(TOPICAL REVISION – SUGGESTED SOLUTION)
RECURRENCE RELATIONS

Question 1 [Solution]

1	**(i)** Solving $18 = 20p + q, \quad 17 = 18p + q$ => $p = \frac{1}{2}, \quad q = 8$. **(ii)** **(Method 1)** For $p \neq 1$, rewrite the RR $x_{n+1} = px_n + q$ as $x_{n+1} - k = p(x_n - k)$, where $k \in \mathbb{R}$. Then $x_{n+1} = p(x_n - k) + k \Rightarrow q = k - kp$ and $k = \frac{q}{1-p}$. Let $u_n = x_n - k$. Then $\frac{u_{n+1}}{u_n} = p$ is a constant and the sequence $\{u_n\}$ is a geometric sequence with first term $u_0 = x_0 - k$ and common ratio p. For the sequence $\{u_n\}$ to be convergent, $-1 < p < 1$. **(Method 2)** Alternatively, limits may be used to explain the condition using the similar approach of rewriting the RR $x_{n+1} = px_n + q$ as $x_{n+1} - k = p(x_n - k)$, where $k \in \mathbb{R}$. $x_n - k = p(x_{n-1} - k) = p^2(x_{n-2} - k) = p^3(x_{n-3} - k) = \cdots = p^n(x_0 - k)$ $\lim_{n\to\infty}(x_n - k) = \lim_{n\to\infty} p^n(x_0 - k) = (x_0 - k)\lim_{n\to\infty} p^n$ For the sequence to be convergent, $-1 < p < 1$ so that $\lim_{n\to\infty} p^n(x_0 - k) = (x_0 - k)\lim_{n\to\infty} p^n = 0$ Then $\lim_{n\to\infty} x_n = k$ **(iii)** Since $k = \frac{q}{1-p} = 16$, $\lim_{n\to\infty} x_n = k = 16$ **(iv)** $x_r = px_{r-1} + q$ $= p(px_{r-2} + q) + q = p^2 x_{r-2} + q(1+p)$ $= p^r x_0 + q(1 + p + p^2 + \ldots + p^{r-1}) = p^r x_0 + q\left(\frac{1-p^r}{1-p}\right)$ $x_r = 16 + 4\left(\frac{1}{2}\right)^r, r \geq 0$

Question 2 [Solution]

2	"KillPest": $k_n = 0.35k_{n-1} + 500$, in the long run, $\lim_{n\to\infty} k_n \approx 769$ "PestKill": $p_n = 0.15p_{n-1} + 650$, in the long run, $\lim_{n\to\infty} p_n \approx 765$ Pestkill will be more effective in the long run since there will be fewer pests on the trees.

Question 3 [Solution]

3	$x_{n+1}x_n + 7x_{n+1} + x_n + 15 = 0$ $x_{n+1}\left(x_n + 7\right) + \left(x_n + 7\right) + 8 = 0$ $\left(\frac{y_{n+2}}{y_{n+1}} - 7\right)\left(\frac{y_{n+1}}{y_n}\right) + \left(\frac{y_{n+1}}{y_n}\right) + 8 = 0$ $\frac{y_{n+2}}{y_n} - 6\left(\frac{y_{n+1}}{y_n}\right) + 8 = 0$ $y_{n+2} - 6y_{n+1} + 8y_n = 0$ Characteristic equation: $m^2 - 6m + 8 = 0$ $m = 2$ or $m = 4$ $\therefore y_n = A(2^n) + B(4^n)$ $\therefore x_n = \frac{A(2^{n+1}) + B(4^{n+1})}{A(2^n) + B(4^n)} - 7$

Question 4 [Solution]

4	$\frac{y_n}{y_{n+1}} = 3\left(\frac{y_{n-1}}{y_n}\right) - 2n + 8$ $\frac{y_n}{y_{n+1}} - n = 3\left(\frac{y_{n-1}}{y_n} - (n-1)\right) + 5$ $u_n = 3u_{n-1} + 5$ Thus $u_n = A(3^n) + B$, where $B = 3B + 5 \quad \Rightarrow \quad B = -\frac{5}{2}$ Now $u_1 = \frac{y_1}{y_2} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$ $u_1 = A(3^1) - \frac{5}{2} \Rightarrow \frac{1}{2} = 3A - \frac{5}{2} \Rightarrow A = 1$ $\therefore u_n = 3^n - \frac{5}{2}$

