Chapter 7D: Complex Numbers IV – Loci

SYLLABUS INCLUDES
H2 Further Mathematics:

- loci of the form $|z-a| \le r$,
- loci of the form $|z-a| = |z-b|$,
- loci of the form $\arg(z-a) = \theta$,

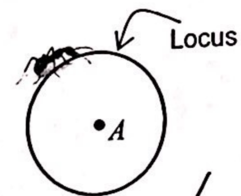
PRE-REQUISITES
- Trigonometry,
- Co-ordinate Geometry,
- Basic Geometrical Properties of Circles and Triangles,
- Complex Numbers I and II.

CONTENT

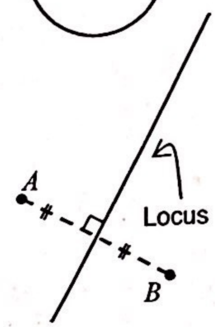
1 Introduction

The **locus** (plural **loci**) of a variable point is defined to be the path traced out by the point moving under certain conditions.

To illustrate, imagine an ant that can only move on this piece of paper. What is the locus of the ant (describe and draw the locus) in each of the following cases?

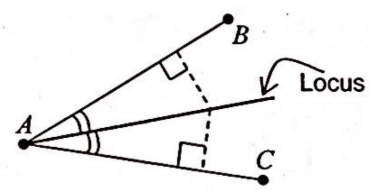


(a) The ant moves such that it is always 1cm from a fixed point A. The locus of the ant is a circle



(b) The ant moves such that it is always equidistant from 2 fixed points A and B. The locus of the ant is ____________________

(c) The ant moves such that it is always equidistant from AB and AC. The locus of the ant is the angle bisector of $\angle BAC$.



Example 1
On a single Argand diagram, sketch and describe the loci given by
(i) $\mathrm{Re}(z)=3$, **(ii)** $\mathrm{Im}(z)=4$, **(iii)** $\mathrm{Re}(z)+\mathrm{Im}(z)=3$.

Solution

Let

(i)

The locus is the vertical line through the point (3,0).

(ii)

The locus is the horizontal line through the point (0,4).

(iii)

The locus is the line with gradient -1 and y-intercept 3.

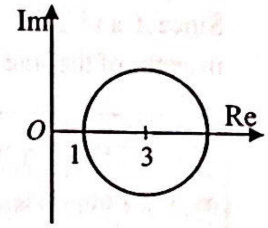
2 Loci of the Form $|z-a|=r$

Given 2 complex numbers z and w, $|z-w|$ is the distance (or length) between the 2 points representing the complex numbers z and w.

Q: *What is the locus of a variable point P representing the complex number z such that* $|z-3|=2$?

Let points P and A represent the complex numbers z and 3 respectively. Then $|z-3|=2$ means that the distance AP is 2.

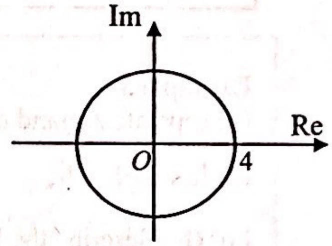
Now A is fixed, while P is a variable point.
So, the point P moves such that its distance from A is always 2.



$\therefore$ The locus of P is a circle with centre (3, 0) and radius 2.

Q: *What is the locus of a variable point P representing the complex number z such that* $|z|=4$?

Since $|z|=4$ means that $OP=4$, where O is the origin, therefore the locus of P is a circle with centre (0, 0) and radius 4.



> If $|z-a|=r$ where P represents the variable complex number z, point A represents the fixed complex number a and $r>0$, the locus of P is a
> **circle with centre A and radius r.**

Example 2
On separate Argand diagrams, sketch the locus given by
(a) $|z-1+\mathrm{i}|=4$, **(b)** $|z-3-4\mathrm{i}|=5$.
For **(b)**, describe the locus and write down the cartesian equation.

