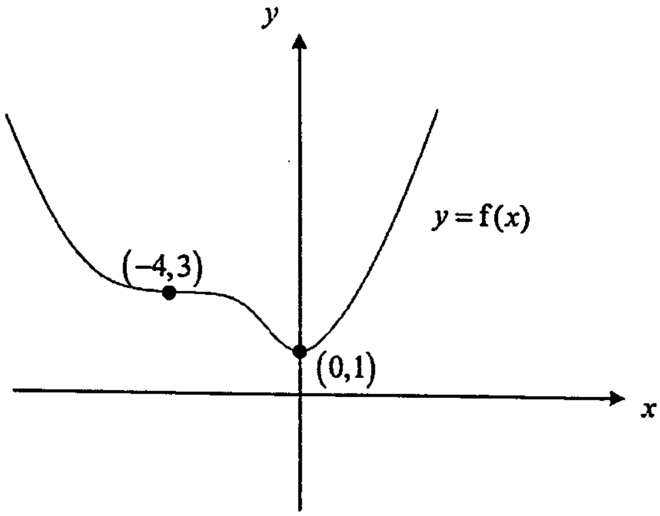
CJC H2 Mathematics 2023 Promotional Exam

1 The diagram below shows the graph of $y = \mathrm{f}(x)$. The curve has two stationary points at $(-4,3)$ and $(0,1)$.



(a) State the range of values of x for which the graph of $y = \mathrm{f}(x)$ is strictly increasing. [1]

(b) Sketch the derivative graph of f, $y = \mathrm{f}'(x)$, indicating clearly the coordinates of any points of intersection with the axes. [2]

2 (a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x} x\tan^{-1}(2x)$. [3]

(b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln\sqrt{\dfrac{x^2+1}{x}}$, leaving your answer as a single fraction. [3]

3 Plane π_1 contains point $A(7, 5, 1)$ and line l with equation $\mathbf{r} = \begin{pmatrix} -5 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \lambda \in$.

(a) Find an equation of π_1 in scalar product form. [3]

Plane π_2 has equation $x + 3y - 3z = 8$.

(b) Find the acute angle between π_1 and π_2. [2]

(c) Find the perpendicular distance from A to π_2. [2]

4 The curve C has equation $y = \dfrac{x-1}{x^2+3x}$.

(a) Sketch the graph of C, stating the exact coordinates of any points of intersection with the axes, any coordinates of turning points and any equation of asymptotes. [4]

(b) By adding a suitable line to the graph drawn in part (a) and labelling its equation, solve $\dfrac{x-1}{2x^2+6x} \geq \dfrac{1}{2}x + 1$. [4]

5 It is given that $y=\sin^{-1}x$, $-1\le x\le 1$.

(a) Show that $\frac{d^2y}{dx^2}=x\left(\frac{dy}{dx}\right)^3$. [3]

(b) Using the result in part (a), find the Maclaurin expansion for y, up to and including the term in x^3. [3]

(c) By substituting $x=0.5$ into the expansion found in part (b), find an approximate value of π. [2]

6 It is given that two lines, l and m, with equations

$$\mathbf{r}=(\mathbf{i}+3\mathbf{k})+\lambda(2\mathbf{i}-4\mathbf{j}+7\mathbf{k})$$

$$\text{and } \mathbf{r}=(2\mathbf{i}+2\mathbf{j}-\mathbf{k})+\mu(3\mathbf{i}-2\mathbf{j}+3\mathbf{k}), \text{ where } \lambda, \mu \in$$

respectively, intersect at the point P.

(a) Verify that the coordinates of P are $(-1, 4, -4)$. [1]

A point A with position vector $\mathbf{i}+3\mathbf{k}$ lies on line l. The point Q on the line m is the closest point from A to the line m.

(b) Find the length of AQ. [3]

The line n is a reflection of the line l in the line m.

(c) Find the position vector of the point of reflection of A in the line m.
Hence, find a cartesian equation of the line n. [5]

7 (a) The curve $y=f(x)$ cuts the axes at (1, 0) and (0, -2). It is given that f^{-1} exists. State, if it is possible to do so, the coordinates of the points where the following curves cut the axes.

(i) $y=f^{-1}(x)$ [1]

(ii) $y=f\left(\frac{1}{2}x\right)$ [1]

(iii) $y=f\left(\frac{1}{2}x+1\right)$ [1]

(b) Two functions g and h are defined as follows:

$$g: x\mapsto\sqrt{x+2},\ x\in\ ,\ x\ge -2,$$

$$h: x\mapsto x^2-4x+5,\ x\in\ ,\ x\ge 0.$$

(i) Show that gh exists and find the exact range of gh. [4]

(ii) Use a non-calculator method to solve $gh(x)=2x$. [3]

8 (a) A sequence $u_1, u_2, u_3, \ldots$ is defined such that $u_1=1$, $u_2=\frac{3}{4}$, $u_3=\frac{7}{16}$, and $u_{n+1}=au_n+b$, where $n\ge 1$, and a and b are rational numbers.
Find the values of a and b. Describe how the sequence behaves. [3]

(b) (i) Express $\frac{2r^3-2r+3}{r^2-1}$ in the form $Ar+\frac{B}{r-1}+\frac{C}{r+1}$, where A, B and C are constants to be determined. [2]

(ii) Hence evaluate $\sum_{r=2}^{n}\frac{2r^3-2r+3}{r^2-1}$ exactly. (You need not simplify your answer.) [5]

9 Mrs Chan wants to set up a rectangular portable playpen for her toddler in her living room. To optimise the space of the living room, she decides that one side of the playpen will be formed by the living room wall so that she only needs to construct three sides of the playpen. Mrs Chan wants to maximise the enclosed area of the playpen while keeping the total length of the three sides of the playpen, excluding the wall, fixed at 360 cm.