	$\dfrac{y_n}{y_{n+1}} = 3^n + n - \dfrac{5}{2}$

Question 5 [Solution]

5

$W_n = 4W_{n-1} + 3I_{n-1} \quad \cdots(1)$

$I_n = I_{n-1} - 2W_{n-1} \quad \cdots(2)$

$\Rightarrow W_n = 4W_{n-1} + 3(I_{n-2} - 2W_{n-2}) \quad \cdots(3)$

From (1), $3I_{n-2} = W_{n-1} - 4W_{n-2}$

Put into (3): $W_n = 4W_{n-1} + (W_{n-1} - 4W_{n-2}) - 6W_{n-2}$

$W_n = 5W_{n-1} - 10W_{n-2}$

$\therefore W_n - 5W_{n-1} + 10W_{n-2} = 0$ (shown)

Auxiliary Equation: $\lambda^2 - 5\lambda + 10 = 0 \Rightarrow \lambda = \dfrac{5 \pm \sqrt{-15}}{2} = \dfrac{5}{2} \pm \dfrac{\sqrt{15}}{2}\mathrm{i}$

$r = \sqrt{\lambda_1 \lambda_2} = \sqrt{10}; \quad \theta = \tan^{-1}\dfrac{\sqrt{15}}{5} = 0.65906$

$W_n = 10^{\frac{n}{2}}\left(A\cos(0.65906n) + B\sin(0.65906n)\right)$

$W_0 = 10 \Rightarrow A = 10$

$W_1 = 4W_0 + 3I_0 \Rightarrow W_1 = 55$

$55 = \sqrt{10}\left(10\cos(0.65906) + B\sin(0.65906)\right)$

$\Rightarrow B = 15.4919$

$W_n = 10^{\frac{n}{2}}\left(10\cos(0.659n) + 15.5\sin(0.659n)\right)$

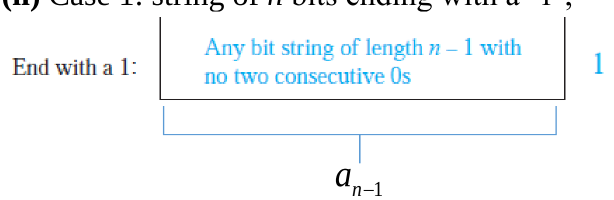
Question 6 [Solution]

6

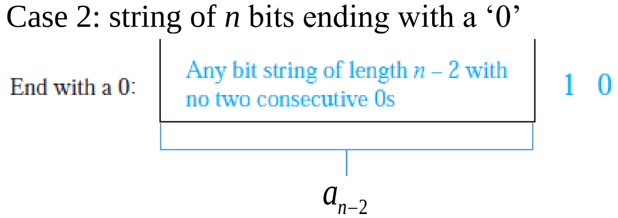
(i) $a_1 = 2$ ['1' or '0']

$a_2 = 3$ ['11', '01' or '10']

(ii) Case 1: string of n bits ending with a '1',

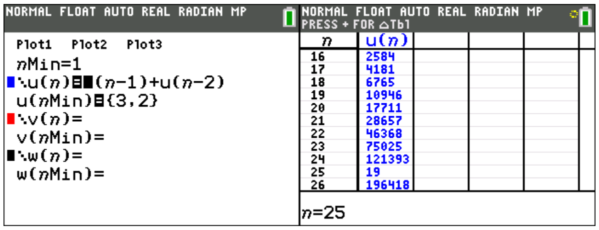


Case 2: string of n bits ending with a '0'



$\therefore a_n = a_{n-1} + a_{n-2}\ , n \geq 3$

(iii) From GC, there are 17711 ways of constructing the string of length 20 bits.



Question 7 [Solution]

7 **(i)** $a_1 = 3, \quad a_2 = 8$ (all possible cases $3^2 - 1$ [RR])

(ii) Considering cases for the colour of the *n*th tile.
If *n*th tile is not red (i.e. gray or green), there will be $2a_{n-1}$ ways.
If *n*th tile is red, we need to ensure that the (*n*-1)th tile must either be green or gray, there will be $2a_{n-2}$ ways.