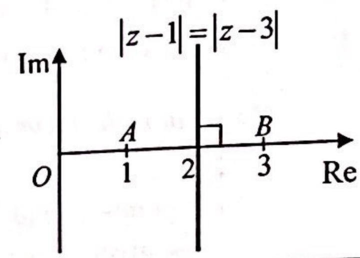
Solution

(a)	(b)

Note: For questions involving loci, circles should be drawn with a compass and lines drawn with a ruler. Remember to use the same scale for the real and imaginary axes.

3 Loci of the Form $|z-a|=|z-b|$

Consider $|z-1|=|z-3|$.
Let P, A and B be points representing the complex numbers z, 1 and 3 respectively. Then $|z-1|=|z-3|$ means $AP=BP$.



Since A and B are fixed points, the locus of P is the perpendicular bisector of the line segment joining $A\ (1, 0)$ and $B\ (3, 0)$.

> If $|z-a|=|z-b|$ where point P represents the variable complex number z, the points A and B represent the fixed complex numbers a and b, the locus of P is the
> **perpendicular bisector of the line segment joining A and B.**

Example 3
On separate Argand diagrams, sketch the locus given by

(a) $|z+\mathrm{i}|=|z-1|$, **(b)** $|z+3|=|z-2+\mathrm{i}|$, **(c)** $\left|\frac{1+\mathrm{i}}{z}-2\right|=2$.

For **(b)**, describe the locus and find the cartesian equation.

Solution

(a)	**(b)** The locus is Let A and B represent the points $(-3, 0)$ and $(2, -1)$ respectively. Gradient of AB is Gradient of the perpendicular bisector of AB is 5. Midpoint of AB has coordinates The cartesian equation of the locus is

Note: To obtain the cartesian equation of the locus, we may also use $z = x + \mathrm{i}y$ where $x,\ y \in \mathbb{R}$. For example 3(b), we have $\left|z-(-3)\right| = \left|z-(2-\mathrm{i})\right|$

$$\begin{aligned}
\left|x+\mathrm{i}y-(-3)\right| &= \left|x+\mathrm{i}y-(2-\mathrm{i})\right| \\
\left|(x+3)+\mathrm{i}y\right| &= \left|(x-2)+\mathrm{i}(y+1)\right| \\
\sqrt{(x+3)^2+y^2} &= \sqrt{(x-2)^2+(y+1)^2} \\
x^2+6x+9+y^2 &= x^2-4x+4+y^2+2y+1 \\
y &= 5x+2
\end{aligned}$$

The cartesian equation for any locus, be it circle, half-line, etc., can be found in this way, by writing z as $x+\mathrm{i}y$ where $x,\ y \in \mathbb{R}$. This is especially helpful when the equation of the given locus is a 'non-standard' one, as in example 1.

(c) $\left|\dfrac{1+\mathrm{i}}{z}-2\right| = 2$

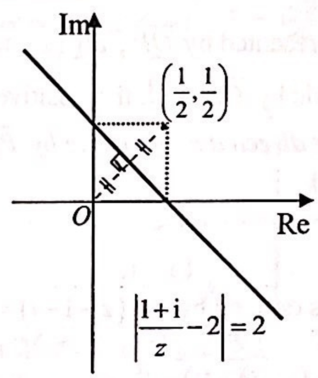
$\left|\dfrac{1+i-2z}{z}\right| = 2$

$\dfrac{\left|(1+i)-2z\right|}{|z|} = 2$

$\left|2z-(1+i)\right| = 2|z|$

$2\left|z-\frac{1}{2}(1+i)\right| = 2|z|$

$\left|z-\frac{1}{2}(\quad)\right|$

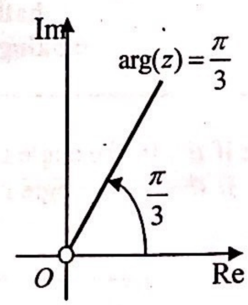


4 Loci of the Form $\arg(z-a)=\theta$

Let P and Q be the points on an Argand diagram representing z and w respectively. Recall that arg(z) is the directed angle made by vector $\overrightarrow{OP}$ with the positive real axis.