(a) Without the use of a calculator, find the maximum enclosed area that her toddler has within the playpen. [6]

Mrs Chan wants to increase the enclosed area that her toddler has within the playpen. To do so, she believes the playpen should be that of an isosceles triangle. Assuming that the total length of the two sides of the playpen, excluding the wall, is still 360 cm, she then aligns the base of the isosceles triangle along the wall of her living room and sets up the playpen.

(b) Using differentiation, find the base and the height of this isosceles triangular playpen such that its area is a maximum. [5]

(c) Comment on Mrs Chan's belief that area enclosed is larger when the playpen is in the shape of an isosceles triangle compared to the rectangle in part **(a)**. [1]

10 A Sports Equipment manufacturer is conducting a series of tests on tennis balls.

(a) In the first test, a tennis ball is released from rest at a height of 180 cm above the ground surface of a tennis court. It is found that the ball always rebounds to $\frac{3}{4}$ of the height from which it falls.

(i) Find an expression for the total distance the ball travels until it is about to touch the floor the $(n + 1)^{\text{th}}$ time, giving your answer in the form $A + B\left(\frac{3}{4}\right)^n$, where A and B are integers to be determined. [4]

(ii) Hence find the number of times the ball has bounced when it has travelled 12m and also the total distance it has travelled before coming to rest. [3]

(b) In the second test, the vertical distance travelled by the same tennis ball during each regular time interval is recorded and is found to follow an arithmetic progression. The distance travelled by the tennis ball in the first 0.1 seconds is shown in the table below:

Time interval (in seconds)	Distance travelled during interval (in cm)
0.00 – 0.02	0.2
0.02 – 0.04	0.6
0.04 – 0.06	1.0
0.06 – 0.08	1.4
0.08 – 0.10	1.8
...	...

Without using a calculator, show that the time taken for the tennis ball to reach the ground surface is 0.6 seconds. [4]

(c) State an assumption, in context, for the calculations in both tests to be valid. [1]

11 A curve C has parametric equations

$$x = 1 - 2t^2, \quad y = t^3 + 1, \quad \text{for } t \in \ .$$

(a) Sketch C, indicating the coordinates of the points of intersection with the axes. [3]

The line $x + y = 2$ cuts C at the points A and B where the x-coordinate of A is greater than the x-coordinate of B.

(b) Find the exact length of AB in the form $k\sqrt{2}$ where k is an integer to be determined. [3]

(c) The point M is the midpoint of AP where P is the point $\left(1-2p^2, p^3+1\right)$. Find a cartesian equation of the curve traced by M as p varies. Leave your answer in the form $(ax+b)^3 = (cy+d)^2$, where a, b, c and d are integers to be determined. [3]

(d) Show that the equation of the normal to C at the point P is

$$y = \left[\mathrm{f}(p)\right]x + p^3 + \frac{8p}{3} + 1 - \frac{4}{3p}$$

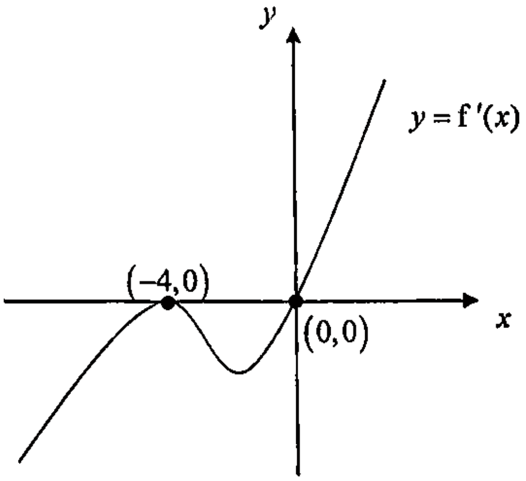
where $\mathrm{f}(p)$ is an expression in p to be determined. [3]

(e) Find the acute angle between the normals to the curve at $x = -7$. [3]

Question 1:

(a) For f to be strictly increasing, $f'(x) > 0$
Therefore, $x > 0$

(b)



Question 2:

(a) $\frac{d}{dx} x\tan^{-1}(2x)$

$= x\left(\frac{1}{1+(2x)^2}\right)(2) + \tan^{-1}(2x)(1)$

$= \frac{2x}{1+4x^2} + \tan^{-1}(2x)$

(b) $\frac{d}{dx} \ln\sqrt{\frac{x^2+1}{x}}$

$= \frac{d}{dx}\frac{1}{2}\left[\ln\left(x^2+1\right) - \ln x\right]$

$= \frac{1}{2}\left[\frac{2x}{x^2+1} - \frac{1}{x}\right]$

$= \frac{1}{2}\left[\frac{2x^2-\left(x^2+1\right)}{x\left(x^2+1\right)}\right]$

$= \frac{x^2-1}{2x\left(x^2+1\right)}$

Alternative:

$$\frac{d}{dx}\ln\sqrt{\frac{x^2+1}{x}} = \frac{\frac{1}{2}\left(\frac{x^2+1}{x}\right)^{-\frac{1}{2}}\left(\frac{x(2x)-\left(x^2+1\right)(1)}{x^2}\right)}{\left(\frac{x^2+1}{x}\right)^{\frac{1}{2}}}$$

$$= \frac{\frac{1}{2}\left(\frac{x^2+1}{x}\right)^{-\frac{1}{2}}\left(\frac{x^2-1}{x^2}\right)}{\left(\frac{x^2+1}{x}\right)^{\frac{1}{2}}}$$

$$= \frac{\frac{1}{2}\left(\frac{x^2-1}{x^2}\right)}{\left(\frac{x^2+1}{x}\right)}$$

$$= \frac{1}{2}\left(\frac{x^2-1}{x\left(x^2+1\right)}\right)$$

Question 3:

