Solutions

Functions

1

$y = e^{x^2-2x} - 2$

$x^2 - 2x = \ln(y+2)$

$(x-1)^2 - 1 = \ln(y+2)$

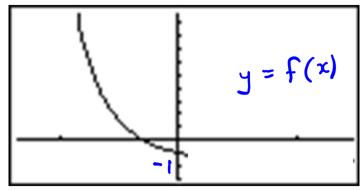
$(x-1)^2 = 1 + \ln(y+2)$

$x = 1 \pm \sqrt{1+\ln(y+2)}$

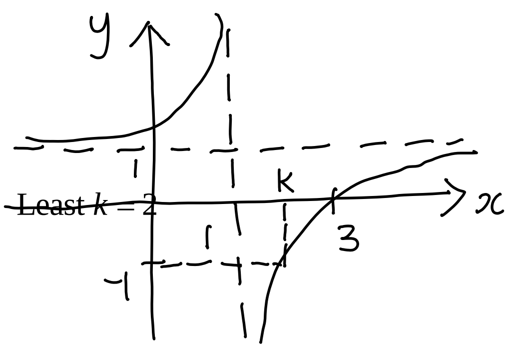
Since $x < 0$, $\quad x = 1 - \sqrt{1+\ln(y+2)}$

$D_{f^{-1}} = R_f = (-1, \infty)$

$\therefore f^{-1} : x \to 1 - \sqrt{1+\ln(x+2)}$, $x > -1$



$R_g \subseteq D_{f^{-1}} \Rightarrow R_g \subseteq (-1, \infty)$



~~Least $k = 2$~~

When $g(x) = -1$

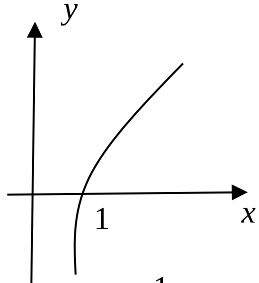
$g(2) = \frac{2-3}{2-1} = -1$

$D_g = (2, \infty)$

$(2, \infty) \xrightarrow{g} (-1, 1) \xrightarrow{f^{-1}} \left(1 - \sqrt{1+\ln 3}, 0\right)$

2

(i)

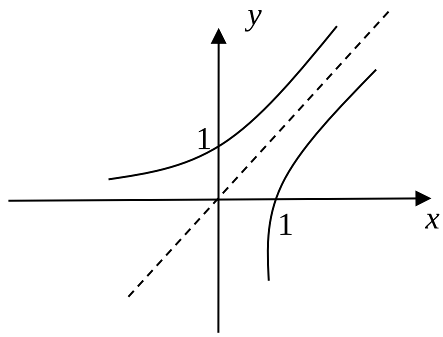


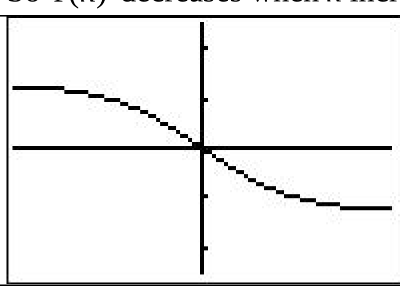
Since any horizontal line $y = k$, $k \in \mathbb{R}$ will cut the graph of f at most once, hence f is one-one. Thus, f^{-1} exists.

(ii) Let $y = x - \frac{1}{x}$

$x = \frac{y}{2} \pm \frac{1}{2}\sqrt{y^2+4}$

But $x > 0$, $\therefore x = \frac{y}{2} + \frac{1}{2}\sqrt{y^2+4}$

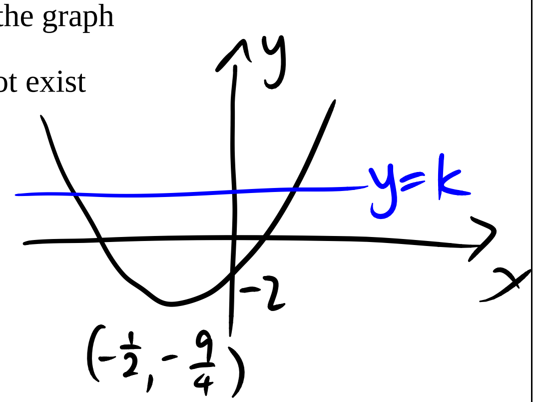
	Hence, $f^{-1}: x \to \frac{x}{2}+\frac{1}{2}\sqrt{x^2+4}$, $x \in (-\infty, \infty)$ (iii) [B1] – correct graph of f^{-1} with $y = x$ [B1] - intercepts  (iv) D_f : $(0,\infty)$ and R_g : $[-1, 1]$. Since $R_g \not\subseteq D_f$, therefore fg does not exist. (v) Largest D_g : $(0, \pi)$ $fg(x) = f(\sin x)$ $= \sin x - \frac{1}{\sin x}$ Hence, $fg: x \to \sin x - \frac{1}{\sin x}$, $x \in (0, \pi)$.

3a	$f(x) = x - 2\tan^{-1}(2x)$ $f'(x) = 1 - \frac{4}{1+4x^2}$ $0 \le x^2 < \frac{3}{4}$ $1 \le 1+4x^2 < 4$ $\frac{1}{4} < \frac{1}{1+4x^2} \le 1$ $1 < \frac{4}{1+4x^2} \le 4$ Since $1 < \frac{4}{1+4x^2} \le 4$, $\therefore 1 - \frac{4}{1+4x^2} < 0$, $f'(x) < 0$ So $f(x)$ decreases when x increases.
	

	$R_f = \left(\frac{\sqrt{3}}{2} - \frac{2\pi}{3}, -\frac{\sqrt{3}}{2} + \frac{2\pi}{3} \right)$ Or $R_f = (-1.23, 1.23)$
	For ff to exist, $R_f \subseteq D_f$ $R_f = \left(\frac{\sqrt{3}}{2} - \frac{2\pi}{3}, -\frac{\sqrt{3}}{2} + \frac{2\pi}{3} \right)$ or $R_f = (-1.23, 1.23)$ $D_f = \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right)$ or $R_g = (-0.866, 0.866)$ $\therefore R_f \not\subset D_f$ ff does not exist.
	Let $y = x - 2\tan^{-1}(2x)$. Then $x = y - 2\tan^{-1}(2y)$
b)	$h(x) = x^2 + 2$ $gh(x) = x^4 - 2x^2 + 5 = (x^2 - 1)^2 + 4$ $g(x+2) = \left[(x^2 + 2) - 3 \right]^2 + 4$ $g(x) = (x-3)^2 + 4$ **Alternatively**, $h(x) = x^2 + 2$ $gh(x) = x^4 - 2x^2 + 5$ $= (x^2+2)^2 - 4x^2 - 4 - 2x^2 + 5$ $= (x^2+2)^2 - 6(x^2+2) + 13$ $\Rightarrow g(x) = x^2 - 6x + 13$

