Question 1 – Topic : Numbers and Proofs

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| **1.** | Let $f(n)=13^{n+1}(6n+7)-1$. By considering $f(k+1)-f(k)$, where $k\in\mathbb{Z}^+$, prove using Mathematical Induction that $f(n)$ is divisible by 18 for every positive integer $n$. | [6] |

**Solution**

Let $P_n$ be the statement "$f(n)$ is divisible by 18", where $n\in\mathbb{Z}^+$.

correct $P_n$, Induction Hypothesis and Inductive Step

When $n=1$,

$f(1)=13^2(6+7)-1=2196=18(122)$ which is divisible by 18.

$\therefore$ $P_1$ is true.

show base case $P_1$ is true

Assume P($k$) is true for some $k\in\mathbb{Z}^+$

i.e. $f(k)=13^{k+1}(6k+7)-1$ is divisible by 18

i.e. $f(k)=18m$ for some $m\in\mathbb{Z}$

To show P($k$+1) is true, i.e. $f(k+1)=18p$ for some $p\in\mathbb{Z}$

Consider

$$
\begin{aligned}
f(k+1)-f(k) &= 13^{k+2}\left[6(k+1)+7\right]-1-\left[13^{k+1}(6k+7)-1\right]\\
&= 13^{k+2}(6k+13)-13^{k+1}(6k+7)\\
&= 13^{k+1}(78k+169-6k-7)\\
&= 13^{k+1}(72k+162)\\
&= 18\left(13^{k+1}\right)(4k+9)
\end{aligned}
$$

form expression for $f(k+1)-f(k)$

$$
\begin{aligned}
f(k+1) &= f(k)+18\left(13^{k+1}\right)(4k+9)\\
&= 18m+18\left(13^{k+1}\right)(4k+9)\\
&= 18p,
\end{aligned}
$$

apply induction hypothesis

where $p=m+\left(13^{k+1}\right)(4k+9)\in\mathbb{Z}$

$\therefore$ $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true

Since $P_1$ is true, and $P_k$ is true $\Rightarrow$ $P_{k+1}$ is true, we conclude by Mathematical Induction that $P_n$ is true for all $n\in\mathbb{Z}^+$.

for proper conclusion

Question 2 – Topic : Functions and Graphs

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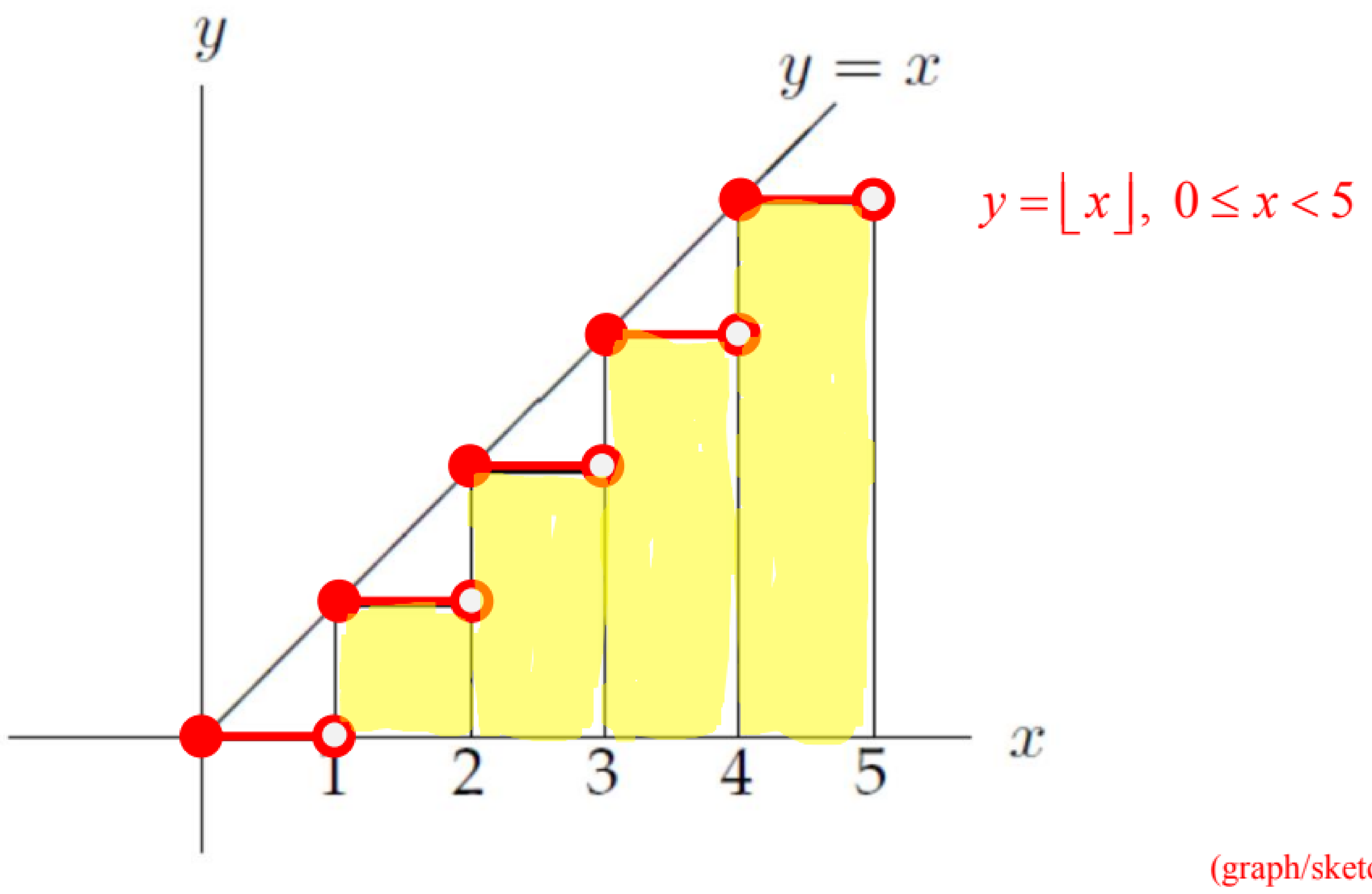
| **2.** | **(a)** | For any real number $x$, the largest integer less than or equal to $x$ is denoted by $\lfloor x \rfloor$.<br>For example, $\lfloor 3.7 \rfloor = 3$ and $\lfloor 4 \rfloor = 4$. | | |
| | | **(i)** | Use a sketch graph of $y = \lfloor x \rfloor$ for $0 \le x < 5$ to evaluate $\int_0^5 \lfloor x \rfloor \mathrm{d}x$. | [2] |
| | | | **Solution**<br><br>(graph/sketch)<br>$\int_0^5 \lfloor x \rfloor \mathrm{d}x =$ Sum of areas of the rectangles shown<br>$= 0 + 1 + 2 + 3 + 4$<br>$= 10$ | |