(a) Another direction vector in π_1 is $\begin{pmatrix}12\\0\\-3\end{pmatrix}$ or $\begin{pmatrix}4\\0\\-1\end{pmatrix}$ or $\begin{pmatrix}-4\\0\\1\end{pmatrix}$

$$\mathbf{n}=\begin{pmatrix}4\\0\\-1\end{pmatrix}\times\begin{pmatrix}4\\2\\1\end{pmatrix}=\begin{pmatrix}2\\-8\\8\end{pmatrix}=2\begin{pmatrix}1\\-4\\4\end{pmatrix}$$

Equation of π_1 is

$$\mathbf{r}\bullet\begin{pmatrix}1\\-4\\4\end{pmatrix}=\begin{pmatrix}7\\5\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-4\\4\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}1\\-4\\4\end{pmatrix}=-9$$

(b) acute angle between π_1 and $\pi_2 = \cos^{-1}\left|\dfrac{\begin{pmatrix}1\\-4\\4\end{pmatrix}\bullet\begin{pmatrix}1\\3\\-3\end{pmatrix}}{\sqrt{1^2+(-4)^2+4^2}\sqrt{1^2+(3)^2+(-3)^2}}\right|$

$$=23.3°\text{ or }0.406\text{ rad}$$

(c) Let B be any point on π_2, e.g (8, 0, 0)

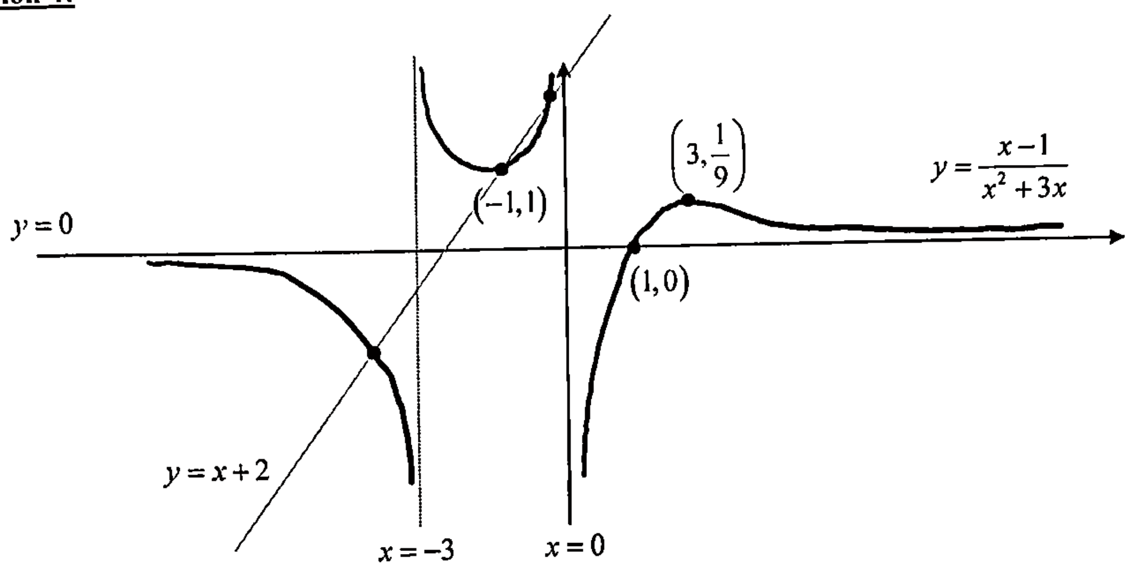
$$\overrightarrow{BA}=\begin{pmatrix}7\\5\\1\end{pmatrix}-\begin{pmatrix}8\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\5\\1\end{pmatrix}$$

perpendicular distance between A and $\pi_2 = \left|\begin{pmatrix}-1\\5\\1\end{pmatrix}\bullet\dfrac{\begin{pmatrix}1\\3\\-3\end{pmatrix}}{\sqrt{1^2+(3)^2+(-3)^2}}\right|$

$$=\frac{11}{\sqrt{19}}\approx 2.52$$

Question 4:

(a)



(b) $y=x+2$

$x\le -3.73$ or $-3<x\le -1$ or $-0.268\le x<0$

Question 5:

(a) $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)$$

$$= x\left[\left(1-x^2\right)^{-\frac{1}{2}}\right]^3$$

$$= x\left(\frac{dy}{dx}\right)^3$$

(b) $$\frac{d^3y}{dx^3} = x(3)\left(\frac{dy}{dx}\right)^2\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3$$

When $x = 0$,

$$y = 0,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = 0,\ \frac{d^3y}{dx^3} = 1$$

Hence $y = x + \frac{1}{6}x^3 + \ldots$

(c) $$\sin^{-1}(0.5) = 0.5 + \frac{1}{6}(0.5)^3 + \ldots$$

$$\frac{\pi}{6} \approx \frac{25}{48}$$

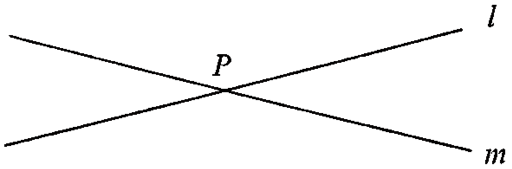
$$\pi \approx \frac{25}{8}$$

$$= 3.125$$

Question 6:

(a) $l:\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\7\end{pmatrix},\ \lambda\in\square$

$m:\mathbf{r}=\begin{pmatrix}2\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\-2\\3\end{pmatrix},\ \mu\in\square$

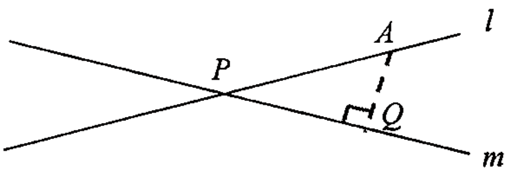


When $\lambda=-1$, $\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-4\\7\end{pmatrix}=\begin{pmatrix}-1\\4\\-4\end{pmatrix}$

When $\mu=-1$, $\begin{pmatrix}2\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\-2\\3\end{pmatrix}=\begin{pmatrix}-1\\4\\-4\end{pmatrix}$

Hence P lies on both lines and is a point of intersection.