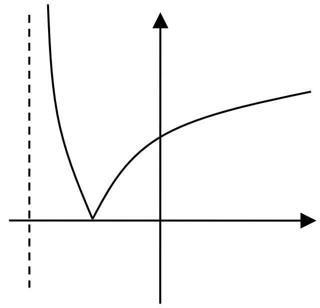
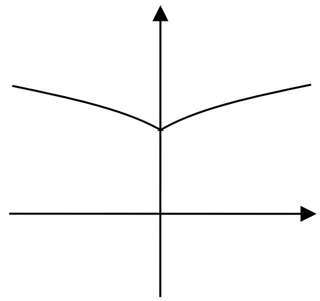
4	(i) $D_f = (-\infty, \infty)$ and $R_g = (2, \infty)$ Since $R_g = (2, \infty) \subset D_f = (-\infty, \infty)$ $\Rightarrow$ function *fg* exists. $fg(x) = f\left[g(x) \right] = \left(\sqrt{x+4} \right)^2 - 1$ $fg(x) = x + 3, \quad x \in \Re^+$ (ii) Since $R_{fg} = (3, \infty) \subset D_g = (0, \infty)$ $\Rightarrow$ function *gfg* exists $h(x) = gfg(x) = g\left[fg(x) \right] = \sqrt{(x+3)+4} = \sqrt{x+7}, \quad x > 0$

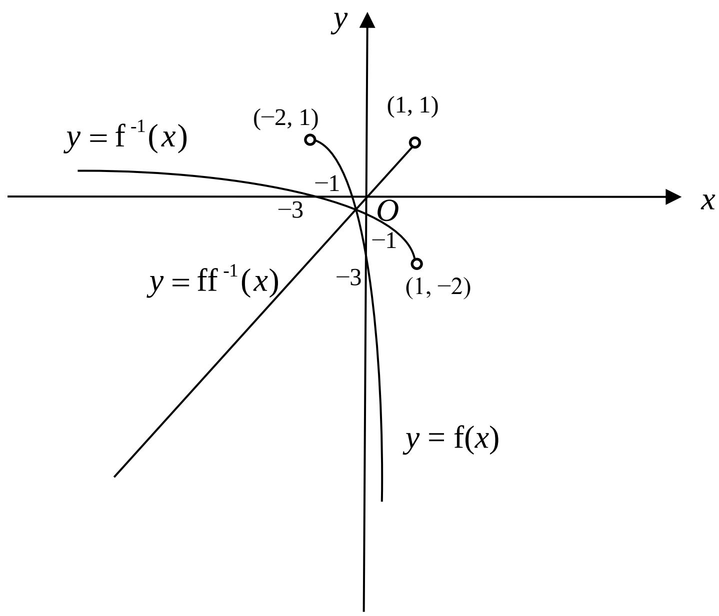
	(iii) $h(x) = g^{-1}g(x), \quad x > 0$ $g[fg(x)] = g^{-1}g(x)$ $\sqrt{x+7} = x, \quad x \in \Re^+$ $x + 7 = x^2$ $x^2 - x - 7 = 0$ $x = \dfrac{1 \pm \sqrt{1-4(1)(-7)}}{2(1)} = \dfrac{1}{2}(1 \pm \sqrt{29})$ $x = \dfrac{1}{2}(1+\sqrt{29}) \quad \text{since } x \in \Re^+$

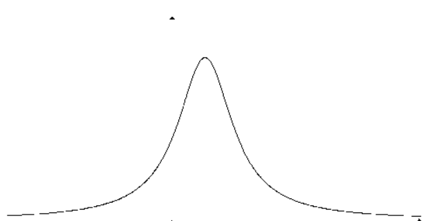
5	**(i)** $\mathrm{g}(x) = \mathrm{f}(x-1)$ The transformation is a translation of +1 unit in the direction of the *x*-axis Alternatively, $\mathrm{g}(x) = 2^{x-1} = \dfrac{1}{2}(2^x) = \dfrac{1}{2}\mathrm{f}(x)$ The transformation is a scaling parallel to the *y*-axis by factor $\dfrac{1}{2}$ **(ii)** A horizontal line $y = k$ ($k > -\dfrac{9}{4}$) cuts the graph at 2 points, h is not one- one , so h^{-1} does not exist  **(iii)** $a = -\dfrac{1}{2}$ $\mathrm{h}: x \mapsto x^2 + x - 2, \quad x \in \mathbb{R}, x > -\dfrac{1}{2}$ Let $y = \mathrm{h}(x) = x^2 + x - 2, \quad x > -\dfrac{1}{2}$ $y = \left(x + \dfrac{1}{2}\right)^2 - \dfrac{9}{4}$ $\left(x + \dfrac{1}{2}\right)^2 = y + \dfrac{9}{4} = \dfrac{4y+9}{4}$ $x = -\dfrac{1}{2} \pm \dfrac{\sqrt{4y+9}}{2}$ $x = -\dfrac{1}{2} + \dfrac{\sqrt{4y+9}}{2}$ (since $x > -\dfrac{1}{2}$)

	$\mathrm{h}^{-1} : x \mapsto \dfrac{-1+\sqrt{4x+9}}{2}, \quad x \in \mathbb{R}, x > -\dfrac{9}{4}$ **(iv)** $R_{\mathrm{f}} = (0, \infty)$ $D_{\mathrm{h}^{-1}} = (-\dfrac{9}{4}, \infty)$ $\therefore \quad R_{\mathrm{f}} \subset D_{\mathrm{h}^{-1}}$. Hence $\mathrm{h}^{-1}\mathrm{f}$ exists. $\mathrm{h}^{-1}\mathrm{f}(x) = \mathrm{h}^{-1}(2^x)$ $= \dfrac{-1+\sqrt{4(2^x)+9}}{2}$ $\mathrm{h}^{-1}\mathrm{f} : x \mapsto \dfrac{-1+\sqrt{4(2^x)+9}}{2}, \quad x \in \mathbb{R}$ $\mathrm{R}_{\mathrm{h}^{-1}\mathrm{f}} = (1, \infty)$