$a_n = 2\left(a_{n-1} + a_{n-2}\right)$

Characteristic equation:

$m^2 - 2m - 2 = 0$

$m = \dfrac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$

$a_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$

Using $a_0 = 1, \quad a_1 = 3$

$A + B = 1$

$A + B + \sqrt{3}\left(A - B\right) = 3$

$A - B = \dfrac{2}{\sqrt{3}}$

$A = \dfrac{1}{2} + \dfrac{1}{\sqrt{3}}, \quad B = \dfrac{1}{2} - \dfrac{1}{\sqrt{3}}$

$a_n = \left(\dfrac{1}{2} + \dfrac{1}{\sqrt{3}}\right)(1+\sqrt{3})^n + \left(\dfrac{1}{2} - \dfrac{1}{\sqrt{3}}\right)(1-\sqrt{3})^n, n \geq 1$

Question 8 [Solution]

8(i)	The characteristic equation for $F_{n+1}=F_n+F_{n-1}$ is $\lambda^2-\lambda-1=0\Rightarrow\lambda=\frac{1\pm\sqrt{5}}{2}$. Hence the general solution is $F_n=A\left(\frac{1+\sqrt{5}}{2}\right)^n+B\left(\frac{1-\sqrt{5}}{2}\right)^n$.
(ii)	Since $F_2=F_1+F_0\Rightarrow F_0=F_2-F_1=0$. Hence we have $A+B=0$ and $A\left(\frac{1+\sqrt{5}}{2}\right)+B\left(\frac{1-\sqrt{5}}{2}\right)=1$, which upon solving gives $A=\frac{1}{\sqrt{5}}$ and $B=-\frac{1}{\sqrt{5}}$. Therefore $F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. We can also formulate the following 2 equations and solve for *A* and *B* (but taking a longer time): $A\left(\frac{1+\sqrt{5}}{2}\right)+B\left(\frac{1-\sqrt{5}}{2}\right)=1$ and $A\left(\frac{1+\sqrt{5}}{2}\right)^2+B\left(\frac{1-\sqrt{5}}{2}\right)^2=1$.
(iii)	$\sum_{n=0}^{\infty}\frac{F_n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ $=\frac{1}{3\sqrt{5}}\sum_{n=0}^{\infty}\left[\left(\frac{1+\sqrt{5}}{6}\right)^n-\left(\frac{1-\sqrt{5}}{6}\right)^n\right]$ $=\frac{1}{3\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{5}}{6}\right)^n-\sum_{n=0}^{\infty}\left(\frac{1-\sqrt{5}}{6}\right)^n\right]$ $=\frac{1}{3\sqrt{5}}\left[\frac{1}{1-\left(\frac{1+\sqrt{5}}{6}\right)}-\frac{1}{1-\left(\frac{1-\sqrt{5}}{6}\right)}\right]$ $=\frac{1}{3\sqrt{5}}\left[\frac{6}{5-\sqrt{5}}-\frac{6}{5+\sqrt{5}}\right]$ $=\frac{1}{3\sqrt{5}}\left[\frac{6\left(5+\sqrt{5}\right)}{25-5}-\frac{6\left(5-\sqrt{5}\right)}{25-5}\right]=\frac{1}{3\sqrt{5}}\left(\frac{12\sqrt{5}}{20}\right)=\frac{1}{5}$.

Question 9 [Solution]

9(i)	Number of shrimp = previous total + new born - dead $x_{n+2} = x_{n+1} + \frac{1}{2}x_{n+1} - \frac{9}{16}x_n$ $= \frac{3}{2}x_{n+1} - \frac{9}{16}x_n$ $x_{n+2} = \frac{3}{2}x_{n+1} - \frac{9}{16}x_n$ From the recurrence relation, the auxiliary eqn is $m^2 - \frac{3}{2}m + \frac{9}{16} = 0$. Since $\left(m - \frac{3}{4}\right)^2 = 0$, the general solution is $x_n = A\left(\frac{3}{4}\right)^n + Bn\left(\frac{3}{4}\right)^n$. Substitute $x_1 = 330$ and $x_2 = 360$, we can solve to get $A = 240$ and $B = 200$. $x_n = 240\left(\frac{3}{4}\right)^n + 200n\left(\frac{3}{4}\right)^n$
(ii)	$x_{11} = 103.05$ Number of shrimps = 103 000 (to nearest thousand)
(iii)	As $n \to \infty$, $\left(\frac{3}{4}\right)^n \to 0$ and $n\left(\frac{3}{4}\right)^n \to 0$. So $x_n \to 0$. The shrimp population will die out.