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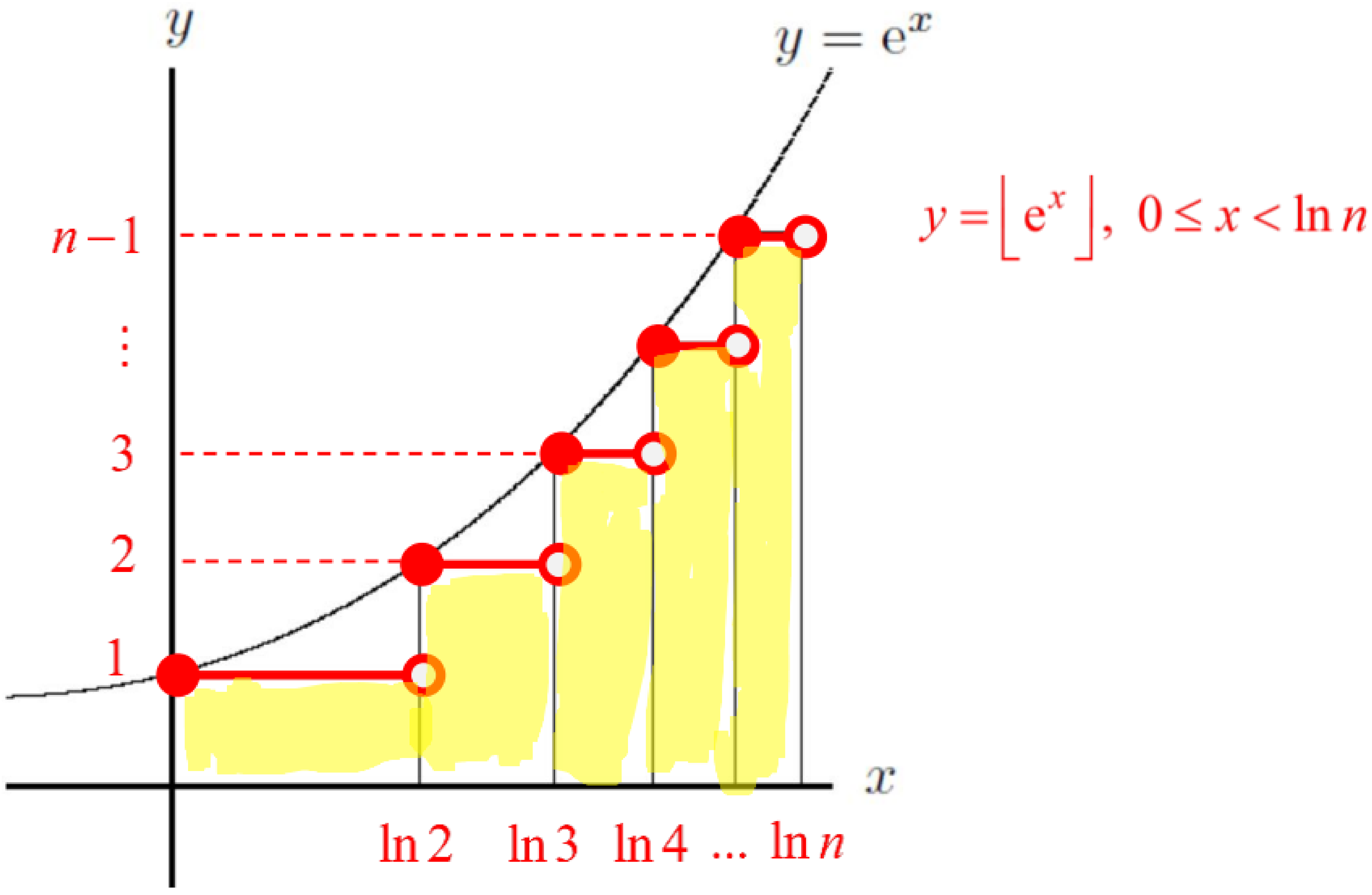
| **2.** | **(a)** | **(ii)** | Use a sketch graph of $y=\lfloor e^x \rfloor$ for $0 \le x < \ln n$, where $n$ is an integer, to show that $\int_0^{\ln n} \lfloor e^x \rfloor \, dx = n\ln n - \ln(n!)$. | [3] |

**Solution**




(graph/sketch)

$\int_0^{\ln n} \lfloor e^x \rfloor \, dx$

$=$ Sum of areas of the rectangles shown

$= 1(\ln 2 - \ln 1) + 2(\ln 3 - \ln 2) + 3(\ln 4 - \ln 3) + \ldots + (n-1)\big(\ln n - \ln(n-1)\big)$

$= \quad -\ln 1 \quad -\ln 2 \quad -\ln 3 \quad \ldots + \quad -\ln(n-1)$

$-\ln n$

$+n\ln n$

$= \quad -\ln(n!) \ + n\ln n$ (shown)

| 2. | (a) | (iii) | Hence, show that $n! \geq n^n e^{1-n}$. | [3] |
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| | | | **Solution**<br>Since $\lfloor e^x \rfloor \leq e^x$,<br>$\int_0^{\ln n} \lfloor e^x \rfloor dx \leq \int_0^{\ln n} e^x dx$<br>$\Rightarrow \; -\ln(n!) + n\ln n \leq \left[ e^x \right]_{x=0}^{x=\ln n}$<br>$= e^{\ln n} - e^0$<br>$= n - 1$ (evaluate definite integral)<br>$\Rightarrow \; \ln(n!) \geq n\ln n + 1 - n$<br>$\ln(n!) \geq \ln n^n + \ln e^{1-n}$<br>$\ln(n!) \geq \ln(n^n e^{1-n})$ (rewrite and take exponential)<br>$n! \geq n^n e^{1-n}$ (shown) | |

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| **2.** | **(b)** | Find the exact value of the integral $\int_{-2}^{2} \sqrt{9-x^2}\left(x^3 \cos\frac{x}{2} + \frac{1}{2}\right) \mathrm{d}x$. | [4] |

**Solution**

$$\int_{-2}^{2} \sqrt{9-x^2}\left(x^3 \cos\frac{x}{2} + \frac{1}{2}\right) \mathrm{d}x$$