6	Let $y = (x+1)^2 + 1$ $\Rightarrow x = -1 \pm \sqrt{y-1}$ Since $x \leq -1$, $\therefore \quad x = -1 - \sqrt{y-1}$ $\therefore \quad \mathrm{g}^{-1}(x) = -1 - \sqrt{x-1}$, $x \geq 1$
	$\mathrm{R}_{\mathrm{g}} = [\ 1, \infty)$ $\mathrm{D}_{\mathrm{f}} = \mathbb{R}$ $\mathrm{R}_{\mathrm{g}} \subseteq \mathrm{D}_{\mathrm{f}}$, $\therefore$ fg exists
	$\mathrm{fg}(x) = 3 - 2\mathrm{e}^{(x+1)^2+2}$; $\quad x \in \mathbb{R}$, $x \leq -1$ $\mathrm{R}_{\mathrm{fg}} = (-\infty,\ 3 - 2\mathrm{e}^2]$

7(a)	fg(x) = gf(x) $\Rightarrow$ f($\ln(\lvert x\rvert+2)$) = g($\ln(x+2)$) $\Rightarrow \ln[\ \ln(\lvert x\rvert+2)+2\ ] = \ln[\ \lvert\ln(x+2)\rvert+2\ ]$ $\Rightarrow \ln(\lvert x\rvert+2) = \lvert\ln(x+2)\rvert$   $y = \lvert\ln(x+2)\rvert$ $y = \ln(\lvert x\rvert+2)$ Consider $\ln(-x+2) = -\ln(x+2)$ $\Rightarrow (-x+2) = \dfrac{1}{x+2}$ $\Rightarrow (2+x)(2-x) = 1$ $\Rightarrow 4-x^2 = 1$ $\Rightarrow x^2 = 3$ $\Rightarrow x = -\sqrt{3}$ (reject $x = \sqrt{3}$) Hence, for fg(x) = gf(x), $x \ge 0$ or $x = -\sqrt{3}$
(b)(i)	g(x) = $\ln(\lvert x\rvert+2)$. Note that both $x=1$ and $x=-1$ are in the domain of g. Clearly $-1 \ne 1$, But g(−1) = g(1) = ln3. Therefore, function g is not 1-1 and inverse of function of g cannot be formed.
(ii)	largest a = 0.
(iii)	Let y = g(x) = $\ln(\lvert x\rvert+2)$. Since $x \le 0$, $y = \ln(-x+2)$ $\Rightarrow -x+2 = e^y$ $\Rightarrow x = 2-e^y$ $\Rightarrow \mathrm{g}^{-1}(y) = 2-e^y$ Hence $\mathrm{g}^{-1} : x \to 2-e^x$, $x \in [\ln 2, \infty)$

8(i)	$R_f = (-\infty, \lambda) \quad D_g = (-\infty, 1)$ Since $R_f \subseteq D_g$, largest λ=1.
(ii)	
(iii)	$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ $1-(x+2)^2 = x$ $x^2+5x+3=0$ $x = \dfrac{-5 \pm \sqrt{13}}{2}$ $= \dfrac{-5+\sqrt{13}}{2} \quad \left(\because \dfrac{-5-\sqrt{13}}{2} \text{ is rejected as } x > -2 \right)$

9(i)	 $R_f = (0, 2]$
(ii)	The horizontal line $y = 1$ cuts the graph of f at two points. $\therefore$ f is not one-one. Thus f does not have an inverse. <u>Alt</u> Since $\mathrm{f}\left(-\frac{3}{5}\right) = \mathrm{f}(1) = \frac{2}{17}$, f is not one-one. Thus f does not have an inverse.
(iii)	Least value of k = 1/5

Let $y = \mathrm{f}(x) = \dfrac{2}{1+(5x-1)^2}, x \geq \dfrac{1}{5}$.

$$1+(5x-1)^2 = \frac{2}{y}$$

$$5x-1 = \pm\sqrt{\frac{2}{y}-1}$$

Since $x \geq \dfrac{1}{5}, 5x-1 \geq 0$.

$$\therefore\ x = \frac{1}{5}\left(1+\sqrt{\frac{2}{y}-1}\right)$$

$$\mathrm{f}^{-1}(x) = \frac{1}{5}\left(1+\sqrt{\frac{2}{x}-1}\right)$$

(iv)

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

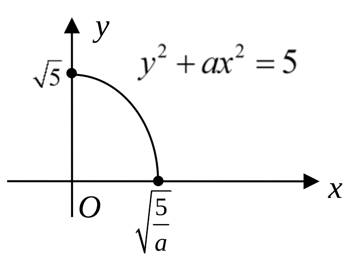
$\Rightarrow\ \mathrm{f}(x) = x$

$\Rightarrow\ \dfrac{2}{1+(5x-1)^2} = x$

$\Rightarrow\ x\left(1+25x^2-10x+1\right) = 2$

$\Rightarrow\ 25x^3-10x^2+2x-2 = 0$

10



Since any horizontal line $y = k$, $0 \leq k \leq \sqrt{5}$ cuts the graph of $y = \mathrm{f}(x)$ once and only once, f is a one-one function. Hence f^{-1} exists.

Let $y = \sqrt{5-ax^2}$, $\quad 0 \leq x \leq \sqrt{\dfrac{5}{a}}$

$$y^2 = 5-ax^2$$

$$x^2 = \frac{5-y^2}{a}$$

$x=\sqrt{\dfrac{5-y^2}{a}}$, since $0\le x\le\sqrt{\dfrac{5}{a}}$

$\therefore \mathrm{f}^{-1}: x\mapsto\sqrt{\dfrac{5-x^2}{a}}$, $\quad 0\le x\le\sqrt{5}$

(i)

$\mathrm{f}^2(x)=x \quad$ for all $x\in\mathbb{R}$, $0\le x\le\sqrt{\dfrac{5}{a}}$

Given: $\mathrm{f}^2(x)=x$

$\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$

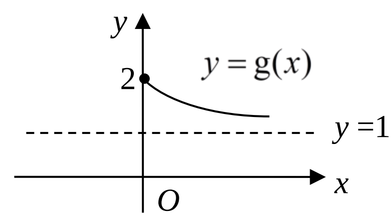
$\Rightarrow \sqrt{5-ax^2}=\sqrt{\dfrac{5}{a}-\dfrac{x^2}{a}}$

$\Rightarrow 5=\dfrac{5}{a},\quad a=\dfrac{1}{a}$

$\therefore a=1$ (shown)

(ii)

Since $\mathrm{R_g}=(1,2]\subset \mathrm{D_f}=[0,\sqrt{5}]$, $\therefore$ fg exists.



