ANGLO-CHINESE JUNIOR COLLEGE
JC1 PROMOTIONAL EXAMINATION

Higher 2

FURTHER MATHEMATICS 9649/01

Paper 1 **4 October 2023**

3 hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 By writing $\cos(5\theta)$ in terms of $\cos\theta$, find exactly all solutions of the equation

$$16x^5 - 20x^3 + 5x - 1 = 0,$$

stating which roots are repeated. [6]

2 ~~Use mathematical induction to show that~~

$$\cancel{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^x \sin\left(\sqrt{3}x\right)\right) = 2^n \mathrm{e}^x \sin\left(\frac{n\pi}{3} + \sqrt{3}x\right)}$$

~~for all positive integers *n*. [6]~~

3 **(i)** Find all solutions of the equation $z^5 + 1 = 0$. [2]

Let $\mathrm{p}(z) = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2$.

(ii) By writing $\mathrm{p}(z)$ as a geometric series, determine the solutions of the equation $\mathrm{p}(z) = 0$. [2]

Let $w = z + \frac{1}{z}$.

(iii) Write $\mathrm{p}(z)$ as a quadratic expression in *w*, and hence find the values of *w* for which $\mathrm{p}(z) = 0$, giving your answers in surd form. [3]

(iv) From **(ii)** and **(iii)**, determine the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ in surd form. [3]

4 A curve C_1 has the equation $3(x+1)^2 + 4y^2 = 12$.

(i) Find the polar equation of C_1 in the form $r = \mathrm{f}(\theta)$. [4]

Another curve C_2 with polar equation $r = 2 + 2\cos\theta$, where $0 \le \theta < 2\pi$, intersects C_1 at points *P* and *Q*.

(ii) Sketch C_1 and C_2 on the same diagram, indicating clearly all axial intercepts. [2]

(iii) Find the area of triangle *OPQ*, where *O* is the origin. [4]

5 A recurrence relation is given by

$$x_{n+2} = 2\sqrt{3}x_{n+1} - 4x_n .$$

(i) Find the general solution of the recurrence relation in the form

$$x_n = R^n \left(A\cos(n\theta) + B\sin(n\theta) \right)$$

where A and B are arbitrary real constants and R and θ are to be determined exactly. [4]

(ii) Show that $x_{n+6} = kx_n$ for all non-negative integers n, where k is a constant to be determined. [3]

(iii) Determine the values of A and B exactly if the initial conditions are

$$x_0 = \tfrac{1}{2}\sqrt{3} \text{ and } x_1 = 1 .$$ [4]

6 **(a)** Describe completely, in geometrical terms, the loci given by

$$|z - 4 + 3\mathrm{i}| = 5 \quad \text{and} \quad |z + 1 - 5\mathrm{i}| = |z - 5 - 5\mathrm{i}|$$

and sketch both loci on the same diagram.

Find, in the form $a + \mathrm{i}b$, the complex numbers representing the points of intersection of the loci, giving the exact values of a and b. [8]

(b) Another complex number w satisfies the inequalities

$$|w - 4 + 3\mathrm{i}| \le 5 \quad \text{and} \quad |w + 1 - 5\mathrm{i}| \le |w - 5 - 5\mathrm{i}|$$

(i) Find the greatest possible value of $|w - 8 - 2\mathrm{i}|$. [3]

(ii) Find the range of values of $\arg(w - 3\mathrm{i})$. [4]

7 In a science experiment, a beaker containing 100 ml of water is prepared. Every hour, the following steps take place:

1. First, 10 g of salt are added into the beaker. The solution is stirred thoroughly so that the salt is dissolved and the concentration is uniform.
2. After that, k ml of the solution in the beaker is removed, and k ml of pure water is added into the beaker.

Let u_n be the amount of salt in the solution in the beaker in grams, after the nth time that steps above have been carried out.

(i) Explain why

$$u_n = \left(1 - \frac{k}{100}\right)\left(u_{n-1} + 10\right)$$

and state the initial conditions of the recurrence relation. [2]

(ii) Solve the recurrence relation and express u_n in terms of n and k. [3]

For parts **(iii)** and **(iv)**, suppose that $k = 10$.

(iii) Find the limiting value L of u_n as n tends to infinity. [2]

(iv) Determine the smallest value of n such that u_n is within 10% of L. [3]

(v) Find the range of values of k such that the limiting value of u_n as n tends to infinity is less than 50. [3]

8 ~~Parabolic mirrors are used because they reflect incoming parallel rays to a single point. In reality, however, it is easier and cheaper to manufacture spherical mirrors, which have a cross-section that is circular instead of parabolic.~~

~~Suppose the cross-section of a circular mirror is given by the equation~~

$$x^2+(y-r)^2=r^2,$$

~~where $r>0$. In particular, the mirror consists only of the portion of the curve which is concave up. The reflective surface is the concave side of the curve, and the light rays initially travel in the negative y-direction.~~

(i) ~~Show that a ray of light which travels along the line $x=k$, where $|k|<r$, is reflected such that it intersects the y-axis at a distance~~

$$r\left[1-\frac{1}{2}\left(1-\frac{k^2}{r^2}\right)^{-\frac{1}{2}}\right]$$

~~from the origin.~~ [5]

~~We approximate the part of the cross-section in the neighbourhood of the origin with a parabola.~~

(ii) ~~Rewrite the equation of the circular cross-section near the origin by expressing y in terms of ascending powers of x, up to and including the term in x^2.~~ [4]

~~Denote the answer to **(ii)** by $\mathrm{p}(x)$.~~

(iii) ~~Find the focus of the parabola $y=\mathrm{p}(x)$.~~ [2]

(iv) ~~For a spherical mirror, the reflected rays are said to converge approximately at a distance half of the radius from the origin. Comment on the suitability of this approximation.~~ [1]

9 ~~A parabola is defined by the parametric equations~~

$$\cancel{x = at^2, \quad y = 2at.}$$

~~The points P and Q on the parabola have parameters p and q respectively.~~

~~(i)~~ ~~The point R lies on the parabola such that the tangent to the parabola at R is parallel to PQ. Find r, the parameter of R, in terms of p and q.~~ ~~[3]~~

~~(ii)~~ ~~The point S lies on the parabola such that the tangent to the parabola at S is parallel to PR. A line through S parallel to the axis of the parabola intersects PQ at T. Show that the coordinates of T are~~

$$\cancel{\left(\frac{a}{4}\left(3p^2+q^2\right), \frac{a}{2}\left(3p+q\right)\right).} \qquad [4]$$

~~(iii)~~ ~~The lines PR and ST intersect at X. Show that $XT = 2SX$.~~ ~~[5]~~

~~(iv)~~ ~~Hence, find the ratio of the areas of ΔPQR and ΔPRS.~~ ~~[3]~~

~~(v)~~ ~~Deduce that the area of the region bounded by the parabola and PQ is $\frac{4}{3}$ times the area of ΔPQR.~~ ~~[2]~~