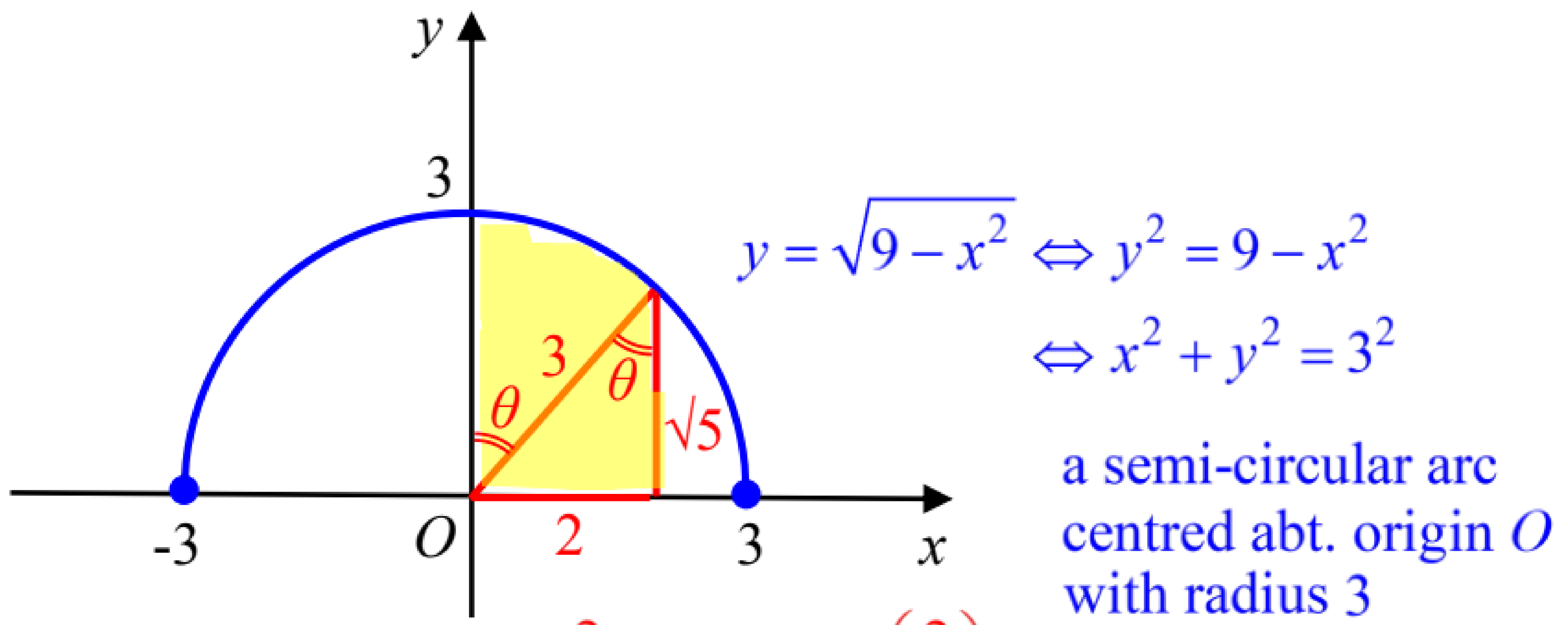
$$= \underbrace{\int_{-2}^{2} \sqrt{9-x^2}\left(x^3 \cos\frac{x}{2}\right) \mathrm{d}x}_{I_1} + \frac{1}{2}\underbrace{\int_{-2}^{2} \sqrt{9-x^2}\, \mathrm{d}x}_{I_2}$$ (break down integral)

Let $\mathrm{f}(x) = \sqrt{9-x^2}\left(x^3 \cos\frac{x}{2}\right)$. Then,

$$\mathrm{f}(-x) = \sqrt{9-(-x)^2}\left((-x)^3 \cos\frac{-x}{2}\right)$$

$$= -\sqrt{9-x^2}\left(x^3 \cos\frac{x}{2}\right) = -\mathrm{f}(x)$$

Since the integrand in $I_1$ is an odd function, $\therefore I_1 = 0$. (use of odd function)




$\sin\theta = \frac{2}{3}, \quad \theta = \sin^{-1}\left(\frac{2}{3}\right)$ (use of area of semi-circle)

$$I_2 = \int_{-2}^{2} \sqrt{9-x^2}\, \mathrm{d}x$$

$$= 2\int_{0}^{2} \sqrt{9-x^2}\, \mathrm{d}x$$

$$= 2\left[(\text{Area of sector}) + (\text{Area of right-}\perp\Delta)\right]$$

$$= 2\left[\frac{1}{2}(3^2)\sin^{-1}\left(\frac{2}{3}\right) + \frac{1}{2}(2)\sqrt{5}\right]$$

$$= 2\left[\frac{9}{2}\sin^{-1}\left(\frac{2}{3}\right) + \sqrt{5}\right]$$

$\therefore$ Required integral $= I_1 + \frac{1}{2}I_2 = \frac{9}{2}\sin^{-1}\left(\frac{2}{3}\right) + \sqrt{5}$



**[Turn over**

Question 3 – Topic : Counting

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| **3.** | An $n$-digit number uses only digits 1, 2 and 3. It does not contain any occurrence of '12' or '21'. Let there be $T_n$ such numbers, with $X_n$ of these having first digit 1 and $Y_n$ having first digit 3. | | |
| | **(i)** | Prove that, for $n \geq 2$,<br>**(a)** $X_n = X_{n-1} + Y_{n-1}$,<br>**(b)** $Y_n = 2X_{n-1} + Y_{n-1}$. | [1]<br>[2] |
| | | **Solution**<br>**(a)** To count $X_n$, consider 2 cases : (explain 2 cases)<br>• the 2nd digit is 1 so there are $X_{n-1}$ ways to write the 2nd to $n$th digits<br>• the 2nd digit is 3 so there are $Y_{n-1}$ ways to write the 2nd to $n$th digits<br>$X_n = X_{n-1} + Y_{n-1}$ (by Addition Principle)<br><br>**(b)** Define $Z_n$ to be the number of $n$-digit numbers from $T_n$ with first digit 2.<br>By symmetry, $Z_n = X_n$<br><br>To count $Y_n$, consider 3 cases : (explain 3 cases)<br>• the 2nd digit is 1 so there are $X_{n-1}$ ways to write the $2^{\text{nd}}$ to $n^{\text{th}}$ digits<br>• the 2nd digit is 2 so there are $Z_{n-1}$ ways to write the $2^{\text{nd}}$ to $n^{\text{th}}$ digits<br>• the 2nd digit is 3 so there are $Y_{n-1}$ ways to write the $2^{\text{nd}}$ to $n^{\text{th}}$ digits<br><br>$Y_n = X_{n-1} + Y_{n-1} + Z_{n-1}$ (by Addition Principle)<br>$= 2X_{n-1} + Y_{n-1}$ (by Symmetry Principle) (use symmetry principle to deduce answer) | |
| **3.** | **(ii)** | Hence, find a recurrence relation for $Y_{n+1}$ in terms of $Y_n$ and $Y_{n-1}$ for $n \geq 2$. | [2] |
| | | **Solution**<br>Using **(b)**, $Y_n = 2X_{n-1} + Y_{n-1}$<br>$\Rightarrow X_{n-1} = \frac{Y_n - Y_{n-1}}{2}$ and $X_n = \frac{Y_{n+1} - Y_n}{2}$<br>Using **(a)**, $X_n = X_{n-1} + Y_{n-1}$<br>$\frac{Y_{n+1} - Y_n}{2} = \frac{Y_n - Y_{n-1}}{2} + Y_{n-1}$ (expression entirely in $Y$)<br>$Y_{n+1} - Y_n = Y_n - Y_{n-1} + 2Y_{n-1}$<br>$Y_{n+1} = 2Y_n + Y_{n-1}$ | |

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| **3.** | **(iii)** | Prove that $Y_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n$ for $n \in \mathbb{Z}^+$. | [5] |

**Solution**

Let $P(n)$ be the statement that $Y_n = \frac{1}{2}\left(1+\sqrt{2}\right)^n + \frac{1}{2}\left(1-\sqrt{2}\right)^n$ for $n \in \mathbb{Z}^+$.