Method 1

$\mathrm{fg}(x)=\sqrt{5-(1+e^{-x})^2},\quad x\ge 0$

$\mathrm{R_{fg}}=[1,2)$

Method 2

$[0,\infty)\xrightarrow{\mathrm{g}}(1,2]\xrightarrow{\mathrm{f}}[1,2)$

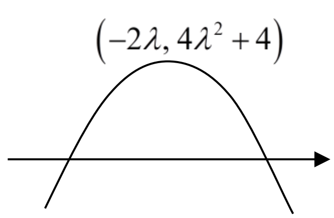
$\mathrm{R_{fg}}=[1,2)$

11

(i) $\mathrm{f}(x)=4-x^2-4\lambda x$

$=-\left[(x+2\lambda)^2-(2\lambda)^2\right]+4$

$=-(x+2\lambda)^2+4\lambda^2+4$



Since horizontal line y = 0 cuts the graph $y=\mathrm{f}(x)$ twice, f is not a one-one function. $\therefore$ f does not have an inverse.

(ii) Largest value of $k = -2\lambda$

Let $y = -\left(x+2\lambda\right)^2 + 4\lambda^2 + 4$

$x = -2\lambda \pm \sqrt{4\lambda^2 + 4 - y}$

Since $x < -2\lambda$, $x = -2\lambda - \sqrt{4\lambda^2 + 4 - y}$

$\therefore \mathrm{f}^{-1} : x \mapsto -2\lambda - \sqrt{4\lambda^2 + 4 - x}, \quad x < 4\lambda^2 + 4$

(iii) $R_{\mathrm{f}^{-1}} = \left(-\infty, -2\lambda\right)$

$D_{\mathrm{g}} = \left(-\infty, 1\right)$

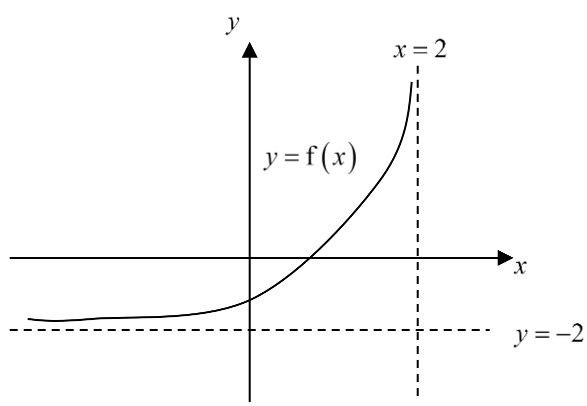
Given $\lambda > -\frac{1}{2}$, $\therefore R_{\mathrm{f}^{-1}} = \left(-\infty, -2\lambda\right) \subset \left(-\infty, 1\right) = D_{\mathrm{g}}$

$\therefore \mathrm{gf}^{-1}$ exists

$\therefore R_{\mathrm{gf}^{-1}} = \left(\ln\left(1+2\lambda\right), \infty\right)$

12

(i) $\mathrm{f}\left(x\right) = -2 - \dfrac{3}{x-2}$



(ii) let $y = \dfrac{1-2x}{x-2}$

$xy - 2y = 1 - 2x$

$x\left(y+2\right) = 1 + 2y$

$x = \dfrac{1+2y}{y+2}$

$\mathrm{f}^{-1} : x \mapsto \dfrac{1+2x}{x+2}, \ x > -2$

iii)

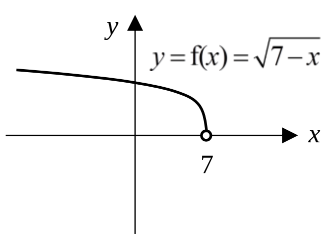
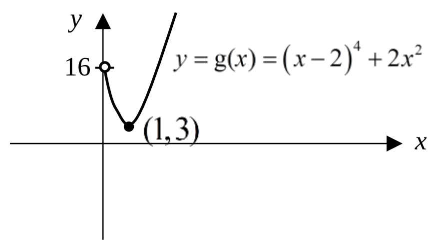
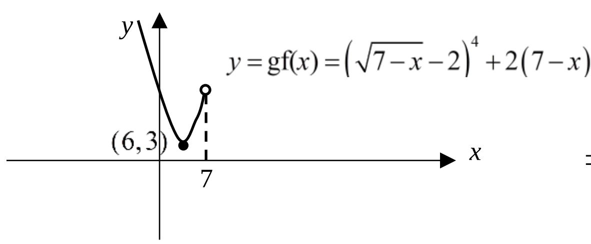
$R_g = \left(-\infty, \infty\right)$ $\qquad D_f = \left(-\infty, 2\right)$

Since $R_g = \left(-\infty, \infty\right) \not\subset D_f = \left(-\infty, 2\right)$, fg does not exist

$R_f = \left(-2, \infty\right)$ $\qquad D_g = \left(-3, \infty\right)$

Since $R_f = \left(-2, \infty\right) \subseteq D_g = \left(-3, \infty\right)$, gf exists

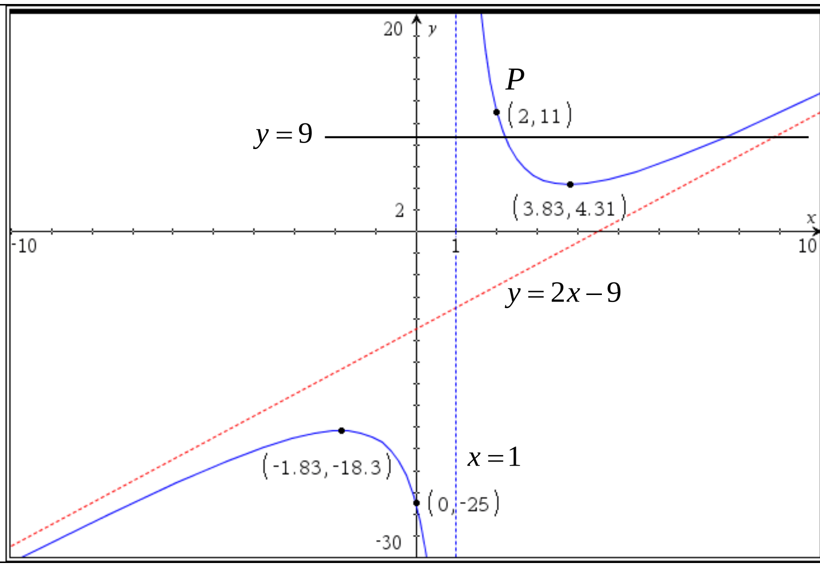
	$\text{gf}(x)=\text{g}\left(\dfrac{1-2x}{x-2}\right)$ $=\ln\left(\dfrac{1-2x}{x-2}+3\right)$ $=\ln\left(\dfrac{1-2x+3x-6}{x-2}\right)=\ln\left(\dfrac{x-5}{x-2}\right),\quad x<2$