Question 10 [Solution]

10(i)	$u_1 = 9$
(ii)	Consider how an acceptable *n*-digit passcode can be obtained from a *n*-1 digits. Case I: If the *n*-1 digit passcode is acceptable, the n^{th} digit cannot be 5 (since that will lead to an odd number of 5s) i.e. there are 9 digits to choose from to be the n^{th} digit. $\therefore$ the number of *n* digits that can be formed from this case is $9u_{n-1}$. Case II: If the *n*-1 digit passcode is not acceptable, the the n^{th} digit must be 5. There are $10^{n-1} - u_{n-1}$ passcodes that are not acceptable. $\therefore$ the number of *n* digits that can be formed from this case is $10^{n-1} - u_{n-1}$. Therefore, $u_n = 9u_{n-1} + 10^{n-1} - u_{n-1} = 8u_{n-1} + 10^{n-1}$

(iii)	$u_n = 8u_{n-1} + 10^{n-1}$ $= 8\left(8u_{n-2} + 10^{n-2}\right) + 10^{n-1}$ $= 8^2 u_{n-2} + 8\cdot 10^{n-2} + 10^{n-1}$ $= 8^3 u_{n-3} + 8^2\cdot 10^{n-3} + 8\cdot 10^{n-2} + 10^{n-1}$ $= 8^4 u_{n-4} + 8^3\cdot 10^{n-4} + 8^2\cdot 10^{n-3} + 8\cdot 10^{n-2} + 10^{n-1}$ By observation, $u_n = 8^{n-1}u_1 + 8^{n-2}\cdot 10^1 + 8^{n-3}\cdot 10^2 + \ldots + 8^2\cdot 10^{n-3} + 8\cdot 10^{n-2} + 10^{n-1}$ $u_n = 9\cdot 8^{n-1} + \dfrac{8^{n-2}\cdot 10\left(1-\left(\frac{10}{8}\right)^{n-1}\right)}{1-\frac{10}{8}}$ $u_n = 9\cdot 8^{n-1} - 4\left(8^{n-2}\cdot 10\left(1-\left(\frac{10}{8}\right)^{n-1}\right)\right)$ $u_n = 9\cdot 8^{n-1} - 40\cdot 8^{n-2} + 4\cdot\dfrac{10^n}{8}$ $u_n = 9\cdot 8^{n-1} - 5\cdot 8^{n-1} + \dfrac{1}{2}\cdot 10^n$ $u_n = 4\cdot 8^{n-1} + \dfrac{1}{2}\cdot 10^n$ $u_n = \dfrac{1}{2}\left(8^n + 10^n\right)$

Question 11 [Solution]

(a) $2u_{n+2} - 3u_{n+1} + u_n = 1,$ where $u_1 = 1, u_2 = 2$

Suppose the sequence u_n converges, $u_n \to L$.

$$2L - 3L + L = 1$$
$$\Rightarrow 0 = 1 \rightarrow\leftarrow$$

$\therefore$ the sequence diverges. (shown)

(b)(i) $v_{n+2} - \sqrt{2}\,v_{n+1} + v_n = 0,$ where $v_0 = 1, v_1 = \sqrt{2}.$

Auxiliary equation: $\lambda^2 - \sqrt{2}\lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{\sqrt{2} \pm \sqrt{2}\,\mathrm{i}}{2} = \cos\frac{\pi}{4} \pm \mathrm{i}\sin\frac{\pi}{4}$$

$$\therefore v_n = A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}$$

$v_0 = 1 \Rightarrow A = 1$

$v_1 = \sqrt{2} \Rightarrow \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} = \sqrt{2} \Rightarrow A + B = 2 \therefore B = 1$

$$\therefore v_n = \cos\frac{n\pi}{4} + \sin\frac{n\pi}{4} = \sqrt{2}\cos\left(\frac{n\pi}{4} - \frac{\pi}{4}\right) \left[\text{or } \sqrt{2}\sin\left(\frac{n\pi}{4} + \frac{\pi}{4}\right)\right]$$

(ii) Possible values of v_n are 0, $\pm 1, \pm\sqrt{2}$.

Question 12 [Solution]

<table>
<tr><td>12</td><td>
(i) $u_{n+1} - u_n = u_n - qu_n - 4$
For the population to grow, $u_{n+1} - u_n > 0$ for each $n \in \mathbb{Z}^+$
In particular, need $u_1 - u_0 = (1-q)u_0 - 4 > 0$, that is $u_0 > \frac{4}{1-q}$ $(1 - q > 0)$.