(b)



Method 1

Length of AQ = shortest distance from A to line m

$$=\frac{\left|\overrightarrow{PA}\times\begin{pmatrix}3\\-2\\3\end{pmatrix}\right|}{\sqrt{3^2+(-2)^2+3^2}}$$

$$=\frac{\left|\left(\begin{pmatrix}1\\0\\3\end{pmatrix}-\begin{pmatrix}-1\\4\\-4\end{pmatrix}\right)\times\begin{pmatrix}3\\-2\\3\end{pmatrix}\right|}{\sqrt{9+4+9}}$$

$$=\frac{1}{\sqrt{22}}\left|\begin{pmatrix}2\\-4\\7\end{pmatrix}\times\begin{pmatrix}3\\-2\\3\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{22}}\left|\begin{pmatrix}2\\15\\8\end{pmatrix}\right|$$

$$=\sqrt{\frac{293}{22}} \text{ units}$$

Method 2

Since Q lies on the line m, $\overrightarrow{OQ}=\begin{pmatrix}2+3\mu\\2-2\mu\\-1+3\mu\end{pmatrix}$, for some $\mu\in\Box$

Since $\overrightarrow{AQ}\perp m$

$$\left(\begin{pmatrix}2+3\mu\\2-2\mu\\-1+3\mu\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}\right)\bullet\begin{pmatrix}3\\-2\\3\end{pmatrix}=0$$

$$\begin{pmatrix}1+3\mu\\2-2\mu\\-4+3\mu\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\3\end{pmatrix}=0$$

$$3+9\mu-4+4\mu-12+9\mu=0$$

$$22\mu=13$$

$$\mu=\frac{13}{22}$$

Hence, $\overrightarrow{OQ}=\begin{pmatrix}\frac{83}{22}\\\frac{9}{11}\\\frac{17}{22}\end{pmatrix}$

Hence, $=\left|\overrightarrow{AQ}\right|=\left|\begin{pmatrix}\frac{83}{22}\\\frac{18}{22}\\\frac{17}{22}\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}\right|=\left|\begin{pmatrix}\frac{61}{22}\\\frac{18}{22}\\-\frac{49}{22}\end{pmatrix}\right|=\sqrt{\left(\frac{61}{22}\right)^2+\left(\frac{18}{22}\right)^2+\left(-\frac{49}{22}\right)^2}=\sqrt{\frac{293}{22}}$

Method 3

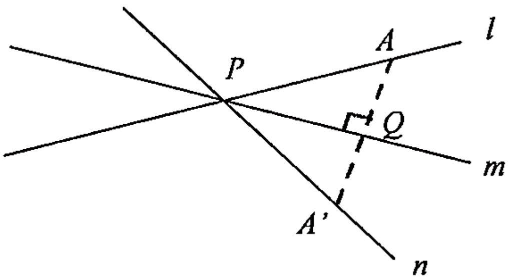
$\overrightarrow{PA}=\begin{pmatrix}2\\-4\\7\end{pmatrix}, PA=\sqrt{69}$

$\overrightarrow{PQ}=\frac{35}{22}\begin{pmatrix}3\\-2\\3\end{pmatrix}, PQ=\frac{35}{\sqrt{22}}$

Hence,

$$\left|\overrightarrow{AQ}\right| = \sqrt{PA^2 - PQ^2}$$
$$= \sqrt{69 - \left(\frac{35}{\sqrt{22}}\right)^2}$$
$$= \sqrt{\frac{293}{22}}$$

(c) **Method 1:**



Let the reflection of A in the line m be A'.

Since Q lies on the line m, $\overrightarrow{OQ} = \begin{pmatrix} 2+3\mu \\ 2-2\mu \\ -1+3\mu \end{pmatrix}$, for some $\mu \in \mathbb{R}$

Since $\overrightarrow{AQ} \perp m$

$$\left(\begin{pmatrix} 2+3\mu \\ 2-2\mu \\ -1+3\mu \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right) \bullet \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1+3\mu \\ 2-2\mu \\ -4+3\mu \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = 0$$

$$3+9\mu-4+4\mu-12+9\mu = 0$$
$$22\mu = 13$$
$$\mu = \frac{13}{22}$$

Hence, $\overrightarrow{OQ} = \begin{pmatrix} \frac{83}{22} \\ \frac{9}{11} \\ \frac{17}{22} \end{pmatrix}$

Using mid-point theorem, $\overrightarrow{OQ} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'}\right)$

Hence, $\overrightarrow{OA'} = 2\overrightarrow{OQ} - \overrightarrow{OA} = 2\begin{pmatrix} \frac{83}{22} \\ \frac{9}{11} \\ \frac{17}{22} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{72}{11} \\ \frac{18}{11} \\ -\frac{16}{11} \end{pmatrix}$