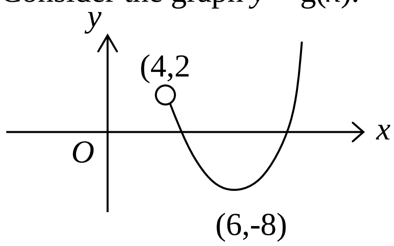
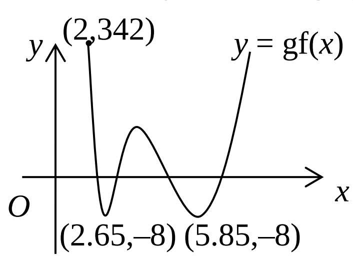
13(a)(i)	$R_{\text{f}}=(0,\infty)\subseteq D_{\text{g}}=(-2,\infty)$ $\therefore$ gf exist. $\text{gf}:x\mapsto\left(\sqrt{7-x}-2\right)^4+2(7-x),\ x\in\mathbb{R},\quad x<7$
(ii)	$\text{D}_{\text{f}}=(-\infty,7)\xrightarrow{\text{f}}\text{R}_{\text{f}}=(0,\infty)\xrightarrow{\text{g}}\text{R}_{\text{gf}}=[3,\infty)$  **<u>Alternative</u>**  $\Rightarrow \text{R}_{\text{gf}}=[3,\infty)$
(b)(i)	$\text{k}(\text{h}(x))=x^2+a$ $\text{h}(x)-5=x^2+a$ $\text{h}(x)=x^2+5+a$
(ii)	Any horizontal line $y=c,\ c\in\mathbb{R}$ cuts the curve $y=\text{h}(x)$ at most once $\Rightarrow$ h is a one-one function $\Rightarrow$ h^{-1} exists. Let $y=\text{h}(x)$ $x=\text{h}^{-1}(y)$ $=\sqrt{y-5-a}$ or $-\sqrt{y-5-a}$ $\left(\text{rej } \because x>\sqrt{5}\right)$ $\text{h}^{-1}(x)=\sqrt{x-5-a}$ $\text{h}^{-1}(x)=\sqrt{x-5-a},\quad x\in\mathbb{R},\quad x>10+a$

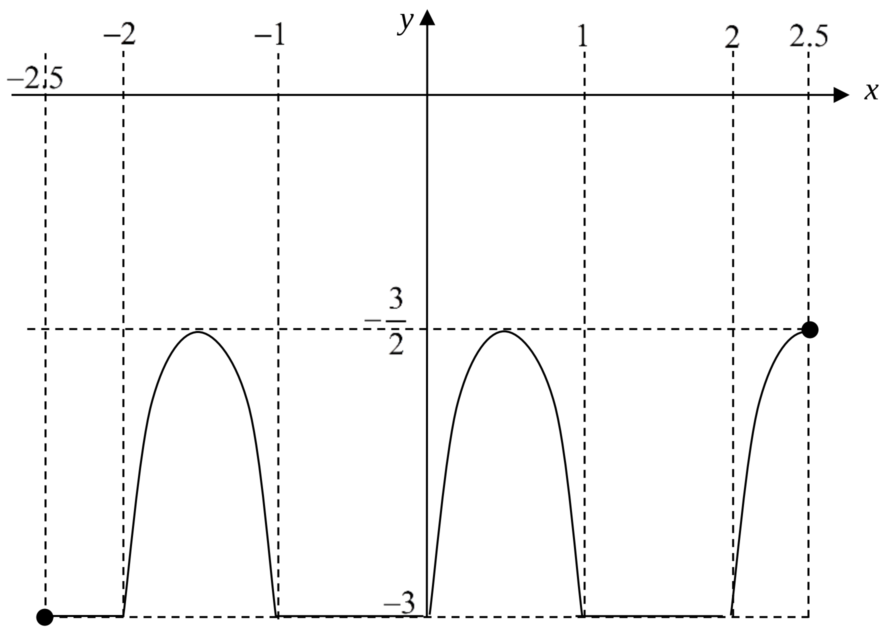
	Alternative $\mathrm{kh}\left(\mathrm{h}^{-1}(x)\right)=\mathrm{k}(x)$ $\left(\mathrm{h}^{-1}(x)\right)^2+a=x-5$ $\mathrm{h}^{-1}(x)=\sqrt{x-5-a}$ or $-\sqrt{x-5-a}$ $\left(\text{rejected } \because R_{\mathrm{h}^{-1}}=D_{\mathrm{h}}=(\sqrt{5},\infty)\right)$ $\mathrm{h}^{-1}(x)=\sqrt{x-5-a}$ $\mathrm{h}^{-1}(x)=\sqrt{x-5-a},\quad x\in\mathbb{R},\quad x>10+a$

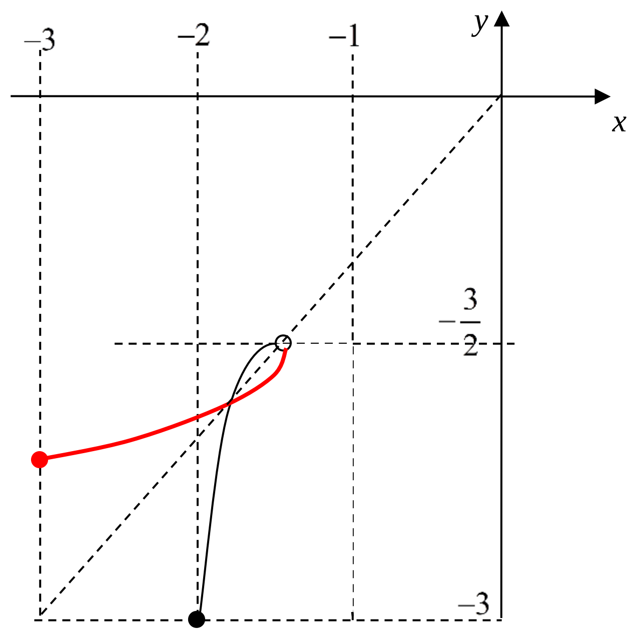
14(i)	$k=-1$
(ii)	Let $y=2-(x+1)^2$ $x+1=\pm\sqrt{2-y}$ $x=-1+\sqrt{2-y}$ (NA) or $x=-1-\sqrt{2-y}$ $(\because x\le -1)$ $\mathrm{f}^{-1}: x\mapsto -1-\sqrt{2-x}$, $x\le 2$
(iii)	$\mathrm{f}\left(x\right)=\mathrm{f}^{-1}\left(x\right)$ $\Rightarrow\ \mathrm{f}\left(x\right)=x$ $\Rightarrow\ 2-\left(x+1\right)^2=x$ $\Rightarrow\ x^2+3x-1=0$ Using GC, $x=-3.303$ (since domain of f is $x\le -1$)

15	$\dfrac{2x^2-11x+25}{x-1}=2x-9+\dfrac{16}{x-1}$



(i)(a)	The line y = 9 cuts the graph of y = f(x) more than once. Hence f is not a one-one function and f^{-1} does not exist. Alternatively, since f(3) = f(5), f is not a one-one function. f^{-1} does not exist.
(b)	From graph, $\lambda > 3.83$, least integer $\lambda = 4$
(ii)	Consider the graph y = g(x).  $D_f = [2, \infty) \xrightarrow{f} R_f = [4.31, \infty) \xrightarrow{g} R_{gf} = [-8, \infty)$ Alternatively, consider graph of gf, with D_f as domain. 