The 1-digit number starting with 3 is 3, and
the 2 digit numbers starting with 3 is 31, 32, 33. (explain LHS)

$Y_1 = 1$ and $\frac{1}{2}\left(1+\sqrt{2}\right)^1 + \frac{1}{2}\left(1-\sqrt{2}\right)^1 = 1$

$Y_2 = 3$ and $\frac{1}{2}\left(1+\sqrt{2}\right)^2 + \frac{1}{2}\left(1-\sqrt{2}\right)^2 = \frac{1}{2}\left(3+2\sqrt{2}\right) + \frac{1}{2}\left(3-2\sqrt{2}\right) = 3$

Therefore, $P(1)$ and $P(2)$ are true. (2 base cases)

Suppose $P(k)$ and $P(k+1)$ are true for some $k \in \mathbb{Z}^+$:

$Y_k = \frac{1}{2}\left(1+\sqrt{2}\right)^k + \frac{1}{2}\left(1-\sqrt{2}\right)^k$

and $Y_{k+1} = \frac{1}{2}\left(1+\sqrt{2}\right)^{k+1} + \frac{1}{2}\left(1-\sqrt{2}\right)^{k+1}$ (induction hypothesis)

We want to prove $P(k+2)$ is true: $Y_{k+2} = \frac{1}{2}\left(1+\sqrt{2}\right)^{k+2} + \frac{1}{2}\left(1-\sqrt{2}\right)^{k+2}$

$$
\begin{aligned}
Y_{k+2} &= 2Y_{k+1} + Y_k \\
&= \left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1} + \frac{1}{2}\left(1+\sqrt{2}\right)^k + \frac{1}{2}\left(1-\sqrt{2}\right)^k \\
&= \frac{1}{2}\left(1+\sqrt{2}\right)^k\left(2+2\sqrt{2}+1\right) + \frac{1}{2}\left(1-\sqrt{2}\right)^k\left(2-2\sqrt{2}+1\right) \\
&= \frac{1}{2}\left(1+\sqrt{2}\right)^k\left(1+\sqrt{2}\right)^2 + \frac{1}{2}\left(1-\sqrt{2}\right)^k\left(1-\sqrt{2}\right)^2 \\
&= \frac{1}{2}\left(1+\sqrt{2}\right)^{k+2} + \frac{1}{2}\left(1-\sqrt{2}\right)^{k+2}
\end{aligned}
$$

(induction step)

Since $P(1)$ and $P(2)$ are true, and
(for $k \in \mathbb{Z}^+$, $P(k)$ and $P(k+1)$ are true $\Rightarrow P(k+2)$ is true),
by the principle of mathematical induction, (for $n \in \mathbb{Z}^+, P(n)$ is true).

(conclusion)

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| | | Alternative inductive step<br>$Y_{k+2} = 2Y_{k+1} + Y_k$<br>$= \left(1+\sqrt{2}\right)^{k+1} + \left(1-\sqrt{2}\right)^{k+1} + \frac{1}{2}\left(1+\sqrt{2}\right)^{k} + \frac{1}{2}\left(1-\sqrt{2}\right)^{k}$<br>$= \left(1+\sqrt{2}\right)^{k+1}\left(1+\frac{1}{2\left(1+\sqrt{2}\right)}\right) + \left(1-\sqrt{2}\right)^{k+1}\left(1+\frac{1}{2\left(1-\sqrt{2}\right)}\right)$<br>$= \left(1+\sqrt{2}\right)^{k+1}\left(1+\frac{-1+\sqrt{2}}{2}\right) + \left(1-\sqrt{2}\right)^{k+1}\left(1-\frac{1+\sqrt{2}}{2}\right)$<br>$= \left(1+\sqrt{2}\right)^{k+1}\left(\frac{1+\sqrt{2}}{2}\right) + \left(1-\sqrt{2}\right)^{k+1}\left(\frac{1-\sqrt{2}}{2}\right)$<br>$= \frac{1}{2}\left(1+\sqrt{2}\right)^{k+2} + \frac{1}{2}\left(1-\sqrt{2}\right)^{k+2}$ | |

| 3. | (iv) | Find and simplify an expression for $T_n$ for $n \in \mathbb{Z}^+$. | [2] |
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| | | **Solution**<br>$T_n = X_n + Z_n + Y_n$<br>$= 2X_n + Y_n$<br>$= 2\left(\frac{Y_{n+1} - Y_n}{2}\right) + Y_n$ (symmetry or use (ii))<br>$= Y_{n+1}$<br>$= \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1} + \frac{1}{2}\left(1-\sqrt{2}\right)^{n+1}$<br><br>Alternative solution<br>For any $n \in \mathbb{Z}^+$, $n \geq 2$,<br>$Y_n = X_{n-1} + Y_{n-1} + Z_{n-1}$ (by Addition Principle, from (ib))<br>$= T_{n-1}$<br>$\therefore T_n = Y_{n+1}$<br>$= \frac{1}{2}\left(1+\sqrt{2}\right)^{n+1} + \frac{1}{2}\left(1-\sqrt{2}\right)^{n+1}$ | |

Question 4 – Topic : Functions and Graphs

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| **4.** | The Bernoulli polynomials, $\mathrm{B}_n(x)$, where $n$ = 0, 1, 2, …, are defined by $\mathrm{B}_0(x)=1$ and, for $n \geq 1$, $\dfrac{\mathrm{dB}_n(x)}{\mathrm{d}x} = n\mathrm{B}_{n-1}(x)$ and $\int_0^1 \mathrm{B}_n(x)\mathrm{d}x = 0$. | | |
| | **(i)** | Show that $\mathrm{B}_4(x) = x^2(x-1)^2 + A$, where $A$ is a constant (that need not be evaluated). | [4] |