(ii)
Given that $u_1 = 10$ and $u_2 = 15$, we have
$$15 = (2-q)(10) - 4$$
which gives $q = 0.1$.

The recurrence relation is then $u_{n+1} = 1.9u_n - 4$.

Using the GC to evaluate the terms, we have $u_{20} = 1000000$ (1 s.f.)

This is highly impractical as the formulation of the recurrence relation did not take into account the physical constraints of the jar. It could be too small to hold such a huge population.

(iii)
The suggestion should not be taken up as the differential equation assumes a model where the population is continuous, which only works better when the population is large enough.
</td></tr>
</table>

Question 13 [Solution]

13 (i)

Day	***Nympha***	***Iuvenus***	***Adultus***
1	2	0	0
2	2	2	0
3	2+9(2)	2	2
4	2+9(2) +9(2) +9(2)	2+9(2)	2+2
5	2+9(2) +9(2) +9(2) +9[2+9(2)] +9 (2+2)	2+9(2) +9(2) +9(2)	2+9(2) + 2+2

Note that

- The new *nympha kaka* on the (n-1)$^{\text{th}}$ day is given by $u_{n-1}-u_{n-2}$
- The total invenus kaka and adultus kaka on the (n-1)$^{\text{th}}$ day is u_{n-2}

  $\therefore \quad u_n = u_{n-1} + \left(u_{n-1} - u_{n-2}\right) + 9u_{n-2}$ [Or equivalently, $u_n = 2\left(u_{n-1} - u_{n-2}\right) + 10u_{n-2}$]

The recurrence relation is $u_n = 2u_{n-1} + 8u_{n-2}, n \geq 3$, $u_1 = 2$, $u_2 = 4$.

(ii) The characteristic equation is

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\therefore \quad x = 4 \text{ or } x = -2$

Therefore, the general solution is $u_n = A(4)^n + B(-2)^n$

Now $u_1 = 2$ and $u_2 = 4$

We have $u_1 = 2 = 4A - 2B ------(1)$

$u_2 = 4 = 16A + 4B ------(2)$

Solving eq (1) and (2) simultaneously, we have $A = \frac{1}{3}$ and $B = -\frac{1}{3}$

$\therefore \quad u_n = \frac{1}{3}(4)^n - \frac{1}{3}(-2)^n$

(iii) Stephen's immune system will not attack the *kaka* causing the death of the micro-organism in his body. (or any other plausible answers)

Question 14 [Solution]

14	**(i)** $(0.85)^4(25)=13.1$ **(ii)** Just before the $(n+1)$ -th dose, amount present in body is $(0.85)^4 u_n$. Thus $u_{n+1}=0.85^4 u_n+25$, i.e. $u_{n+1}=0.522u_n+25$. **(iii)** C.F.: $u_n=A(0.522)^n$ P.S.: $u_n=B$. Since $B=0.522B+25 \Rightarrow$ B=52.3$ G.S.: $u_n=A(0.522)^n+52.3$ Since $u_1=25$, thus $A=-52.3$. Thus $u_n=52.3(1-0.522^n)$ **(iv)** Need $0.522u_n>20$, thus $0.522\times 52.3(1-0.522^n)>20$ $\Rightarrow 0.522^n<0.61759$ $\Rightarrow n>2.0288$ $\Rightarrow n\geq 3$ Thus the interrogation can begin after the 3rd dose. (v) Note that $u_n\to 52.3$ as $n\to\infty$. Thus amount of serum is body is always less than 55 regardless of the number of doses. Thus there is no maximum length of time.

Question 15 [Solution]