16(i)	
(ii)	$\int_{-\frac{3}{2}}^{1} \lvert f(x) \rvert \, dx$ $= 3 - \frac{3}{2}\int_{-2}^{-1} \frac{3}{(2x+3)^2 - 2} \, dx$ …….(*) $= 3 - \frac{3}{2}\left[\frac{3}{2\sqrt{2}} \cdot \frac{1}{2}\ln\left\lvert\frac{2x+3-\sqrt{2}}{2x+3+\sqrt{2}}\right\rvert\right]_{-2}^{-1}$ $= 3 - \frac{9}{8\sqrt{2}}\left\{\ln\left\lvert\frac{1-\sqrt{2}}{1+\sqrt{2}}\right\rvert - \ln\left\lvert\frac{-1-\sqrt{2}}{-1+\sqrt{2}}\right\rvert\right\}$ $= 3 - \frac{9}{8\sqrt{2}}\left\{\ln\left\lvert\frac{\sqrt{2}-1}{1+\sqrt{2}}\right\rvert - \ln\left\lvert\frac{1+\sqrt{2}}{\sqrt{2}-1}\right\rvert\right\}$ $= 3 - \frac{9}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{1+\sqrt{2}}\right)$
(iii)	Greatest value of $a = -\frac{3}{2}$

(iv)	
(v)	Since the curves intersect at the line y = x, solving $\text{h}(x) = \text{h}^{-1}(x)$ is equivalent to solving $\text{h}(x) = x$. $\frac{3}{(2x+3)^2-2} = x$ $3 = x\left(4x^2+12x+9-2\right)$ $4x^3+12x^2+7x-3=0$ Using GC, $x=-1.781$ or $x=-1.5$ (rej) or $x=0.2808$ (rej)

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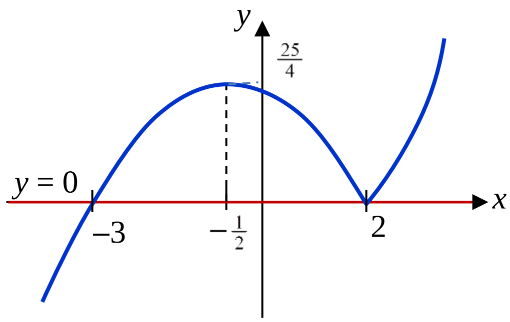
17 **(i)** Since f(–3) = f(2) = 0 where $-3, 2 \in D_f$, f is not one-one.

Thus f^{-1} does not exist.

OR **(i)** Since the horizontal line
$y = 0$, cuts the graph of
$y = f(x)$ at more than 1 point,
f is not one to on, thus f^{-1} does not exist.



(ii) Since $k \le x \le 2$, $f(x) = -(x-2)(x+3)$. Least $k = -\frac{1}{2}$.

Let $y = f(x)$, where $-\frac{1}{2} \le x \le 2$.

$$y = -(x+3)(x-2) = -(x^2+x-6) = -\left(x+\frac{1}{2}\right)^2 + \frac{25}{4}$$

$$\left(x+\frac{1}{2}\right)^2 = \frac{25}{4} - y \quad \Rightarrow x = -\frac{1}{2} \pm \sqrt{\frac{25-4y}{4}}$$

Since $-\frac{1}{2} \le x \le 2$, $\quad x = -\frac{1}{2} + \frac{1}{2}\sqrt{25-4y}$

$$\therefore \ f^{-1}: x \mapsto -\frac{1}{2} + \frac{1}{2}\sqrt{25-4x}, \quad x \in \mathbb{R}, \ 0 \le x \le \frac{25}{4}.$$

Alternative Method

Let $y = f(x)$, where $-\frac{1}{2} \le x \le 2$.

$$y = -(x+3)(x-2) = -x^2 - x + 6$$

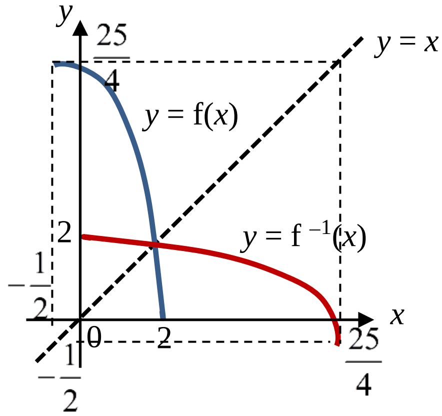
$$x^2 + x + y - 6 = 0$$

$$x = \frac{-1 \pm \sqrt{1-4(y-6)}}{2} = \frac{-1 \pm \sqrt{25-4y}}{2}$$

Since $-\frac{1}{2} \le x \le 2$, $\quad x = \frac{-1+\sqrt{25-4y}}{2}$

$$\therefore \ f^{-1}: x \mapsto \frac{-1+\sqrt{25-4x}}{2}, \quad x \in \mathbb{R}, \ 0 \le x \le \frac{25}{4}.$$

(iii)



$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = x$

$-x^2 - x + 6 = x$

$x^2 + 2x - 6 = 0$

$x = \dfrac{-2 \pm \sqrt{28}}{2} = -1 \pm \sqrt{7}$

From the diagram, $x \geq 0$, $\therefore\ x = -1 + \sqrt{7}$

(iv) Range of f = $\left[0,\ \dfrac{25}{4}\right]$, Domain of g^{-1} = Range of g = $(-1,\ 7)$

Since $R_{\mathrm{f}} \subseteq D_{\mathrm{g}^{-1}}$, $\therefore$ $\mathrm{g}^{-1}\mathrm{f}$ exists.