**Solution**

$\mathrm{B}_0(x)=1$

$$\frac{\mathrm{dB}_1(x)}{\mathrm{d}x} = \mathrm{B}_0(x) = 1 \quad \Rightarrow \quad \mathrm{B}_1(x) = x + k, \text{ where } k \text{ is a constant.}$$

$$\because \int_0^1 \mathrm{B}_1(x)\mathrm{d}x = 0, \ \left[\frac{x^2}{2} + kx\right]_{x=0}^{x=1} = 0, \ \frac{1}{2} + k = 0, \ k = -\frac{1}{2}.$$

$$\therefore \ \mathrm{B}_1(x) = x - \frac{1}{2}.$$

$$\frac{\mathrm{dB}_2(x)}{\mathrm{d}x} = 2\mathrm{B}_1(x) = 2x - 1 \quad \Rightarrow \quad \mathrm{B}_2(x) = x^2 - x + k', \text{ where } k' \text{ is a constant.}$$

$$\because \int_0^1 \mathrm{B}_2(x)\mathrm{d}x = 0, \ \left[\frac{x^3}{3} - \frac{x^2}{2} + k'x\right]_{x=0}^{x=1} = 0, \ \frac{1}{3} - \frac{1}{2} + k' = 0, \ k' = \frac{1}{6}.$$

$$\therefore \ \mathrm{B}_2(x) = x^2 - x + \frac{1}{6}.$$

$$\frac{\mathrm{dB}_3(x)}{\mathrm{d}x} = 3\mathrm{B}_2(x) = 3x^2 - 3x + \frac{1}{2} \quad \Rightarrow \quad \mathrm{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x + k'', \text{ where } k'' \text{ is a constant.}$$

$$\because \int_0^1 \mathrm{B}_3(x)\mathrm{d}x = 0, \ \left[\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} + k''x\right]_{x=0}^{x=1} = 0, \ \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + k'' = 0, \ k'' = 0.$$

$$\therefore \ \mathrm{B}_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

$$\frac{\mathrm{dB}_4(x)}{\mathrm{d}x} = 4\mathrm{B}_3(x) = 4x^3 - 6x^2 + 2x \quad \Rightarrow \quad \mathrm{B}_4(x) = x^4 - 2x^3 + x^2 + k''', \text{ where } k''' \text{ is a constant.}$$

$$\therefore \ \mathrm{B}_4(x) = x^2(x-1)^2 + A, \text{ where } A \text{ is a constant.}$$

(shown)

| 4. | (ii) | Show that, for $n \geq 2$, $\mathrm{B}_n(1) - \mathrm{B}_n(0) = 0$. | [2] |
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| | | **Solution**<br>For any integer $n \geq 2$,<br>$\mathrm{B}_n(1) - \mathrm{B}_n(0) = \left[\mathrm{B}_n(x)\right]_{x=0}^{x=1}$<br>$= \int_0^1 \frac{\mathrm{dB}_n(x)}{\mathrm{d}x}\,\mathrm{d}x$ (rewrite expression)<br>$= \int_0^1 n\mathrm{B}_{n-1}(x)\,\mathrm{d}x$<br>$= n\left(\int_0^1 \mathrm{B}_{n-1}(x)\,\mathrm{d}x\right)$ (apply conditions in the definition)<br>$= n(0) \quad \because n-1 \geq 1$<br>$= 0$ (shown). | |

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| **4.** | **(iii)** | Show that $\mathrm{B}_n(x+1)-\mathrm{B}_n(x)=nx^{n-1}$ for all positive integers $n$. | [4] |

**Solution**

For any positive integer $n$, let $P_n$ denote the statement $\mathrm{B}_n(x+1)-\mathrm{B}_n(x)=nx^{n-1}$.

<u>Base Case</u> :

$$\begin{aligned}\text{LHS} &= \mathrm{B}_1(x+1)-\mathrm{B}_1(x)\\ &=\left[(x+1)-\frac{1}{2}\right]-\left[x-\frac{1}{2}\right]\\ &=1\end{aligned}$$

$\text{RHS}=(1)x^{1-1}=x^0=1$

$\therefore\ \mathrm{B}_1(x+1)-\mathrm{B}_1(x)=(1)x^{1-1}$. i.e. $P_1$. (show base case)

<u>Induction Step</u> :

Consider any positive integer $k$.

Suppose $P_k$. i.e. $\mathrm{B}_k(x+1)-\mathrm{B}_k(x)=kx^{k-1}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{B}_{k+1}(x+1)-\mathrm{B}_{k+1}(x)\right]$$

$$=(k+1)\ \mathrm{B}_k(x+1)\ (1)-(k+1)\ \mathrm{B}_k(x)$$

$$=(k+1)\ \left[\mathrm{B}_k(x+1)-\mathrm{B}_k(x)\right]$$

$$=(k+1)\ \left[kx^{k-1}\right]$$ (use induction hypothesis)

$$=(k+1)\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x^k\right]$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[(k+1)x^k\right]$$

$\therefore \mathrm{B}_{k+1}(x+1)-\mathrm{B}_{k+1}(x)=(k+1)x^k+C$, where $C$ is a constant

When $x=0$,

$$\underbrace{\mathrm{B}_{k+1}(1)-\mathrm{B}_{k+1}(0)}_{=0\text{ (result of part ii)}}=\underbrace{(k+1)(0)^k}_{=0}+C$$ (complete induction step)

$\Rightarrow\quad C=0$

$\therefore \mathrm{B}_{k+1}(x+1)-\mathrm{B}_{k+1}(x)=(k+1)x^k$, i.e. $P_{k+1}$.

Since $P_k$ is true, and (for any $k\in\mathbb{Z}^+$, $P_k\Rightarrow P_{k+1}$) (conclusion)

By the Principle of Mathematical Induction, (for any $n\in\mathbb{Z}^+$, $P_n$ is true).

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| **4.** | **(iv)** | Hence, for any positive integer $N$, show that $\sum_{m=1}^{N} m^{n-1} = \frac{1}{n}\left[\mathrm{B}_n(N+1) - \mathrm{B}_n(1)\right]$, and deduce that $\sum_{m=1}^{N} m^3 = \left(\frac{N(N+1)}{2}\right)^2$. | [3] |

**Solution**

$$\sum_{m=1}^{N} m^{n-1} = \frac{1}{n}\sum_{m=1}^{N} nm^{n-1}$$

$$= \frac{1}{n}\sum_{m=1}^{N}\left[\mathrm{B}_n(m+1) - \mathrm{B}_n(m)\right] \qquad \text{(rewrite summand as a difference)}$$

$$\begin{aligned} = \frac{1}{n}\Big[\ & \mathrm{B}_n(2) - \mathrm{B}_n(1) \\ & + \mathrm{B}_n(3) - \mathrm{B}_n(2) \\ & + \mathrm{B}_n(4) - \mathrm{B}_n(3) \\ & \quad \vdots \\ & + \mathrm{B}_n(N) - \mathrm{B}_n(N-1) \\ & + \mathrm{B}_n(N+1) - \mathrm{B}_n(N)\ \Big] \end{aligned} \qquad \text{(apply method of difference)}$$

$$= \frac{1}{n}\left[\mathrm{B}_n(N+1) - \mathrm{B}_n(1)\right] \qquad \text{(shown)}$$