Direction vector of line n is $\overrightarrow{PA'} = \begin{pmatrix} \frac{72}{11} \\ \frac{18}{11} \\ -\frac{16}{11} \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{83}{11} \\ -\frac{26}{11} \\ \frac{28}{11} \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 83 \\ -26 \\ 28 \end{pmatrix}$

Hence, equation of line n is $\mathbf{r}=\begin{pmatrix}-1\\4\\-4\end{pmatrix}+\beta\begin{pmatrix}83\\-26\\28\end{pmatrix},\ \beta\in\mathbb{R}$ or $\mathbf{r}=\begin{pmatrix}\frac{72}{11}\\\frac{18}{11}\\-\frac{16}{11}\end{pmatrix}+\beta\begin{pmatrix}83\\-26\\28\end{pmatrix},\ \beta\in\mathbb{R}$

Cartesian equation is $\frac{x+1}{83}=\frac{4-y}{26}=\frac{z+4}{28}$

$\frac{11x-72}{83}=\frac{18-11y}{26}=\frac{11z+16}{28}$, $\frac{x-\frac{72}{11}}{83}=\frac{\frac{18}{11}-y}{26}=\frac{z+\frac{16}{11}}{28}$

Method 2: Alternative Method (using projection to find $\overrightarrow{OQ}$

$$\overrightarrow{PQ}=\left(\overrightarrow{PA}\bullet\frac{1}{\sqrt{22}}\begin{pmatrix}3\\-2\\3\end{pmatrix}\right)\frac{1}{\sqrt{22}}\begin{pmatrix}3\\-2\\3\end{pmatrix}$$

$$=\frac{1}{22}\left(\begin{pmatrix}2\\-4\\7\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\3\end{pmatrix}\right)\begin{pmatrix}3\\-2\\3\end{pmatrix}$$

$$=\frac{1}{22}(6+8+21)\begin{pmatrix}3\\-2\\3\end{pmatrix}$$

$$\overrightarrow{PQ}=\frac{35}{22}\begin{pmatrix}3\\-2\\3\end{pmatrix}$$

$$\overrightarrow{OQ}-\overrightarrow{OP}=\frac{35}{22}\begin{pmatrix}3\\-2\\3\end{pmatrix}$$

$$\overrightarrow{OQ}=\frac{35}{22}\begin{pmatrix}3\\-2\\3\end{pmatrix}+\overrightarrow{OP}=\begin{pmatrix}\frac{105}{22}\\-\frac{70}{22}\\\frac{105}{22}\end{pmatrix}+\begin{pmatrix}-1\\4\\-4\end{pmatrix}=\begin{pmatrix}\frac{83}{22}\\\frac{18}{22}\\\frac{17}{22}\end{pmatrix}$$

Using mid-point theorem, $\overrightarrow{OQ}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)$

Hence, $\overrightarrow{OA'}=2\overrightarrow{OQ}-\overrightarrow{OA}=2\begin{pmatrix}\frac{83}{22}\\\frac{9}{11}\\\frac{17}{22}\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}\frac{72}{11}\\\frac{18}{11}\\-\frac{16}{11}\end{pmatrix}$

Direction vector of line n is $\overrightarrow{PA'} = \begin{pmatrix} \frac{72}{11} \\ \frac{18}{11} \\ -\frac{16}{11} \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{83}{11} \\ -\frac{26}{11} \\ \frac{28}{11} \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 83 \\ -26 \\ 28 \end{pmatrix}$

Hence, equation of line n is $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 83 \\ -26 \\ 28 \end{pmatrix}$, $\beta \in \mathbb{R}$ or $\mathbf{r} = \begin{pmatrix} \frac{72}{11} \\ \frac{18}{11} \\ -\frac{16}{11} \end{pmatrix} + \beta \begin{pmatrix} 83 \\ -26 \\ 28 \end{pmatrix}$, $\beta \in \mathbb{R}$

Cartesian equation is $\frac{x+1}{83} = \frac{4-y}{26} = \frac{z+4}{28}$

$\frac{11x-72}{83} = \frac{18-11y}{26} = \frac{11z+16}{28}$, $\frac{x-\frac{72}{11}}{83} = \frac{\frac{18}{11}-y}{26} = \frac{z+\frac{16}{11}}{28}$

<u>Method 3:</u> <u>Find PA' Using Ratio Theorem</u>

$\overrightarrow{PQ} = \left(\overrightarrow{PA} \bullet \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} \right) \frac{1}{\sqrt{22}} \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

$= \frac{1}{22} \left(\begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} \right) \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

$= \frac{1}{22}(6+8+21) \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

$\overrightarrow{PQ} = \frac{35}{22} \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$

Using mid-point theorem, $\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{PA} + \overrightarrow{PA'}\right)$

Hence, $\overrightarrow{PA'} = 2\overrightarrow{PQ} - \overrightarrow{PA} = 2\left(\frac{35}{22}\right)\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{83}{11} \\ -\frac{26}{11} \\ \frac{28}{11} \end{pmatrix}$

Hence, equation of line n is $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 83 \\ -26 \\ 28 \end{pmatrix}$, $\beta \in \mathbb{R}$ or $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} \frac{83}{11} \\ -\frac{26}{11} \\ \frac{28}{11} \end{pmatrix}$, $\beta \in \mathbb{R}$

Cartesian equation is $\frac{x+1}{83}=\frac{4-y}{26}=\frac{z+4}{28}$

AEF: $\frac{11x-72}{83}=\frac{18-11y}{26}=\frac{11z+16}{28}$, $\frac{x-\frac{72}{11}}{83}=\frac{\frac{18}{11}-y}{26}=\frac{z+\frac{16}{11}}{28}$

Question 7:

(a) **(i)** (0,1) and (-2,0)
(ii) (2, 0) and (0, -2)
(iii) (0,0) only

(b) **(i)** By graphical method or by completing the square, $R_h = [1, \infty)$
Since $R_h = [1, \infty) \subseteq [-2, \infty)$,
gh exists

Using mapping method:
$D_{gh} = D_h = [0, \infty) \xrightarrow{h} [1, \infty) = R_h = D_g \xrightarrow{g} [\sqrt{3}, \infty) = R_{gh}$

(ii) $gh(x) = 2x \Rightarrow \sqrt{x^2 - 4x + 5 + 2} = 2x$
Squaring both sides and simplifying, we obtain

$$\Rightarrow 3x^2 + 4x - 7 = 0 \Rightarrow x = 1, -\frac{7}{3}$$

Since $D_{gh} = D_h = [0, \infty)$, x can only be 1.