Q: *What is the locus of a point P representing z such that* $\arg(z) = \dfrac{\pi}{3}$?

Since $\arg(z)$ is the directed angle $\overrightarrow{OP}$ makes with the positive real axis, the locus of P is the half-line starting at, but not including, the origin and making a directed angle of $\dfrac{\pi}{3}$ with the positive real axis.



Note:

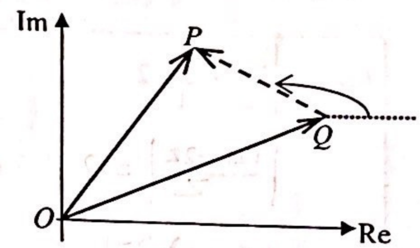
- The argument of $0+0\mathrm{i}$ is not defined, hence $(0,0)$ is excluded from the locus.
- The locus consists of the half line rather than the whole line because points on the 'other half' of the line represent complex numbers whose arguments are $-\frac{2\pi}{3}$.

Q: What is the equation of the locus represented by the other half of the line?
A: The equation of the locus is $\arg(z)=-\frac{2\pi}{3}$

In general, $\arg(z-w)$ is the directed angle made by the vector $\overrightarrow{QP}$ with the positive real axis.

Why?

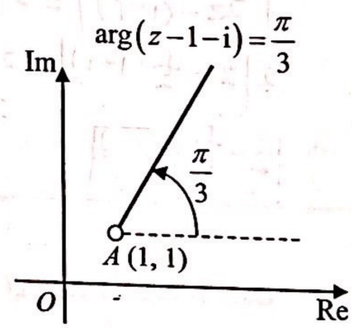
Since $z-w$ is represented by $\overrightarrow{QP}$, $\arg(z-w)$ is the directed angle made by $\overrightarrow{QP}$ with the positive real axis. *(different from the directed angle made by $\overrightarrow{PQ}$ with the positive real axis.)*



Consider the locus defined by $\arg(z-1-\mathrm{i})=\frac{\pi}{3}$.

The equation $\arg(z-(1+\mathrm{i}))=\frac{\pi}{3}$ means the directed angle that $\overrightarrow{AP}$ makes with the positive real axis is $\frac{\pi}{3}$, where A and P represent $1+\mathrm{i}$ and z respectively.

So the locus of P is the half-line starting at, but not including, the point $(1,1)$ and making a directed angle of $\frac{\pi}{3}$ with the positive real axis.



If $\arg(z-a)=\theta$ where the point P represents the variable complex number z and point A represents the fixed complex number a, the locus of P is the

half-line starting at, but not including, A and making a directed angle θ with the positive real axis.

Note: If $\theta>0$, the angle is measured in the anti-clockwise direction from the positive real axis. If $\theta<0$, the angle is measured in the clockwise direction.

Example 4

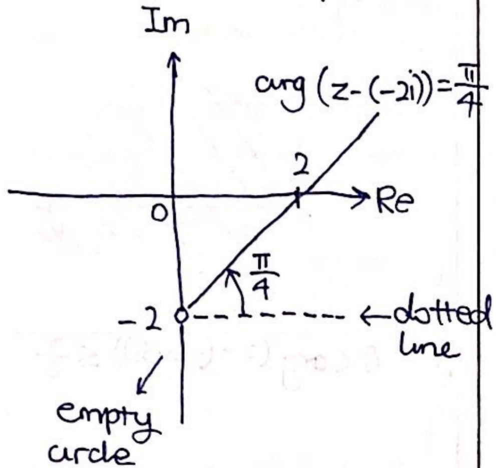
On separate Argand diagrams, sketch the locus given by

(a) $\arg(z+2\mathrm{i})=\frac{\pi}{4}$, **(b)** $\arg(1+\mathrm{i}-z)=\frac{\pi}{4}$ **(c)** $\arg(z-(2+\mathrm{i}))=\arg(3-\mathrm{i})$,

(d) $\arg(z-(-2))=\tan^{-1}\left(\frac{3}{2}\right)$. Describe this locus and find its cartesian equation.