If $n = 4$, then

$$\sum_{m=1}^{N} m^3 = \frac{1}{4}\left[\mathrm{B}_4(N+1) - \mathrm{B}_4(1)\right]$$

$$= \frac{1}{4}\left(\left[x^2(x-1)^2 + A\right]\Big|_{x=N+1} - \left[x^2(x-1)^2 + A\right]\Big|_{x=1}\right)$$

$$= \frac{1}{4}\left(\left[(N+1)^2(N)^2 + A\right] - A\right) \qquad \text{(apply } n=4\text{)}$$

$$= \frac{1}{4}\left[N(N+1)\right]^2$$

$$= \left[\frac{N(N+1)}{2}\right]^2 \qquad \text{(deduced)}$$

Question 5 – Topic : Counting

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| **5.** | **(a)** | New Chang Lee sells 5 types of puffs: curry, sardine, black-pepper chicken, tuna and yam.<br><br>Mr. Ong wants to order a total of 26 puffs such that each type is included and there is an even number of puffs of each type. Find the number of ways he can make the order. | [4] |
| | | **Solution**<br><br>This is equivalent to distributing 26 identical objects into 5 distinct boxes, such that each box has some object and an even number of objects. (describe distribution – problem identical object, distinct boxes)<br><br>Which is equivalent to distributing 13 pairs of identical objects into 5 distinct boxes, such that each box has at least 1 pair of objects. (address even)<br><br>Number of ways $=\binom{13-1}{5-1}$ (expression)<br><br>$=495$ | |
| **5.** | **(b)** | 4 married couples are randomly seated at a round table with 8 chairs.<br>By using the principle of inclusion and exclusion, find the probability that no wife sits next to her husband. | [4] |
| | | **Solution**<br><br>Total no. of circular arrangements without restrictions $=7!=5040$ (total)<br><br>Let $A_i$ be the event where the $i^{\text{th}}$ couple is sitting together, for $i=1,2,3,4$.<br><br>No. of circular arrangements where at least one couple sits together,<br>$=\left\|A_1\cup A_2\cup A_3\cup A_4\right\|$<br><br>$=\sum_{1\le i\le 4}\left\|A_i\right\|-\sum_{1\le i<j\le 4}\left\|A_i\cap A_j\right\|+\sum_{1\le i<j<k\le 4}\left\|A_i\cap A_j\cap A_k\right\|-\sum_{1\le i<j<k<m\le 4}\left\|A_i\cap A_j\cap A_k\cap A_m\right\|$<br><br>$=\binom{4}{1}\left\|A_1\right\|-\binom{4}{2}\left\|A_1\cap A_2\right\|+\binom{4}{3}\left\|A_1\cap A_2\cap A_3\right\|-\binom{4}{4}\left\|A_1\cap A_2\cap A_3\cap A_4\right\|$ (PIE)<br><br>$=\binom{4}{1}(6!\times 2^1)-\binom{4}{2}(5!\times 2^2)+\binom{4}{3}(4!\times 2^3)-\binom{4}{4}(3!\times 2^4)$ No. of circular arrangements for $(A_i)$, $(A_i\cap A_j)$, $(A_i\cap A_j\cap A_k)$ $(A_i\cap A_j\cap A_k\cap A_m)$<br><br>$=3552$<br><br>Required probability<br>$=1-\frac{3552}{5040}$<br>$=\frac{31}{105}$ | |

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| **5.** | **(c)** | Let $a_1, a_2, \ldots, a_{10}$ be a sequence of 10 natural numbers.<br>By considering the sums $a_1, a_1+a_2, \ldots, a_1+a_2+\ldots+a_{10}$, and using the pigeonhole principle, prove that there is a sequence of $n$ consecutive term(s) whose sum is divisible by 10, for some $1 \le n \le 10$. | [5] |

**Solution**

Let $S_1 = a_1$,

$S_2 = a_1 + a_2$,

$\vdots$

$S_{10} = a_1 + a_2 + \ldots + a_{10}$.

Case 1: One of the $S_i$ is divisible by 10 (Consider trivial case)

- Then $a_1, \ldots, a_i$ is the required sequence.

Case 2: None of the $S_i$ is divisible by 10.

- For all $1 \le i \le 10$, $S_i$ has a remainder of $1, 2\ldots, 9$ when divided by 10. (reduce to 9 pigeonholes)
- By pigeonhole principle, there exists $1 \le j < k \le 10$ such that (PP with 10 pigeons and 9 pigeonholes)

$$S_k \equiv S_j \pmod{10}$$
$$a_1 + \ldots + a_k \equiv (a_1 + \ldots + a_j) \pmod{10}$$
$$(a_1 + \ldots + a_k) - (a_1 + \ldots + a_j) \equiv 0 \pmod{10}$$
$$a_{j+1} + \ldots + a_k \equiv 0 \pmod{10}$$

(RHS zero)

(completing the proof)

Then $a_{j+1}, \ldots, a_k$ is the required sequence.

Question 6 (MS) – Topic : Numbers and Proofs

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| **6.** | **(a)** | Let $p$ be a prime number and $r,s \in \mathbb{Z}$ such that $0 < r,s < p$. | | |
| | | **(i)** | For any $a \in \mathbb{Z}$, show that $ra \equiv sa \pmod{p}$ if and only if $r = s$. | [3] |
| | | | **Solution**<br>$(\Leftarrow) r = s$<br>$(r-s)a = 0$<br>$(r-s)a \equiv 0 \pmod{p}$<br>$ra \equiv sa \pmod{p}$<br><br>$(\Rightarrow) ra \equiv sa \pmod{p}$<br>$p \mid (r-s)a$<br>$p \mid (r-s)$ $\because$ euclid's lemma<br>$pq = (r-s)$ for some $q \in \mathbb{Z}$<br>Since $0 < r,s < p$,<br>we have $-p < r-s < p$ implying $q = 0$<br>$\therefore r-s = 0 \Rightarrow r = s$<br><br>$\therefore ra \equiv sa \pmod{p}$ if and only if $r = s$. | |
| **6.** | **(a)** | **(ii)** | By considering the product $a \times 2a \times 3a \times \ldots \times (p-1)a$, prove that for any $a \in \mathbb{Z}$ but not divisible by $p$, $a^{p-1} \equiv 1 \pmod{p}$ (Fermat's Little Theorem). | [2] |
| | | | **Solution**<br>$a \times 2a \times 3a \times \ldots \times (p-1)a \equiv (p-1)! \pmod{p}$<br>$(p-1)!a^{p-1} \equiv (p-1)! \pmod{p}$<br>$a^{p-1} \equiv 1 \pmod{p}$ | |
| **6.** | **(a)** | **(iii)** | Hence, show that for any integers $a$, $b$, $p$ divides $ab^p - a^p b$. | [2] |
| | | | **Solution**<br>$ab^p \equiv ab^{p-1}b \equiv ab \pmod{p}$<br>$a^p b \equiv a^{p-1}ab \equiv ab \pmod{p}$ (use Fermat's little theorem for $a$ and $b$)<br>$ab^p - a^p b \equiv ab - ab \equiv 0 \pmod{p}$<br>$\therefore p \mid (ab^p - a^p b)$ | |
| **6.** | **(b)** | Let $m,n,k \in \mathbb{Z}^+$ and $\gcd(m,n) = 1$. | | [7] |