(v) $\mathrm{g}(x) = \dfrac{24}{x^2+3} - 1$, $x \in \mathbb{R},\ x > 0$

$\mathrm{g}'(x) = -\dfrac{24(2x)}{(x^2+3)^2} < 0$ for $x > 0$ since $(x^2+3)^2 > 0$

Hence, g is a strictly decreasing function.

$\mathrm{g}^{-1}(\mathrm{f}(x)) < 1 \Rightarrow \mathrm{f}(x) > \mathrm{g}(1)$ (since g is a strictly decreasing fn)

$\Rightarrow \mathrm{f}(x) > 5$

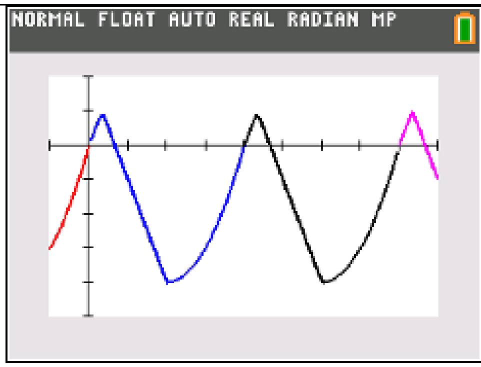
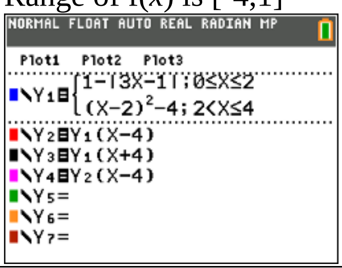
Since $\mathrm{f}^{-1}(5) = -\dfrac{1}{2} + \dfrac{1}{2}\sqrt{25 - 4(5)} = \dfrac{\sqrt{5}-1}{2}$,

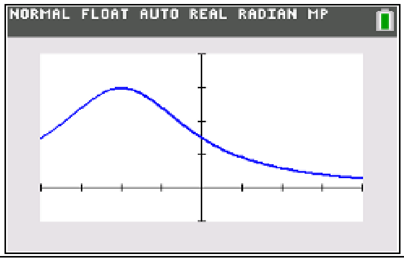
from the graph of $y = \mathrm{f}(x)$,

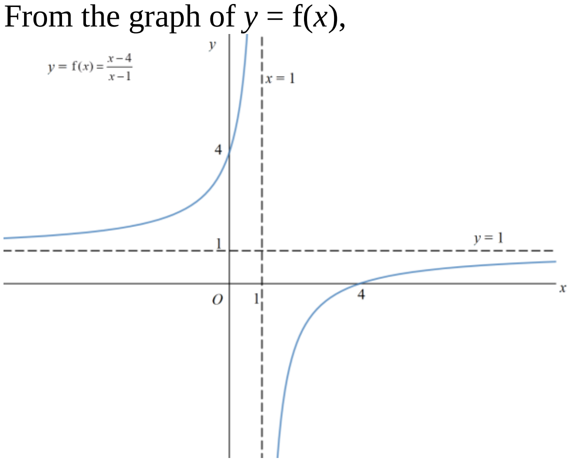
$-\dfrac{1}{2} \leq x < \dfrac{\sqrt{5}-1}{2}$

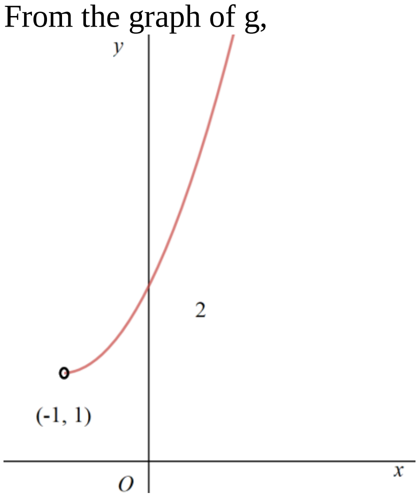
OR

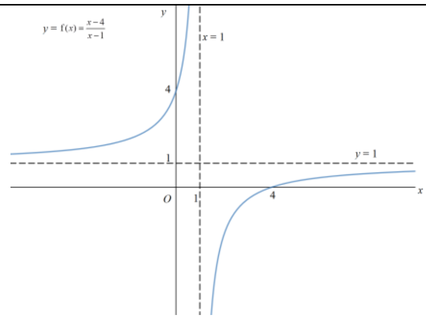
	$g^{-1}(f(x))<1 \Rightarrow f(x)>g(1)$ (since g is a decreasing function) $\Rightarrow f(x)>5$ $\Rightarrow -x^2-x+6>5$ $\Rightarrow x^2+x-1<0$ $\Rightarrow \frac{-1-\sqrt{5}}{2}<x<\frac{-1+\sqrt{5}}{2}$ Since $-\frac{1}{2}\le x\le 2$, $\therefore -\frac{1}{2}\le x<\frac{-1+\sqrt{5}}{2}$

18(i) MS	 f(9)= $f(5+4)=f(5)=f(1+4)=f(1)=1-\lvert 3-1\rvert=-1$ f(-1)= $f(-1+4)=f(3)=(3-2)^2-4=-3$ Coordinates of end-points (-1,-3) and (9,-1) Range of f(x) is [-4,1] 
1(ii)	A: Reflection in the y-axis $y=g(-x)=\frac{1}{1+(-x-1)^2}$ $\quad -2\le -x\le 2$ $=\frac{1}{1+(x+1)^2}$ $\quad -2\le x\le 2$ B: Scaling parallel to the x-axis by a factor of 2

	$y=\mathrm{g}\left(-\frac{x}{2}\right)=\frac{1}{1+\left(\frac{x}{2}+1\right)^2} \quad -2\le\frac{x}{2}\le 2$ $=\frac{4}{4+(x+2)^2} \quad -4\le x\le 4$ C: Scaling parallel to the y-axis by a factor of 3 $\mathrm{h}(x)=3\mathrm{g}\left(-\frac{x}{2}\right)=\frac{12}{4+(x+2)^2} \quad -4\le x\le 4$
1(iii)	Range of f(x) is [-4,1] Domain of h(x) is [-4,4] Since $R_{\mathrm{f}}\subset D_{\mathrm{h}}$, hf($x$) exists. Range of hf($x$) = $\left[\frac{12}{13},3\right]$ 

19(i)	$\mathrm{f}(x)=\frac{x-4}{x-1}=1-\frac{3}{x-1}$ From the graph of y = f(x),  Since <u>every horizontal line</u> $y=h, h\in\mathbb{R}, y\ne 1$ cuts the graph of f at <u>exactly 1 point</u>, f is an one-one function. Therefore f^{-1} exists.
(ii)	Let $y=1-\frac{3}{x-1}$ $\frac{3}{x-1}=1-y$ $x-1=\frac{3}{1-y}$