15(i)	$x_n = 1.06x_{n-1} - 120,\ n > 1,\ \ x_1 = 4380$
(ii)	$x_n = 1.06x_{n-1} - 120$ $\quad = 1.06(1.06x_{n-2} - 120) - 120$ $\quad = 1.06^2 x_{n-2} - 1.06(120) - 120$ $\quad = 1.06^3 x_{n-3} - 1.06^2(120) - 1.06(120) - 120$ $\quad \vdots$ $\quad = 1.06^{n-1}x_1 - 1.06^{n-2}(120) - \cdots - 1.06(120) - 120$ $\quad = 1.06^{n-1}x_1 - \left[\dfrac{120(1.06^{n-1}-1)}{1.06-1}\right]$ $= 1.06^{n-1}(4380) - 2000(1.06^{n-1}-1)$ $= 1.06^{n-1}(4380) - 1.06^{n-1}(2000) + 2000$ $= 2380(1.06^{n-1}) + 2000$ $\therefore x_n = 2380(1.06^{n-1}) + 2000, n \geq 1$
	$2380(1.06^{n-1}) + 2000 \geq 6500$ $\Rightarrow 1.06^{n-1} \geq 1.890756303$ $\Rightarrow n - 1 \geq 10.932$ $\Rightarrow n \geq 11.932$ Least integer n is 12. Earliest time: End of August 2019
(iii)	Let y_n denote the number of functional smartphones manufactured by factory Y at the end of n^{th} month starting from September 2018. $y_n = \left(\dfrac{100 + k - 3.5}{100}\right)y_{n-1}$ $\therefore y_n = \left(\dfrac{96.5 + k}{100}\right)y_{n-1},\quad n > 1,\quad y_1 = 4000$
(iv)	$y_n = \left(\dfrac{96.5 + k}{100}\right)y_{n-1}$

	$=\left(\dfrac{96.5+k}{100}\right)^2 y_{n-2}$ $\vdots$ $=\left(\dfrac{96.5+k}{100}\right)^{n-1} y_1$ $=4000\left(\dfrac{96.5+k}{100}\right)^{n-1}$ $4000\left(\dfrac{96.5+k}{100}\right)^{11} \geq 6500$ $\Rightarrow \left(\dfrac{96.5+k}{100}\right)^{11} \geq \dfrac{6500}{4000}$ $\Rightarrow \dfrac{96.5+k}{100} \geq 1.049748678$ $\Rightarrow k \geq 8.01256$ Least value of k is 8.01 (3 s.f.).
	The recurrence relation model for factory Y is <u>not sustainable</u> in the long run if $\dfrac{96.5+k}{100} < 1 \Rightarrow k < 3.5$, resulting in a nett loss of functional smartphones every month.

Question 16 [Solution]

16(i)	$c_1 = 100$ and $c_2 = 200$ $c_3 = 200+200+20-0.01(200) = 418$
(ii)	For $n \geq 3$, $c_n = 0.99c_{n-1} + 20 + c_{n-1}$ $\Rightarrow c_n = 1.99c_{n-1} + 20$ Let $c_n = A(1.99)^n + B$ for $n \geq 2$. $A(1.99)^n + B = 1.99\left[A(1.99)^{n-1} + B\right] + 20$ $\Rightarrow A(1.99)^n + B = A(1.99)^n + 1.99B + 20$ $\Rightarrow 0.99B = -20$ $\Rightarrow B = -\dfrac{2000}{99}$

	$418 = A(1.99)^3 - \frac{2000}{99}$ $\Rightarrow A = 55.6$ (3 s.f.) $\therefore c_n = 55.6(1.99)^n - \frac{2000}{99}$ for $n \geq 2$.
(iii)	For $n \geq 3$, $d_n = d_{n-1} + 2d_{n-1} - (d_{n-1} - d_{n-2})$ $\Rightarrow d_n = 2d_{n-1} + d_{n-2}$ $\Rightarrow d_n - 2d_{n-1} - d_{n-2} = 0$ (Shown) Characteristic equation: $m^2 - 2m - 1 = 0$ $\Rightarrow m = \frac{2 \pm \sqrt{4+4}}{2}$ $\Rightarrow m = \frac{2 \pm 2\sqrt{2}}{2}$ $\Rightarrow m = 1+\sqrt{2}$ or $1-\sqrt{2}$ $\therefore d_n = C(1+\sqrt{2})^n + D(1-\sqrt{2})^n$ for $n \geq 1$ Since $d_1 = 200$ and $d_2 = 400$, $200 = C(1+\sqrt{2}) + D(1-\sqrt{2})$ $\Rightarrow 200 = (C+D) + \sqrt{2}(C-D)$ -------- (1) $400 = C(1+\sqrt{2})^2 + D(1-\sqrt{2})^2$ $\Rightarrow 400 = C(3+2\sqrt{2}) + D(3-2\sqrt{2})$ $\Rightarrow 400 = 3(C+D) + 2\sqrt{2}(C-D)$ -------- (2) $(1)\times 3-(2)$: $200 = \sqrt{2}(C-D)$ $\Rightarrow C-D = 100\sqrt{2}$ -------- (3) Substitute (3) into (1): $200 = (2D + 100\sqrt{2}) + \sqrt{2}(100\sqrt{2})$ $\Rightarrow 2D = -100\sqrt{2}$ $\Rightarrow D = -50\sqrt{2}$ $\therefore C = 100\sqrt{2} - 50\sqrt{2} = 50\sqrt{2}$ $\therefore d_n = 50\sqrt{2}(1+\sqrt{2})^n - 50\sqrt{2}(1-\sqrt{2})^n$ for $n \geq 1$.