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| | | Prove that $n+m \mid n^2+km^2$ if and only if $n+m \mid k+1$. | |
| | | **Solution**<br>$(\Rightarrow) n+m \mid n^2+km^2$<br>$(n+m)q = n^2+km^2$ for some $q \in \mathbb{Z}$<br>$(n+m)q+m^2-n^2 = (k+1)m^2$<br>$(n+m)(q+m-n) = (k+1)m^2$<br>$\therefore (n+m) \mid (k+1)m^2$<br>Note that $\gcd(n+m,m) = \gcd(n,m) = 1$<br>By Euclid's lemma, $(n+m) \mid (k+1)$<br><br>$(\Leftarrow)(n+m) \mid k+1$<br>we have $(n+m) \mid km^2+m^2$<br>and $(n+m) \mid (n+m)(n-m) = n^2-m^2$<br>$\Rightarrow (n+m) \mid km^2+m^2+n^2-m^2 = n^2+km^2$ | |

Question 7 (MS) – Topic : Inequalities

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| **7.** | **(a)** | **(i)** | Show that $x+\frac{1}{y}\geq 2\sqrt{\frac{x}{y}}$ for $x,\ y>0$. | [1] |
| | | | **Solution**<br><br>For any $x,y\in\mathbb{R}^{+}$, $\frac{1}{y}\in\mathbb{R}^{+}$.<br>By the AM-GM inequality,<br>$\frac{x+\frac{1}{y}}{2}\geq\sqrt{x\left(\frac{1}{y}\right)}$ (apply AM-GM ineq.)<br>$x+\frac{1}{y}\geq 2\sqrt{\frac{x}{y}}$ (shown) | |
| **7.** | **(a)** | **(ii)** | Hence, show that<br>$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\ldots\left(x_{49}+\frac{1}{x_{50}}\right)\left(x_{50}+\frac{1}{x_1}\right)\geq 2^{50}$, for $x_1,\ x_2,\ x_3,\ \ldots,\ x_{50}>0$. | [2] |
| | | | **Solution**<br><br>For any $x_1,\ x_2,\ x_3,\ \ldots,\ x_{50}>0$,<br>Using part (i), $x_i+\frac{1}{x_j}\geq 2\sqrt{\frac{x_i}{x_j}},\ i\in\mathbb{Z}^{+},\ i,j\geq 1$.<br>$\text{LHS}=\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\ldots\left(x_{49}+\frac{1}{x_{50}}\right)\left(x_{50}+\frac{1}{x_1}\right)$<br>$\geq\left(2\sqrt{\frac{x_1}{x_2}}\right)\left(2\sqrt{\frac{x_2}{x_3}}\right)\ldots\left(2\sqrt{\frac{x_{49}}{x_{50}}}\right)\left(2\sqrt{\frac{x_{50}}{x_1}}\right)$ [use result of part (ai)]<br>$=2^{50}\sqrt{\frac{x_1x_2x_3\ldots x_{50}}{x_2x_3\ldots x_{50}x_1}}$<br>$=2^{50}$ (shown) | |

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| **7.** | **(a)** | **(iii)** | Deduce the positive solutions of the system of equations $$x_1+\frac{1}{x_2}=8$$ $$x_2+\frac{1}{x_3}=\frac{1}{2}$$ $$x_3+\frac{1}{x_4}=8$$ $$\vdots$$ $$x_{49}+\frac{1}{x_{50}}=8$$ $$x_{50}+\frac{1}{x_1}=\frac{1}{2}.$$ | [6] |
| | | | **Solution** $$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\cdots\left(x_{49}+\frac{1}{x_{50}}\right)\left(x_{50}+\frac{1}{x_1}\right)=8^{25}\times\left(\frac{1}{2}\right)^{25}$$ [connect part (ii)] $$=2^{75-25}=2^{50}$$ For the inequality of (ii) to hold at equality, it must be that these inequalities hold at equality : $$\begin{array}{cccc} x_1+\frac{1}{x_2}\ge 2\sqrt{\frac{x_1}{x_2}} & x_1=\frac{1}{x_2} & x_1+\frac{1}{x_2}=2x_1=8 & x_1=4 \\ x_2+\frac{1}{x_3}\ge 2\sqrt{\frac{x_2}{x_3}} & x_2=\frac{1}{x_3} & x_2+\frac{1}{x_3}=2x_2=\frac{1}{2} & x_2=\frac{1}{4} \\ \vdots \quad \Rightarrow & \vdots \quad \Rightarrow & \vdots \quad \Rightarrow & \vdots \\ x_{49}+\frac{1}{x_{50}}\ge 2\sqrt{\frac{x_{49}}{x_{50}}} & x_{49}=\frac{1}{x_{50}} & x_{49}+\frac{1}{x_{50}}=2x_{49}=8 & x_{49}=4 \\ x_{50}+\frac{1}{x_1}\ge 2\sqrt{\frac{x_{50}}{x_1}} & x_{50}=\frac{1}{x_1} & x_{50}+\frac{1}{x_1}=2x_{50}=\frac{1}{2} & x_{50}=\frac{1}{4} \end{array}$$ Sub $x_1=4,\ x_{50}=\frac{1}{4}$ into $x_{50}+\frac{1}{x_1}=\frac{1}{2}$ <br> Then $\text{LHS}=x_{50}+\frac{1}{x_1}=\frac{1}{2}=\text{RHS}$ (checked) | |

| 7. | (b) | (i) | Let $x, y, z$ be positive real numbers satisfying $xyz = 1$.<br>Determine, with proof, the minimum value of<br>$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}.$$ | [4] |
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**Solution**

From Cauchy-Schwarz inequality, we have

$$\left[\left(\frac{x}{\sqrt{y+z}}\right)^2+\left(\frac{y}{\sqrt{z+x}}\right)^2+\left(\frac{z}{\sqrt{x+y}}\right)^2\right]\left[\left(\sqrt{y+z}\right)^2+\left(\sqrt{z+x}\right)^2+\left(\sqrt{x+y}\right)^2\right]\ge(x+y+z)^2$$

$$\left[\left(\frac{x}{\sqrt{y+z}}\right)^2+\left(\frac{y}{\sqrt{z+x}}\right)^2+\left(\frac{z}{\sqrt{x+y}}\right)^2\right](y+z+z+x+x+y)\ge(x+y+z)^2$$

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\ge\frac{(x+y+z)^2}{2(x+y+z)}=\frac{x+y+z}{2}$$

From AM – GM inequality,

$$\frac{1}{2}(x+y+z)\ge\frac{1}{2}\left(3\sqrt[3]{xyz}\right)$$
$$=\frac{3}{2}$$

Hence,

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\ge\frac{3}{2}$$

Equality holds when $x=y=z$.

$\Rightarrow xyz=x^3=1,\quad x=y=z=1.$

Check : $\left.\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\right|_{x=y=z=1}=\frac{1^2}{1+1}+\frac{1^2}{1+1}+\frac{1^2}{1+1}=\frac{3}{2}$

Hence, the minimum value of $\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}$ is $\frac{3}{2}$.