Question 8:

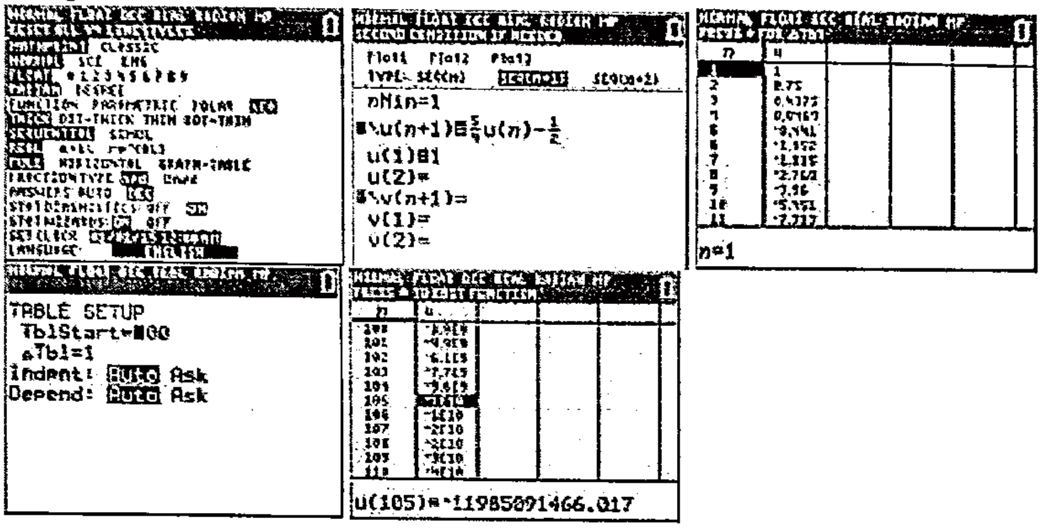
(a) $\dfrac{3}{4} = a + b$ ---------- (1)

$\dfrac{7}{16} = \dfrac{3}{4}a + b$ ---------- (2)

Solving (1) and (2) using GC, $a = \dfrac{5}{4}, b = -\dfrac{1}{2}$.

So, $u_{n+1} = \dfrac{5}{4}u_n - \dfrac{1}{2}$

Using GC:



$u_1 = 1$, $u_2 = \frac{3}{4}$, $u_3 = \frac{7}{16}$, $u_4 = \frac{3}{64}, \ldots$

The sequence decreases and diverges.

(b) **(i)** $\dfrac{2r^3 - 2r + 3}{r^2 - 1} = \dfrac{2r\left(r^2 - 1\right) + 3}{r^2 - 1} = 2r + \dfrac{3}{r^2 - 1}$

or use long-division

$$\begin{array}{r} 2r \\ r^2 - 1 \overline{\big) 2r^3 - 2r + 3} \\ (-)\ 2r^3 - 2r \quad \\ \hline 3 \end{array}$$

So, $\dfrac{2r^3 - 2r + 3}{r^2 - 1} = 2r + \dfrac{3}{r^2 - 1}$

Let $\dfrac{3}{(r-1)(r+1)} = \dfrac{B}{r-1} + \dfrac{C}{r+1}$

By cover-up rule, $B = \dfrac{3}{2}$ and $C = -\dfrac{3}{2}$

Hence, $\dfrac{2r^3 - 2r + 3}{r^2 - 1} = 2r + \dfrac{3}{2(r-1)} - \dfrac{3}{2(r+1)}$

So, $A=2, B=\frac{3}{2}$ and $C=-\frac{3}{2}$

Alternative Method

$$\frac{1}{(r-1)(r+1)}=\frac{a}{r-1}+\frac{b}{r+1}$$

$r=0, -1=-a+b$

$r=2,\ \frac{1}{3}=\ a+\frac{1}{3}b$

Solving the equations simultaneously, $a=\frac{1}{2}, b=-\frac{1}{2}$

$$\therefore \frac{1}{(r-1)(r+1)}=\frac{1}{2}\left[\frac{1}{r-1}-\frac{1}{r+1}\right].$$

$$\frac{2r^3-2r+3}{r^2-1}=2r+\frac{3}{r^2-1}$$

$$=2r+\frac{3}{2}\left[\frac{1}{r-1}-\frac{1}{r+1}\right]$$

(ii) $$\sum_{r=2}^{n}\frac{2r^3-2r+3}{r^2-1}=\sum_{r=2}^{n}\left(2r+\frac{3}{2}\left[\frac{1}{r-1}-\frac{1}{r+1}\right]\right)$$

$$=2\sum_{r=2}^{n}r+\frac{3}{2}\sum_{r=2}^{n}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$$

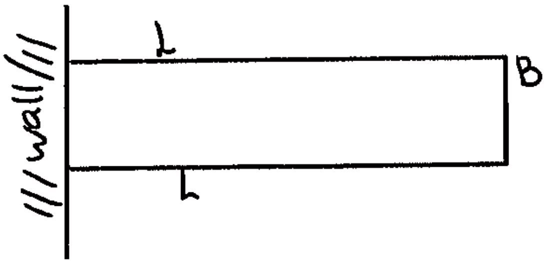
$$=2\left(\frac{n-1}{2}\right)(2+n)+\frac{3}{2}\left\{\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ \vdots\ +\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\right\}$$

$$=(n-1)(n+2)+\frac{3}{2}\left[\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right]$$

$$=(n-1)(n+2)+\frac{9}{4}-\frac{3}{2n}-\frac{3}{2(n+1)}$$

Question 9:

(a) Let the length and breadth of the playpen be L and B respectively.



