$ is closer to 1, the formula $\frac{1}{E} = e + fw$ is a better model.
(iv)	The least squares regression line of $\frac{1}{E}$ on w is appropriate since w is the independent variable. Using GC, the equation of the least squares regression line of $\frac{1}{E}$ on w is $\frac{1}{E} = 0.000029587529w + 0.0044040033$ When $E = 0.13 \text{ kg} = 130\text{g}$, we have $\frac{1}{130} = 0.000029587529w + 0.0044040033$ $\therefore \quad w = 111.1381891 = 111.14 \text{ ml}$
13 (i)	Let A be no. of accidents along NS line per week. $A \sim \text{Po}\left(\frac{m}{4}\right)$ $\text{P}(A \geq 1) = 1 - \text{P}(A = 0) = 1 - \text{e}^{-\frac{m}{4}} = 0.25$ $\text{e}^{-\frac{m}{4}} = 0.75$ $\Rightarrow \; m = -4\ln\frac{3}{4} \quad \text{or} \quad 4\ln\frac{4}{3}$
(ii)	$A \sim \text{Po}\left(4\ln\frac{4}{3}\right)$ $\text{P}(A = 2)\, p(1-p) * 2$ $= 0.0621p(1-p)$
(iii)	Let B be no. of accidents along EW line per week. $B \sim \text{Po}\left(\frac{1}{2}\right)$ $A + B \sim \text{Po}(0.5 + 0.28768)$ $\text{P}(B = 0 \mid A + B = 3)$

	$=\dfrac{\mathrm{P}(B=0)\,\mathrm{P}(A=3)}{\mathrm{P}(A+B=3)}$ $=0.0487$
(iv)	Let C be the number of accidents along the EW line per month. $C \sim \mathrm{Po}(2)$ $1-\mathrm{P}(C\le 4)=1-0.94735=0.052653$ P(more than 52 months with at most 4 railway accidents) = P(less than 8 months with more than 4 railway accidents) Let X be no. of months, out of 60, with more than 4 accidents. $X \sim \mathrm{B}(60, 0.052653)$ Since n = 60 is large and p = 0.052653 is small such that np = 3.1592 < 5. $X \sim \mathrm{Po}(3.1592)$ approx $\mathrm{P}(X<8)=\mathrm{P}(X\le 7)=0.984$ Note: If we use P(more than 52 months with at most 4 railway accidents), than $X \sim \mathrm{B}(60, 0.94735)$ and since p is large, Poisson approximation will fail.