(Pure Mathematics) Chapter 1: Equations

Objectives

At the end of the chapter, you should be able to:

(a) solve quadratic equations using factorisation, completing the square, formula, sketching of graph and a GC;

(b) understand and use the conditions for a quadratic equation to have (i) two real and distinct roots, (ii) two real and equal roots, (iii) no real roots;

(c) understand and use the conditions for a quadratic equation to be always positive (or always negative);

(d) formulate a quadratic equation from a problem situation and interpret the solution in the context of the problem;

(e) solve a pair of simultaneous equations, one linear and one quadratic, by substitution;

(f) formulate a system of linear equations from a problem situation;

(g) find the solution of a system of linear equations using a Graphing Calculator.

Content

References

1. New Additional Mathematics, Ho Soo Thong (Msc, Dip Ed), Khor Nyak Hiong (Bsc, Dip Ed)

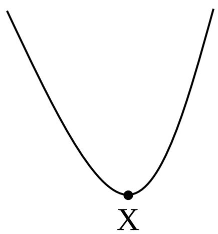
2. New Syllabus Additional Mathematics (7th Edition), Shinglee Publishers Ptd Ltd

1.1 Quadratic Functions

The expression $\mathrm{f}(x) = ax^2 + bx + c$, where $a \neq 0$, is called a quadratic function.

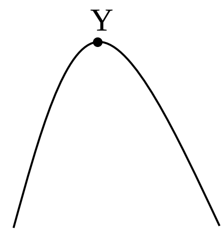
When the graph of the function $y = ax^2 + bx + c$ is drawn, two types of graphs are obtained, depending on the value of *a*.

$a > 0$



U shape

$a < 0$



Inverted – U shape

Note:

1. *a* is the coefficient of x^2
   b is the coefficient of x
   c is the coefficient of x^0 (also known as the constant term or the term independent of *x*)

2. If $a > 0$, the curve has a minimum point at X.
   For example, in $y = 2x^2 - 7x + 4$, $a = 2$, $b = -7$, $c = 4$. Since $a = 2 > 0$, the curve is U shape and has a minimum point.

   If $a < 0$, the curve has a maximum point at Y.
   For example, in $y = -x^2 - 6x + 9$, $a = -1$, $b = -6$, $c = 9$. Since $a = -1 < 0$, the curve is Inverted – U shape and has a maximum point.

3. The shape of a quadratic function is symmetrical about its minimum or maximum point.

4. $\mathrm{f}(x)$ is a function in terms of *x*. f is the name of the function. Go to the following link to know more about function: https://www.mathsisfun.com/sets/function.html.
   (read up to the section "Relating")

1.1.1 Solving Quadratic Equations

The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

There are five methods of solving a quadratic equation, namely:

(a) Factorisation
(b) Completing the Square
(c) Formula
(d) Graphical Method
(e) GC

(a) Factorisation

Method
1. Bring all the terms to one side of the equation such that the other side is 0.
2. Use Cross Factorisation Method to factorize the equation.

Example 1
Solve the quadratic equation $3x^2 + x - 2 = 0$ using factorization.

Solution:

$3x^2 + x - 2 = 0$

$(3x-2)(x+1) = 0$

$3x - 2 = 0$ or $x + 1 = 0$

$x = \frac{2}{3}$ or $x = -1$

True or False?

$(3x-2)(x+1) = 2$

$\Rightarrow 3x - 2 = 2 \quad \text{or} \quad x + 1 = 2$

Ans: False (the result is only true when the right hand side is 0)

Note:

1. The solution of the quadratic equation, i.e. $x = \frac{2}{3}$ or $x = -1$ are also called the **roots** of the quadratic equation.

2. $(3x-2)$ and $(x+1)$ are called **factors** of the quadratic expression $3x^2 + x - 2$.

(b) Completing the Square

A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ can be expressed to the form $a(x-p)^2 + q = 0$ by completing the square.

Note:
The coefficient of x^2 must be 1 before completing the square method can be carried out.

Example 2
Solve the quadratic equation $2x^2 + 8x + 1 = 0$ by completing the square.

Solution:

$2x^2 + 8x + 1 = 0$

$2(x^2 + 4x) + 1 = 0$

$2[x^2 + 4x + 2^2 - 2^2] + 1 = 0$

$2[(x+2)^2 - 2^2] + 1 = 0$

$2(x+2)^2 = 7$

$x + 2 = \pm\sqrt{\frac{7}{2}} \Rightarrow x = -2 \pm \sqrt{\frac{7}{2}}$

Alternative Method:

$2x^2 + 8x + 1 = 0$

$x^2 + 4x + \frac{1}{2} = 0 \Rightarrow x^2 + 4x + 2^2 - 2^2 + \frac{1}{2} = 0$

$(x+2)^2 - 2^2 + 0.5 = 0 \Rightarrow (x+2)^2 = 3.5$

$x + 2 = \pm\sqrt{\frac{7}{2}} \Rightarrow x = -2 \pm \sqrt{\frac{7}{2}}$

(c) Formula

The general formula of solving a quadratic equation of the form $ax^2+bx+c=0$, $a\neq 0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 3
Solve the quadratic equation $x^2+7x-3=0$ using formula.