Perimeter, $2L + B = 360$

$\Rightarrow B = 360 - 2L$

Area enclosed, $A = LB$

$= L(360 - 2L)$

$= -2L^2 + 360L$

$\frac{dA}{dL} = 360 - 4L$

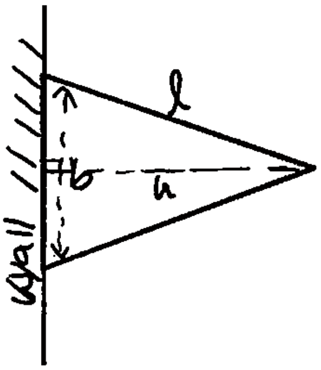
For max/min values, set $\frac{dA}{dL} = 0 \Rightarrow 360 - 4L = 0$

$\Rightarrow L = 90 \text{ cm}$

$\frac{d^2A}{dL^2} = -4 < 0$ for all L, thus when $L = 90$ cm, area is a maximum.

Thus $A = 90(360 - 2(90)) = 16200 \text{ cm}^2$

(b) Let the base, height and slant height of the playpen be b, h and l respectively.



Perimeter. $l + l = 360$

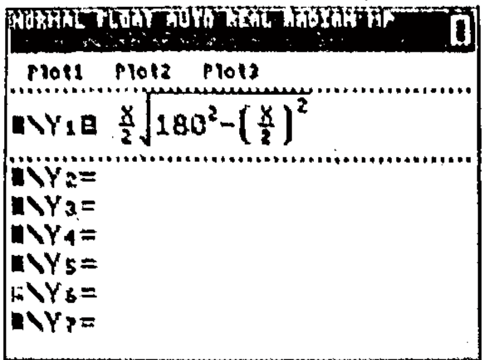
$\Rightarrow l = 180$

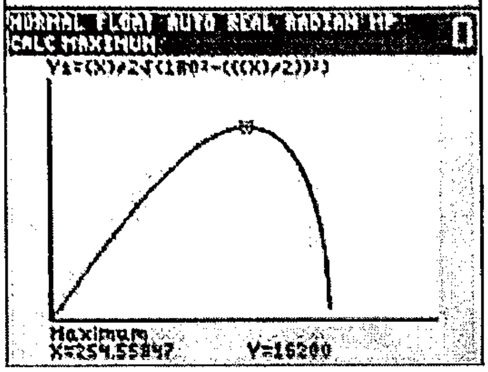
Also, $l = \sqrt{\left(\frac{b}{2}\right)^2 + h^2}$

$\Rightarrow h = \sqrt{180^2 - \left(\frac{b}{2}\right)^2}$

16

Area enclosed $= \frac{1}{2}bh = \frac{1}{2}b\sqrt{180^2 - \left(\frac{b}{2}\right)^2}$





OR via differentiation

$$\frac{\mathrm{d}A}{\mathrm{d}b} = \frac{1}{2}\sqrt{180^2 - \left(\frac{b}{2}\right)^2} + \frac{1}{2}b\left(\frac{1}{2}\right)\left(180^2 - \left(\frac{b}{2}\right)^2\right)^{-\frac{1}{2}}(-2)\left(\frac{b}{2}\right)\left(\frac{1}{2}\right)$$

$$\frac{\mathrm{d}A}{\mathrm{d}b} = \frac{\frac{1}{2}\left(180^2 - \left(\frac{b}{2}\right)^2\right) - \frac{1}{8}b^2}{\sqrt{180^2 - \left(\frac{b}{2}\right)^2}}$$

$$\frac{1}{4}b^2 = \frac{1}{2}\left(180^2\right)$$

$$b = 180\sqrt{2}$$

To determine if $b = 180\sqrt{2}$ gives a max A:

b	$\left(180\sqrt{2}\right)^-$	$b = 180\sqrt{2}$	$\left(180\sqrt{2}\right)^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	> 0	0	< 0
Tangent to curve	/	—	\
Nature of Stationary points		Maximum	

Thus $b = 254.56$ cm ≈ 255 cm (to 3 sf)

With height $= h = \sqrt{180^2 - \left(\frac{b}{2}\right)^2} = 127.29 \approx 127$ cm (to 3 sf)

Alternative method

Let $A = \frac{1}{2}(180)^2 \sin\theta$

$\frac{dA}{d\theta} = 16200s\cos\theta$

At max, $\cos\theta = 0$

$\theta = 90$

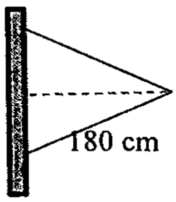
$\frac{d^2A}{d\theta^2} = -16200\sin\theta < 0 \quad (0 < \theta < 180)$

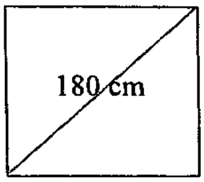
base $= \sqrt{180^2 + 180^2 - 2(180^2)\cos 90^\circ}$

$= 255$ (3 s.f.)