Solution

(a) $\arg(z-(-2\mathrm{i}))=\frac{\pi}{4}$



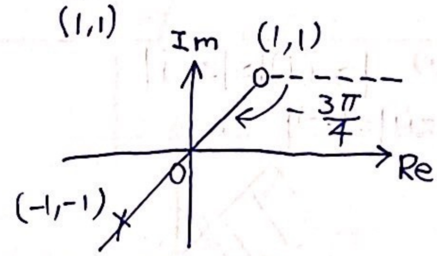
(b) $\arg(1+\mathrm{i}-z)=\frac{\pi}{4}$

$\arg(1-(z-1-\mathrm{i}))=\frac{\pi}{4}$

$\arg(-1)+\arg(z-1-\mathrm{i})=\frac{\pi}{4}$

$\pi+\arg(z-(1+\mathrm{i}))=\frac{\pi}{4}$

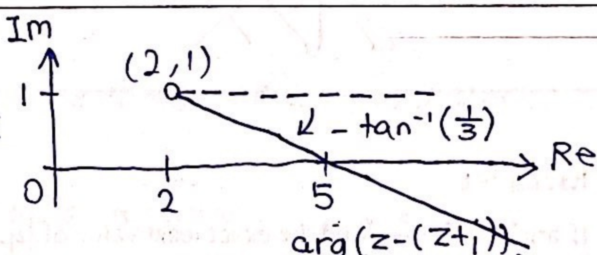
$\arg(z-\underbrace{(1+\mathrm{i})}_{(1,1)})=-\frac{3\pi}{4}$



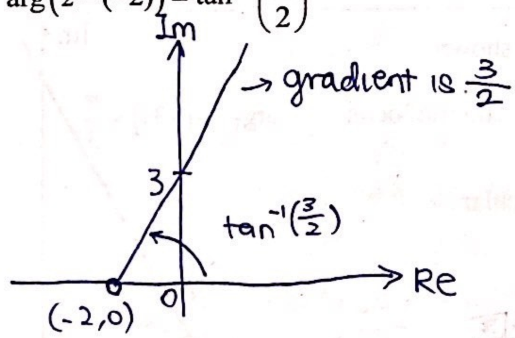
$\arg(z-(1+\mathrm{i}))=-\frac{3\pi}{4}$

(c) $\arg(z-(2+\mathrm{i}))=\arg(3-\mathrm{i})=-\tan^{-1}\left(\frac{1}{3}\right)$

Note that the half-line has gradient $-\frac{1}{3}$



(d) $\arg(z-(-2))=\tan^{-1}\left(\frac{3}{2}\right)$



The locus is a half-line starting at, but not inc the point $(-2,0)$ and making a directed angle of ~~an~~ $\tan^{-1}\left(\frac{3}{2}\right)$ with the positive real axis

The cartesian equation of the locus is

$y=\frac{3}{2}x+3,\ x>-2$

5 Problems Involving One Locus

Example 5 $z-(-2)$

Indicate, on separate Argand diagrams, the region defined by the following inequalities:

(a) $3<\mathrm{Re}(z+2)\le 5$, **(b)** $|z-\mathrm{i}|<2$,

(c) $|z+1|\le|z-\mathrm{i}|$, **(d)** $0<\arg(z+2+3\mathrm{i})\le\frac{\pi}{6}$.