(iv)	From GC,

Total for	Moonhub	POMO
End May 2020	$\approx 6.6922\times10^{6}$	$\approx 6.6923\times10^{6}$
End June 2020	$\approx 1.62\times10^{7}$	$\approx 1.33\times10^{7}$

Thus, the total number of customers who sign up for the new mobile plan under Moonhub will first exceed the total number of customers under POMO is at the end of June 2020.

Question 17 [Solution]

17 **(i)**

Characteristic equation is

$p\lambda^2 - \lambda + (1-p) = 0$

$$\lambda = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm \sqrt{4p^2-4p+1}}{2p} = \frac{1 \pm (2p-1)}{2p}$$

Hence $\lambda = 1$ or $\frac{1-p}{p}$.

Since $p \neq \frac{1}{2}$, there are two distinct roots.

$R_k = F(1)^k + G\left(\frac{1-p}{p}\right)^k$ for some constants F and G.

Using the boundary conditions $R_0 = 0$ and $R_N = 1$, we get

$R_0 = 0 = F + G$

$R_N = 1 = F + G\left(\frac{1-p}{p}\right)^N$

On solving, we obtain $F = \frac{1}{1-\left(\frac{1-p}{p}\right)^N}, G = \frac{-1}{1-\left(\frac{1-p}{p}\right)^N}$

Hence, $R_k = \frac{1}{1-\left(\frac{1-p}{p}\right)^N} - \frac{1}{1-\left(\frac{1-p}{p}\right)^N}\left(\frac{1-p}{p}\right)^k = \frac{1-\left(\frac{1-p}{p}\right)^k}{1-\left(\frac{1-p}{p}\right)^N}$.

	(ii) P (Achieve \$20 with initial \$10) $$= \frac{1-\left(\frac{1-\frac{9}{19}}{\frac{9}{19}}\right)^{10}}{1-\left(\frac{1-\frac{9}{19}}{\frac{9}{19}}\right)^{20}} = \frac{1-\left(\frac{10}{9}\right)^{10}}{1-\left(\frac{10}{9}\right)^{20}} = 0.259 \text{ (3 sf)}$$ P (Achieve \$120 with initial \$100) $$= \frac{1-\left(\frac{1-\frac{9}{19}}{\frac{9}{19}}\right)^{100}}{1-\left(\frac{1-\frac{9}{19}}{\frac{9}{19}}\right)^{120}} = \frac{1-\left(\frac{10}{9}\right)^{100}}{1-\left(\frac{10}{9}\right)^{120}} = 0.122 \text{ (3 sf)}$$ Since P (Achieve \$20 with initial \$10) $>$ P (Achieve \$120 with initial \$100), it is more likely for Ben to achieve \$20 with an initial \$10. **(iii)** If $p=\frac{1}{2}$, then $\lambda=1$ is a repeated root for the characteristic equation. Hence $R_k = Dk(1)^k + E(1)^k = Dk+E$ for some constants D and E. Using the boundary conditions $R_0=0$ and $R_N=1$, we get $R_0 = 0 = E$ $R_N = 1 = DN + E \Rightarrow D = \frac{1}{N}$. Hence $R_k = \frac{k}{N}$

Question 18 [Solution]

18	$m^2 - m - 1 = 0$ $m = \frac{1 \pm \sqrt{5}}{2}$ Since $\psi < \phi$, $\therefore \psi = \frac{1-\sqrt{5}}{2}$ and $\phi = \frac{1+\sqrt{5}}{2}$