	$x=1+\frac{3}{1-y}$ $x=1-\frac{3}{y-1}$ Since $y=\mathrm{f}(x),\ x=\mathrm{f}^{-1}(y)$. $\therefore \mathrm{f}^{-1}(y)=1-\frac{3}{y-1}$ Hence $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1},\ x\in\mathbb{R}, x\neq 1$ Since the $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-\infty,1)\cup(1,\infty)=D_{\mathrm{f}}$ Since $\mathrm{f}^{-1}=\mathrm{f}$, $\mathrm{f}^{2}(x)=\mathrm{ff}^{-1}(x)=x,\ldots,\mathrm{f}^{100}(x)=x$ $\mathrm{f}^{101}(101)=\mathrm{f}\left(\mathrm{f}^{100}(101)\right)$ $=\mathrm{f}(101)$ $=\frac{101-4}{101-1}=\frac{97}{100}$
(iii)	$\mathrm{g}: x\mapsto x^2+2x+2,\ x\in\mathbb{R}, x>-1$ $\mathrm{g}(x)=(x+1)^2+1$ From the graph of g,  Range of g = $(1,\infty)$ Domain of f = $\mathbb{R}\setminus\{1\}$ Since $(1,\infty)\subseteq\mathbb{R}\setminus\{1\}$ i.e. Range of g $\subseteq$ Domain of f Therefore the composite function fg exists. To find range of composite function fg:

	 $(-1,\infty)\xrightarrow{\text{g}}(1,\infty)\xrightarrow{\text{f}}(-\infty,1)$ Range of fg is $(-\infty,1)$.

20(i)	$\text{f}(x)=\dfrac{a}{2-x}, \quad x\neq 0, 2$ From graph, $R_{\text{f}}=\left\{x\in\mathbb{R}: x\neq 0, \dfrac{a}{2}\right\}$ $\text{f}^2(x)=\text{f}(\text{f}(x))=\dfrac{a}{2-\dfrac{a}{2-x}}=\dfrac{a(2-x)}{4-a-2x}$ $\text{f}^2: x\mapsto \dfrac{a(2-x)}{4-a-2x}, x\neq 0, 2$ Let $y=\dfrac{a}{2-x}, \; x\neq 0, 2$ $2-x=\dfrac{a}{y}\Rightarrow x=2-\dfrac{a}{y}$ $\therefore \text{f}^{-1}: x\mapsto 2-\dfrac{a}{x}, x\neq 0, \dfrac{a}{2}$	
20(ii)	For $\text{f}^2(x)=\text{f}^{-1}(x)$, for all $x\neq 0, 2, \dfrac{a}{2}$ $\dfrac{a(2-x)}{4-a-2x}=2-\dfrac{a}{x}\Rightarrow\dfrac{ax-2a}{2x-4+a}=\dfrac{2x-a}{x}$ $ax^2-2ax=(2x-4+a)(2x-a)$ $ax^2-2ax=4x^2+(-2a-8+2a)x+a(4-a)$ By comparing coefficient, $a=4$.	
20(iii)	$\text{f}^2(x)=\text{f}^{-1}(x)\Rightarrow \text{f}^3(x)=x$ Therefore $\text{f}^{2021}(x)=\text{f}^2\,\text{f}^{2019}(x)=\text{f}^2(x)=\dfrac{2x-4}{x}=2-\dfrac{4}{x}$.	

21. ASRJC/2022/I/Q8

(a) Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2, \quad x < 0,$$
$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad x > 0.$$

(i) Explain why the composite function gf exists. [1]

(ii) Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly. [3]

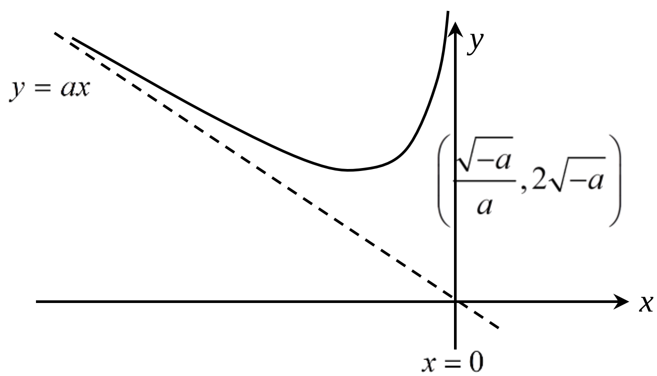
(b) For real values a, the function h is defined by

$$\mathrm{h}: x \mapsto ax - \frac{1}{x}, \quad x < 0.$$

(i) If a is negative, explain clearly with a well-labelled diagram, why h^{-1} does not exist. [4]

(ii) If a = 1, find h^{-1} in similar form. [3]

8	Solution	
	(ai) $\mathrm{R_f} = (0, \infty)$ and $\mathrm{D_g} = (0, \infty)$	
	Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, the composite function gf exists.	
	(ii) Let $\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$	
	$\mathrm{g}^{-1}(3) = \mathrm{f}(k) = k^2$	
	$\mathrm{g}(k^2) = 3$	
	$\frac{1}{k^2} = 3$	
	$k = -\frac{\sqrt{3}}{3} \left(\because \mathrm{D_f} = (-\infty, 0)\right)$	
	(bi) $\mathrm{h}(x) = ax - \frac{1}{x}$ $\mathrm{h}'(x) = a + \frac{1}{x^2}$ For $a + \frac{1}{x^2} = 0 \Rightarrow x = \frac{-1}{\sqrt{-a}} = \frac{\sqrt{-a}}{a} \left(\because x < 0\right)$	

	$\therefore \mathrm{h}\left(\frac{\sqrt{-a}}{a}\right)=\sqrt{-a}+\sqrt{-a}=2\sqrt{-a}$	
		
	Since the horizontal line $y=2\sqrt{-a}+1$ cuts the curve twice, the function is not a 1-1 function and so h^{-1} does not exist.	
	(ii) Let $y=\mathrm{h}(x)=x-\frac{1}{x}$	
	$y=x-\frac{1}{x}$ $yx=x^2-1$	
	$x^2-xy-1=0$	
	$x=\frac{y\pm\sqrt{y^2+4}}{2}$	
	$x=\frac{y-\sqrt{y^2+4}}{2}\quad(\because x<0)$	
	$\mathrm{h}^{-1}: x\mapsto\frac{x-\sqrt{x^2+4}}{2},\quad x\in\mathbb{R}$	