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| **7.** | **(b)** | **(ii)** | Hence, prove that if $a,b,c$ are positive real numbers satisfying $abc=1$, then $$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$ | [2] |
| | | | **Solution**<br>Take $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$.<br>$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=\frac{1}{a^2\left(\frac{1}{b}+\frac{1}{c}\right)}+\frac{1}{b^2\left(\frac{1}{c}+\frac{1}{a}\right)}+\frac{1}{c^2\left(\frac{1}{a}+\frac{1}{b}\right)}$$ $$=\frac{1}{a^2\left(\frac{b+c}{bc}\right)}+\frac{1}{b^2\left(\frac{c+a}{ac}\right)}+\frac{1}{c^2\left(\frac{a+b}{ab}\right)}$$ $$=\frac{abc}{a^3(b+c)}+\frac{abc}{b^3(c+a)}+\frac{abc}{c^3(a+b)}$$ $$=\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$ Also, $xyz=\frac{1}{a}\frac{1}{b}\frac{1}{c}=\frac{1}{abc}=1$ (Multiply *abc* and use *abc* = 1) | |

Question 8 (MS) – Topic : Numbers and Proof

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| **8.** | **(a)** | For each positive integer $r$, let $$a_r = \frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots,$$ $$b_r = \frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots.$$ | | |
| | | **(i)** | Find $b_r$ in terms of $r$. | [2] |
| | | | **Solution** $$b_r = \underbrace{\frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots}_{\text{Geometric Series}}$$ $$= \frac{\frac{1}{r+1}}{1-\frac{1}{r+1}} \quad \text{(sum-to-infinity of GP)}$$ $$= \frac{1}{r+1} \times \frac{r+1}{r}$$ $$= \frac{1}{r}$$ | |
| **8.** | **(a)** | **(ii)** | Deduce that $0 < a_r < \frac{1}{r}$. | [2] |
| | | | **Solution** Since $r > 0$, it is clear that $a_r > 0$. $$\frac{1}{(r+1)(r+2)} < \frac{1}{(r+1)^2}$$ $$\frac{1}{(r+1)(r+2)(r+3)} < \frac{1}{(r+1)^3}$$ $$\dots$$ $$\therefore \underbrace{\frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots}_{a_r} < \underbrace{\frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots}_{b_r}$$ $$\therefore \quad a_r < b_r = \frac{1}{r}$$ Thus $0 < a_r < \frac{1}{r}$. (shown) | |

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| **8.** | **(a)** | **(iii)** | Show that $a_r = r!\mathrm{e} - \lfloor r!\mathrm{e} \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of $x$. | [3] |
| | | | **Solution**<br><br>For $r \in \mathbb{Z}^+$,<br>$\mathrm{e} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \frac{1}{(r+1)!} + \frac{1}{(r+2)!} \ldots,$<br>$r!\mathrm{e} = r! + r! + \frac{r!}{2!} + \frac{r!}{3!} + \ldots + \frac{r!}{r!} + \frac{r!}{(r+1)!} + \frac{r!}{(r+2)!} + \ldots$<br>$= r! + r! + \frac{r!}{2!} + \frac{r!}{3!} + \ldots + \frac{r!}{r!} + \left[ \frac{1}{(r+1)} + \frac{1}{(r+2)(r+1)} + \ldots \right]$<br>$= r! + r! + \frac{r!}{2!} + \frac{r!}{3!} + \ldots + \frac{r!}{r!} + a_r$<br>As $0 < a_r < \frac{1}{r} \leq 1$, $0 < a_r < 1$,<br>$r! + r! + \frac{r!}{2!} + \frac{r!}{3!} + \ldots + \frac{r!}{r!}$ is the integer part of $r!\mathrm{e}$. [apply result of (ii)]<br>$r! + r! + \frac{r!}{2!} + \frac{r!}{3!} + \ldots + \frac{r!}{r!} = \lfloor r!\mathrm{e} \rfloor$<br>$\therefore r!\mathrm{e} = \lfloor r!\mathrm{e} \rfloor + a_r$<br>$a_r = r!\mathrm{e} - \lfloor r!\mathrm{e} \rfloor$ (shown) | |
| **8.** | **(a)** | **(iv)** | Hence show that e is irrational. | [3] |
| | | | **Solution**<br><br><u>Assume</u> to the contrary that e is rational, (proof by contradiction)<br>i.e. there exist positive integers $k$, $m$ such that $\mathrm{e} = \frac{k}{m}$.<br>Then, $m!\mathrm{e} = \frac{k}{m} \times m! = k(m-1)! \in \mathbb{Z}$.<br>Thus $m!\mathrm{e}$ is an integer.<br>So $\lfloor m!\mathrm{e} \rfloor = m!\mathrm{e}$, $m!\mathrm{e} - \lfloor m!\mathrm{e} \rfloor = 0$, $\therefore a_m = 0$ [apply result of (aiii)]<br>But this contradicts the result in (ii) where $a_m > 0$. [apply result of (aii) and obtain a contradiction]<br>Going back to the <u>assumption</u>, $\therefore$ e must be irrational. | |

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| **8.** | **(b)** | Given that a sequence $y_1, y_2, y_3, \ldots$ is defined by $$y_1 = 2,\ y_{n+1} = \frac{y_n}{2} + \frac{1}{y_n}\ \text{, for all } n \in \mathbb{N}.$$ Show that the sequence converges.<br>Find the exact limit of the sequence. | [5] |

**Solution**

$$y_1 = 2,\ y_{n+1} = \frac{y_n}{2} + \frac{1}{y_n}$$

$$y_{n+1}^2 = \left(\frac{y_n}{2} + \frac{1}{y_n}\right)^2$$

$$= \frac{y_n^2}{4} + \frac{1}{y_n^2} + 1$$

$$= \left(\frac{y_n}{2} - \frac{1}{y_n}\right)^2 + 2$$

$$y_{n+1}^2 \geq 2$$

Since $y_{n+1} > 0$ for $n \in \mathbb{N}$, $y_{n+1} \geq \sqrt{2}$. (show sequence is bounded below)

$\therefore$ The sequence $y_1, y_2, y_3, \ldots$ is bounded below by $\sqrt{2}$.

$$y_{n+1} - y_n = \frac{1}{y_n} - \frac{y_n}{2} = \frac{2 - y_n^2}{2y_n} \leq 0 \quad (\text{as } y_n^2 \geq 2 \text{ and } y_n > 0).$$ (show sequence is decreasing)

$\therefore$ The sequence $y_1, y_2, y_3, \ldots$ is a decreasing sequence.

By Monotone Convergence Theorem, the sequence $y_1, y_2, y_3, \ldots$ converges.

(apply MCT)

Let $L = \lim_{n\to\infty} y_n$. Then

$$\lim_{n\to\infty} y_{n+1} = L \text{ , and } \lim_{n\to\infty}\left(\frac{y_n}{2} + \frac{1}{y_n}\right) = \frac{L}{2} + \frac{1}{L}$$

$$L = \frac{L}{2} + \frac{1}{L}$$

$$\frac{L}{2} - \frac{1}{L} = 0$$

$$L^2 - 2 = 0$$

$L = \pm\sqrt{2}$ (reject $L = -\sqrt{2}$ as $L \geq 0$, $\because$ the sequence $y_1, y_2, y_3, \ldots$ is positive.)

The limit is $\sqrt{2}$.