	General solution $f_n = c_1\psi^n + c_2\phi^n$ Substitute $f_0 = 1$, $$c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2$$ Substitute $f_1 = 1$, $$c_1\psi + c_2\phi = 1 \Rightarrow (1 - c_2)\psi + c_2\phi = 1$$ Solving, $$c_2 = \frac{1-\psi}{\phi-\psi} \text{ and } c_1 = \frac{\phi-1}{\phi-\psi}$$ $$c_2 = \frac{\sqrt{5}+1}{2\sqrt{5}} \text{ and } c_1 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$ Therefore, $$f_n = \frac{\sqrt{5}+1}{2\sqrt{5}}\phi^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\psi^n$$ $$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\phi^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\psi^n = \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \psi^{n+1}\right)$$
(i)	$$g_{n+1} = \frac{f_{n+1}}{f_n} = \frac{f_n + f_{n-1}}{f_n} = 1 + \frac{f_{n-1}}{f_n} = 1 + \frac{1}{g_n}$$ $$g_{n+1} - \phi = \left(1 + \frac{1}{g_n}\right) - \left(1 + \frac{1}{\phi}\right) = \frac{1}{g_n} - \frac{1}{\phi} = \frac{\phi - g_n}{\phi\, g_n}$$
(i) (a)	Since $\phi > 0$, $\phi\, g_n > 0$ and so, $0 < g_n < \phi \Rightarrow \phi - g_n > 0 \Rightarrow g_{n+1} > \phi$ and similarly, $g_n > \phi \Rightarrow \phi - g_n < 0 \Rightarrow g_{n+1} < \phi$
(i) (b)	$\lvert g_n - \phi\rvert = \left\lvert\frac{g_{n-1} - \phi}{\phi\, g_{n-1}}\right\rvert$ $\quad\because g_n - \phi = \frac{\phi - g_{n-1}}{\phi g_{n-1}}$ $\le \frac{1}{\phi}\lvert g_{n-1} - \phi\rvert$ $\quad\because g_{n-1} \ge 1$ $\le \frac{1}{\phi^2}\lvert g_{n-2} - \phi\rvert$ $\quad\because \phi > 1$ $\le \cdots$ $\le \frac{1}{\phi^{n-1}}\lvert g_1 - \phi\rvert = (\phi - 1)\phi^{1-n}$
(i) last	Now, we have $0 \le \lvert g_n - \phi\rvert \le (\phi - 1)\phi^{1-n}$.

	Since $\phi > 1$, $\lim_{n\to\infty}\left((\phi-1)\phi^{1-n}\right) = \lim_{n\to\infty}\left(\frac{\phi-1}{\phi^{n-1}}\right) = 0$ By squeeze theorem, $\lim_{n\to\infty}\lvert g_n - \phi\rvert = 0$ Thus, $\lim_{n\to\infty} g_n = \phi$
(ii)	$g_{n+2} = 1+\frac{1}{g_{n+1}} = \frac{g_{n+1}+1}{g_{n+1}} = \frac{1+\frac{1}{g_n}+1}{1+\frac{1}{g_n}} = \frac{2g_n+1}{g_n+1}$ $g_{n+2} - g_n = \frac{2g_n+1}{g_n+1} - g_n = \frac{2g_n+1-g_n^2-g_n}{g_n+1}$ $= \frac{-g_n^2+g_n+1}{g_n+1}$ $g_{n+2}-\phi = \frac{2g_n+1}{g_n+1} - \frac{2\phi+1}{\phi+1}$ $= \frac{2\phi g_n + 2g_n + \phi + 1 - 2g_n\phi - g_n - 2\phi - 1}{(g_n+1)(\phi+1)}$ $= \frac{g_n-\phi}{(g_n+1)(\phi+1)}$ It is obvious that $\phi+1>0$ and $g_n+1>0\ \forall n\in\mathbb{Z}^+$. If $0<g_n<\phi$, then $-g_n^2+g_n+1>0 \Rightarrow g_{n+2}-g_n>0$ $\Rightarrow g_{n+2} > g_n$ Also, $g_n - \phi < 0 \Rightarrow g_{n+2}-\phi<0 \Rightarrow g_{n+2}<\phi$ If $g_n>\phi$, then then $-g_n^2+g_n+1<0 \Rightarrow g_{n+2}-g_n<0$ $\Rightarrow g_{n+2}<g_n$ Also, $g_n-\phi>0 \Rightarrow g_{n+2}-\phi>0 \Rightarrow g_{n+2}>\phi$ Thus, we have $g_n < g_{n+2} < \phi$ if $0<g_n<\phi$ $\phi < g_{n+2} < g_n$ if $g_n > \phi$
	The sequence g_n will oscillate about the number ϕ and eventually tend to the limit ϕ.