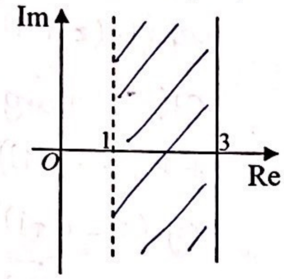
Solution

(a) $3<\mathrm{Re}(z+2)\le 5$

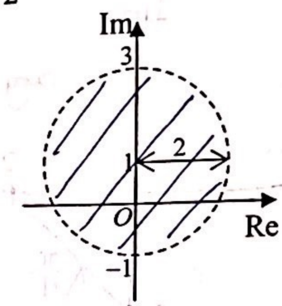
$3 < x+2 \le 5$

$1 < x \le 3$

left bound is exclusive of 1

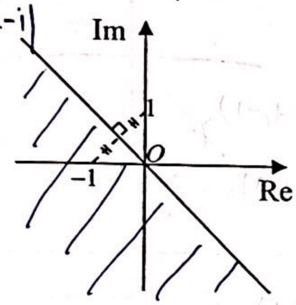


(b) $|z-\mathrm{i}|<2$

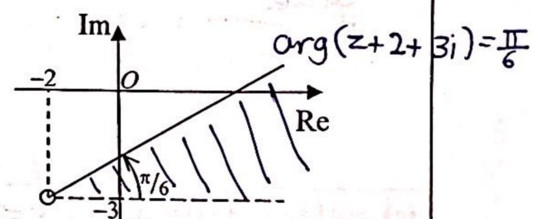


(c) $|z-(-1)|\le|z-\mathrm{i}|$

$|z+1|=|z-\mathrm{i}|$



(d) $0<\arg(z-(-2-3\mathrm{i}))\le\frac{\pi}{6}$



Example 6

If $\arg(z+3)=\frac{\pi}{3}$, find the exact least value of $|z|$.

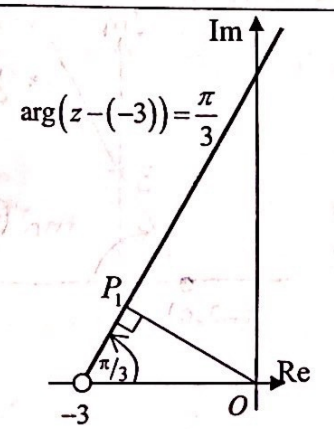
Solution

$\arg(z-(-3))=\frac{\pi}{3}$. The locus is the half-line as shown.

$|z|=OP$, where O is the origin and P is a point on the locus.

Least value of $|z|$ occurs when OP is perpendicular to the half-line, i.e. when $P=P_1$.

Least value of $|z|$ is $OP_1 = 3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$



Example 7

If z is a complex number such that $\left|z-\sqrt{2}-\mathrm{i}\right|=1$, find in exact form, the greatest and least values of (i) $|z-3\mathrm{i}|$, (ii) $\arg(z-\mathrm{i})$.

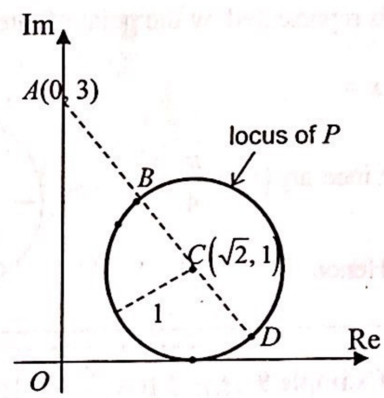
$(\sqrt{2}, 1)$

Solution

Let the point P represent the complex number z in the equation $\left|z-(\sqrt{2}+\mathrm{i})\right|=1$.

(i) *Recall that* $|z-3\mathrm{i}|$ *is the distance between the variable point* P *and the fixed point* $A\,(0, 3)$.

Greatest value of $|z-3\mathrm{i}|$ is given by distance AD as in diagram, with AD passing through the centre of circle C.