Solution:

$x^2+7x-3=0$ Comparing with $ax^2+bx+c=0$, $a=1, b=7, c=-3$	$x=\frac{-7\pm\sqrt{7^2-4(1)(-3)}}{2}$ $=\frac{-7\pm\sqrt{61}}{2}$ $=\frac{-7+\sqrt{61}}{2}$ or $\frac{-7-\sqrt{61}}{2}$

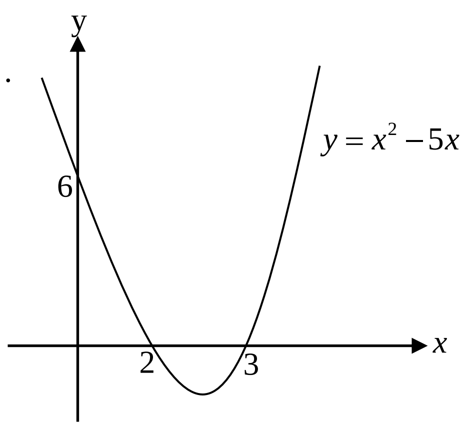
Note:
The expression b^2-4ac in the general formula is known as the **discriminant** of the quadratic equation as it determines the nature (type) of roots that a quadratic equation has.

(d) Graphical Method

Example 4
Solve the quadratic equation $x^2-5x+6=0$ using graphical method.

Solution:

From the graph, $x=2$ or $x=3$.



Why are the x-intercepts the solution for $x^2-5x+6=0$?

Ans: The x-intercepts are the points of intersection between the line $y=0$ and the curve $y=x^2-5x+6$.

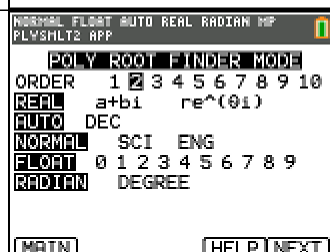
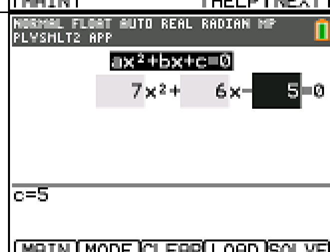
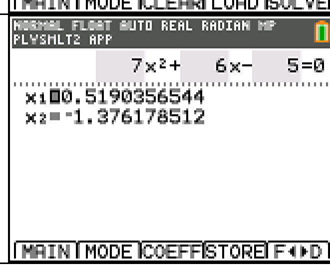
Note:
The x – intercepts of the curve $y=x^2-5x+6$ are the solutions of the quadratic equation $x^2-5x+6=0$.

(e) GC

Example 5

Solve the quadratic equation $7x^2+6x-5=0$ using a calculator.

Solution:

Steps	Screenshot	Remarks
Press  Select PlySmlt2		
Press [1] to select POLYNOMIAL ROOT FINDER		
Select 2 in the ORDER option Press [GRAPH] (NEXT) to go to the next screen		ORDER prefers to the order of the polynomial that you are solving. (i.e. highest degree of the polynomial)
Enter the coefficients of the quadratic equation (i.e. a, b and c) and select "+ " or "– " according to the equation that you are solving.		
Press [GRAPH] (SOLVE) to solve the equation Note: Press F◄►D will sometimes convert decimal to fraction		The solutions are given by x_1 and x_2.

From GC, $x=0.519$ or $x=-1.38$

Note:
GC may not give exact values. If the question requires exact answer, do not use GC but you can always use GC to check your answer.

Exercise 1

1. Solve the following equations using the stated method in the bracket.

   (a) $2x^2 - 11x = -12$ (factorisation)

   (b) $3x^2 - 8x + 1 = 0$ (completing the square)

   (c) $-x^2 - 6x + 1 = 0$ (formula)

   (d) $2x^2 - 13x + 14 = 0$ (GC)

Solution:

(a) $2x^2 - 11x = -12$ $2x^2 - 11x + 12 = 0$ $(2x-3)(x-4) = 0$ $2x - 3 = 0$ or $x - 4 = 0$ $x = \frac{3}{2}$ or $x = 4$	(b) $3x^2 - 8x + 1 = 0$ $3\left[x^2 - \frac{8}{3}x\right] + 1 = 0$ $3\left[x^2 - \frac{8}{3}x + \left(-\frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right)^2\right] + 1 = 0$ $3\left(x - \frac{4}{3}\right)^2 - \frac{16}{3} + 1 = 0$ $\left(x - \frac{4}{3}\right)^2 = \frac{13}{9}$ $x - \frac{4}{3} = \pm\sqrt{\frac{13}{9}}$ $x = \frac{4-\sqrt{13}}{3}$ or $x = \frac{4+\sqrt{13}}{3}$ Alternative Method (b) $3x^2 - 8x + 1 = 0$ $x^2 - \frac{8}{3}x + \frac{1}{3} = 0$ $x^2 - \frac{8}{3}x + \left(-\frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right)^2 + \frac{1}{3} = 0$ $\left(x - \frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right)^2 + \frac{1}{3} = 0$ $\left(x - \frac{4}{3}\right)^2 = \frac{13}{9}$ $x - \frac{4}{3} = \pm\frac{\sqrt{13}}{3} \Rightarrow x = \frac{4}{3} \pm \frac{\sqrt{13}}{3}$
(c) $-x^2 - 6x + 1 = 0$ $x = \frac{6 \pm \sqrt{(-6)^2 - 4(-1)(1)}}{2(-1)}$ $x = \frac{6 \pm \sqrt{40}}{-2}$ $x = \frac{6 \pm 2\sqrt{10}}{-2}$ $x = -3 - \sqrt{10}$ or $x = -3 + \sqrt{10}$	
(d) Using GC, $x = 5.14$ or $x = 1.36$	

Answer

1	(a) $x = 1\frac{1}{2}$ or $x = 4$, (b) $x = \frac{4 \pm \sqrt{13}}{3}$ (c) $x = -(3 \pm \sqrt{10})$ (d) $x = 5.14$ or $x = 1.36$