Alterative method

The max area for a rectangle with fixed perimeter is a square.
So, the max area for an isosceles triangle with fixed equal sides 180cm is also a square with diagonal 180 cm.





Hence, height of the isosceles triangle = side of the square $= \frac{180}{\sqrt{2}} \approx 127.28$ cm

Base of the isosceles triangle $= 2 \times \frac{180}{\sqrt{2}} \approx 255$ cm

(c) Mrs Chan's belief is not valid as the area enclosed when the base of the playpen is in the shape of an isosceles triangle is 16200cm^2 (from GC) which is the same that calculated when the base of the playpen was rectangular in shape.

Question 10:

(a) (i) Total distance $= 180+2\left(\frac{3}{4}\right)180+2\left(\frac{3}{4}\right)^2 180+...+2\left(\frac{3}{4}\right)^n 180$

$$= 180+2\left(\frac{3}{4}\right)180\left[1+\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)^2+...+\left(\frac{3}{4}\right)^{n-1}\right]$$

$$= 180+270\left[\frac{1[1-\left(\frac{3}{4}\right)^n]}{1-\frac{3}{4}}\right]$$

$$= 1260-1080\left(\frac{3}{4}\right)^n$$

So $A = 1260$, $B = -1080$

(ii) Consider $1260-1080\left(\frac{3}{4}\right)^n \geq 1200$

Using GC or taking ln on both sides, $n \leq 10.1$ or

n	$1260-1080\left(\frac{3}{4}\right)^n$
9	1178.9
10	1199.2
11	1214.4

So the ball bounced 10 times before travelling of 12m.
Total distance travelled before coming to rest is 1260 cm.

(b) Length of each regular time interval = 0.02 seconds

Let u_n cm denote the vertical distance travelled by the ball during the nth regular time interval.

$\because$ $u_1, u_2, u_3, ...$ follows an arithmetic progression, with first term $u_1 = 0.2$ and common difference $0.6-0.2=0.4$,

$\therefore$ **Total** distance (in cm) travelled during the first n regular time intervals,

$$S_n = \frac{n}{2}(2\times 0.2+(n-1)\times 0.4) = 180$$

$$\Rightarrow 0.2n^2 = 180$$

$$n^2 = 900$$

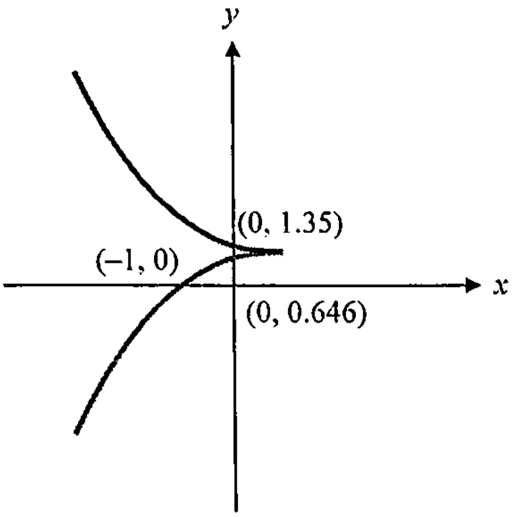
$$n = 30 \quad (\because n \in \mathbb{Z}^+)$$

$\therefore$ Total amount of time it takes for the tennis ball to reach the ground surface
$= 30$ regular time intervals of 0.02 seconds each
$= (30\times 0.02)$ seconds
$= 0.6$ seconds

(c) The floor on which the tennis ball bounces must be flat so that the ball bounces vertically upwards consistently. OR
The ball's bounces are not affected by any external forces like strong winds. OR
The ball's bounces are not affected by the dimensions of the ball.

Question 11:

(a)



(b) At intersection,

$(1-2t^2)+(t^3+1)=2$

$t^3-2t^2=0$

$t^2(t-2)=0$

$t=0$ or $t=2$

At $t=0$, $(x,y)=(1,1)$

At $t=2$, $(x,y)=(-7,9)$

So, $A(1,1)$ and $B(-7,9)$

Exact length

$=\sqrt{[1-(-7)]^2+[1-9]^2}$

$=\sqrt{128}$

$=8\sqrt{2}$

(c) Point A is $(1,1)$

Point P is $\left(1-2p^2, p^3+1\right)$

The midpoint of AP is $\left(\frac{1-2p^2+1}{2}, \frac{p^3+1+1}{2}\right) = \left(1-p^2, \frac{p^3}{2}+1\right)$

$x = 1-p^2$

$y = \frac{p^3}{2}+1$

$p^6 = (1-x)^3 = (2y-2)^2$

The required equation is $(1-x)^3 = (2y-2)^2$

(d) $\frac{dy}{dx}$

$= \frac{dy/dt}{dx/dt}$

$= \frac{3t^2}{-4t}$

$= -\frac{3}{4}t$

Equation of normal to C at P

$y-\left(p^3+1\right) = \frac{4}{3p}\left[x-\left(1-2p^2\right)\right]$

$y = \frac{4x}{3p}+p^3+\frac{8p}{3}+1-\frac{4}{3p}$ (shown)

(e) At $x=-7$,

$-7 = 1-2t^2$

$2t^2 = 8$

$t^2 = 4$

$t = \pm 2$

At $t=2$, gradient of the normal to the curve $= \frac{4}{3(2)} = \frac{2}{3}$

At $t=-2$, gradient of the normal to the curve $= \frac{4}{3(-2)} = -\frac{2}{3}$

Required angle

$= \tan^{-1}\left(\frac{2}{3}\right) + \tan^{-1}\left|\frac{2}{-3}\right|$

$= 67.4°$ or 1.18 rad