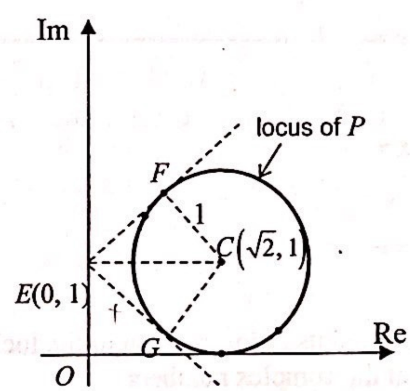
$$AD = AC + CD$$
$$= \sqrt{(0-\sqrt{2})^2+(3-1)^2} + 1$$
$$= \sqrt{6} + 1$$

Least value of $|z-3\mathrm{i}|$ is given by distance AB, with B diametrically opposite to D.

$$AB = AC - BC = \sqrt{6} - 1$$

(ii) *Recall that* $\arg(z-\mathrm{i})$ *is the directed angle that* $\overline{EP}$ *makes with the positive real axis, where E is the fixed point (0, 1).*

EF and EG are tangents to the circle at F and G respectively.



$\frac{\pi}{4}$

$-\frac{\pi}{4}$

6 Problems Involving Two Loci

Example 8

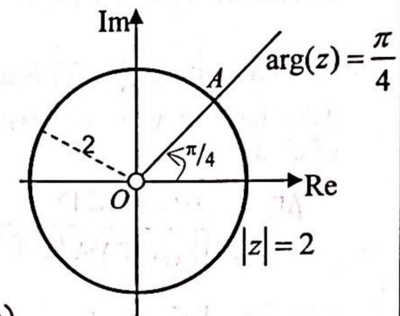
On a single Argand diagram, sketch the loci given by

(i) $|z| = 2$, **(ii)** $\arg(z) = \frac{\pi}{4}$.

Hence, or otherwise, find the complex number z in the form $x + \mathrm{i}y$ that satisfies both **(i)** and **(ii)**.

Solution

The complex number $z = x + \mathrm{i}y$ satisfying both **(i)** and **(ii)** is represented by the point of intersection A of the 2 loci.



$x =$

Since $\arg(z) = \frac{\pi}{4}$,

Hence

centre is (3,-3)

$|z-(3-3i)| = 2$

Example 9

Sketch the loci given by $|z-1| = |z+\mathrm{i}|$ and $|z-3+3\mathrm{i}| = 2$ on the same diagram.

Obtain, in the form $a+\mathrm{i}b$, the complex numbers represented by the points of intersection of the loci, giving the exact values of a and b.

Solution

It is important to observe that the perpendicular bisector passes through $(0,0)$ *and the centre of the circle* $(3,-3)$. *Note also the bisector has Cartesian equation* $y = -x$.

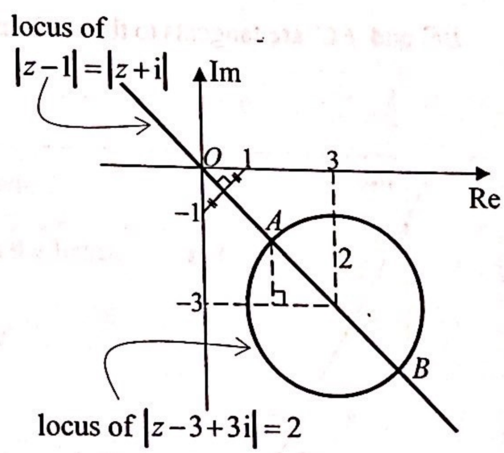
At A, $x =$

$y =$

At B, $x =$

$y =$

Hence the points of intersection of the loci represent the complex numbers



$\frac{dy}{dx}$

$y = \cos t$

Example 10

In each of the following cases, show in an Argand diagram, the set of points which satisfy the constraints imposed:

$x = \sin t$

$\frac{dy}{dt} \Big/ \frac{dx}{dt}$

$y^2 + xy = 0$

Differentiate w r t x

(a) $|z-2| \le 1$ and $\mathrm{Im}(z) \ge 0$;

(b) $|z-2\mathrm{i}| \le 3$ and $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$;

(c) $\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{2}$ and $\mathrm{Re}(z) \le 2$.

$\frac{d}{dy} y^2 \cdot \frac{dy}{dx} + \frac{d}{dy} y \frac{dy}{dx} = 0$

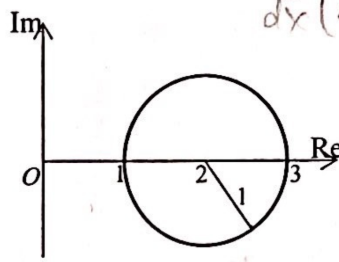
$2y \frac{dy}{dx} + \frac{dy}{dx} = 0$

$\frac{dy}{dx}(2y+1) = 0$?

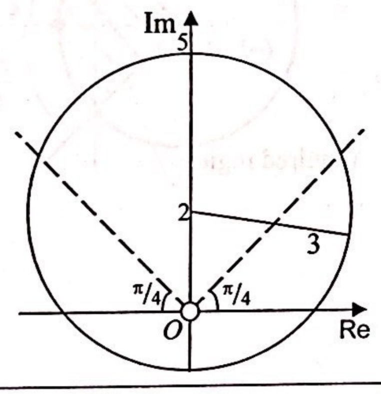
$\frac{dy}{dx} = 0$

Solution

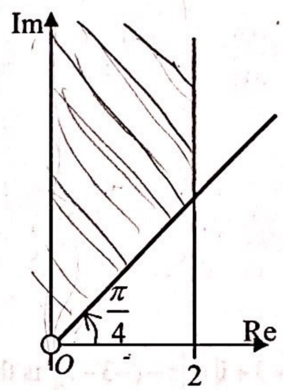
(a) $|z-2| \le 1$ and $\mathrm{Im}(z) \ge 0$



(b) $|z-2\mathrm{i}| \le 3$ and $\frac{\pi}{4} < \arg(z) < \frac{3\pi}{4}$



(c) $\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{2}$ and $\mathrm{Re}(z) \le 2$



vol cone: $\frac{1}{3}\pi r^2 h$

SA cone: $\pi r(r + l)$

vol sphere: $\frac{4}{3}\pi r^3$

SA sphere: $4\pi r^2$

Example 11

Show, on an Argand diagram, the set of points which satisfy

$$|iz-1+i| \le |1-i| \quad \text{and} \quad \frac{\pi}{4} \le \arg(iz-1+i) \le \frac{3\pi}{4}.$$

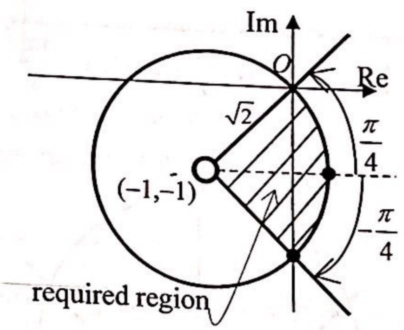
Find **(i)** the exact greatest value of $|z+3+i|$,

(ii) the range of values of $\arg(z+3+i)$, leaving your answer to 3 significant figures.

Solution

$|iz-1+i| \le |1-i|$

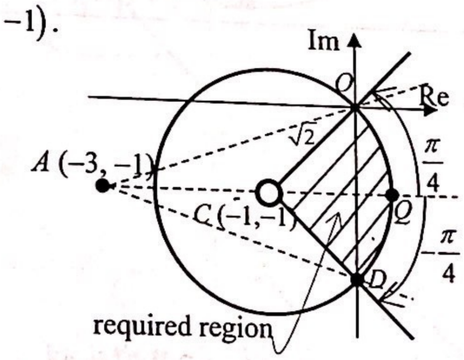
$\frac{\pi}{4} \le \quad \arg(iz-1+i) \quad \le \frac{3\pi}{4}$



(i) $|z+3+i| = |z-(-3-i)|$ is the distance between a point P on the locus and the point $A(-3,-1)$.

Greatest $|z+3+i|$

$= AQ$, where AQ passes through C



(ii) $\arg(z+3+i)$ is the directed angle that $\overrightarrow{AP}$ makes with the positive real axis.

$\angle OAC = \angle DAC =$

SUMMARY