Paper 2

1. **(i)** Express $\dfrac{7x+9}{2x^2-9x-5}$ in partial fractions. [3]

(ii) Hence find $\displaystyle\int_6^7 \frac{7x+9}{2x^2-9x-5}\,dx$ [3]

1. (i)	$\dfrac{7x+9}{2x^2-9x-5}=\dfrac{7x+9}{(2x+1)(x-5)}$ Let $\dfrac{7x+9}{2x^2-9x-5}=\dfrac{A}{x-5}+\dfrac{B}{2x+1}$ $7x+9=A(2x+1)+B(x-5)$ Let $x=5$, $44=11A$ $A=4$ Let $x=-\dfrac{1}{2}$, $7\left(-\dfrac{1}{2}\right)+9=B\left(-\dfrac{1}{2}-5\right)$ $\dfrac{11}{2}=-\dfrac{11}{2}B$ $B=-1$ $\dfrac{7x+9}{2x^2-9x-5}=\dfrac{4}{x-5}-\dfrac{1}{2x+1}$	
(ii)	$\displaystyle\int_6^7 \frac{7x+9}{2x^2-9x-5}\,dx$ $=\displaystyle\int_6^7\left(\frac{4}{x-5}-\frac{1}{2x+1}\right)dx$ $=\left[4\ln(x-5)-\dfrac{1}{2}\ln(2x+1)\right]_6^7$ $=\left(4\ln 2-\dfrac{1}{2}\ln 15\right)-\left(4\ln 1-\dfrac{1}{2}\ln 13\right)$ $=2.70$	**(6 marks)**

2. Given that $f(x) = 2x^3 - 3x^2 - 3x + 22$ is exactly divisible by $2x^2 + bx + c$,

(i) find the value of b and of c, [4]

(ii) show that $2x^3 - 3x^2 - 3x + 22 = 0$ has only one real root, [3]

(iii) find the remainder when $f(x)$ is divided by $(2x+3)$. [2]

2.	**(i)**	$f(x) = 2x^3 - 3x^2 - 3x + 22$ $f(-2) = 2(-2)^3 - 3(-2)^2 - 3(-2) + 22$ $= 0$ $(x+2)$ is a factor. Let $f(x) = 2x^3 - 3x^2 - 3x + 22 = (x+2)(2x^2 + bx + c)$ Comparing coefficient of x^2, $b + 4 = -3$ $b = -7$ $c = 11$	
	(ii)	$f(x) = 0$ $(x+2)(2x^2 - 7x + 11) = 0$ $x = -2$, $2x^2 - 7x + 11 = 0$ Discriminant $= (-7)^2 - 4(2)(11)$ $= -39 < 0$ $2x^2 - 7x + 11 = 0$ has no real roots. Hence, $f(x) = 0$ has only one real roots.	
	(iii)	$f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 - 3\left(-\frac{3}{2}\right)^2 - 3\left(-\frac{3}{2}\right) + 22$ $= 13$	**(9 marks)**

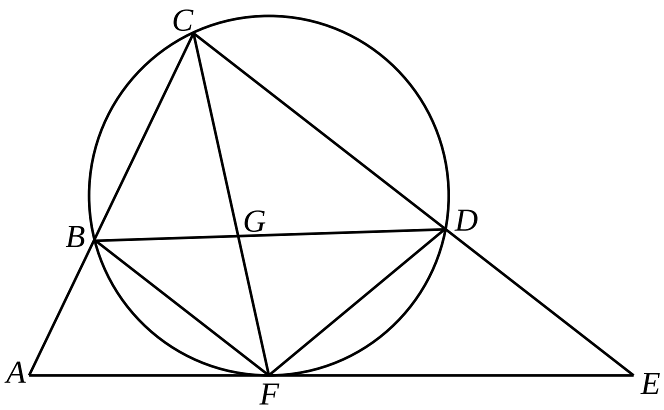
3. The roots of the quadratic equation $2x^2 - 3x - 4 = 0$ are α and β.

(i) Without using a calculator, show that $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2} = \dfrac{99}{32}$. [5]

(ii) Hence find the quadratic equation whose roots are $\dfrac{\alpha}{\beta^2}$ and $\dfrac{\beta}{\alpha^2}$. [2]

3. (i)	$2x^2 - 3x - 4 = 0$ $\alpha + \beta = \dfrac{3}{2}$ $\alpha\beta = -2$ $\dfrac{\alpha}{\beta^2} + \dfrac{\beta}{\alpha^2} = \dfrac{\alpha^3 + \beta^3}{\alpha^2\beta^2}$ $= \dfrac{(\alpha+\beta)(\alpha^2 + \beta^2 - \alpha\beta)}{\alpha^2\beta^2}$ $= \dfrac{(\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]}{\alpha^2\beta^2}$ $= \dfrac{\frac{3}{2}\left[\left(\frac{3}{2}\right)^2 - 3(-2)\right]}{(-2)^2}$ $= \dfrac{\frac{3}{2}\left(\frac{9}{4} + 6\right)}{4}$ $= \dfrac{99}{32}$	
(ii)	$\left(\dfrac{\alpha}{\beta^2}\right)\left(\dfrac{\beta}{\alpha^2}\right) = \dfrac{1}{\alpha\beta}$ $= -\dfrac{1}{2}$ Quadratic equation, $x^2 - \dfrac{99}{32}x - \dfrac{1}{2} = 0$ or $32x^2 - 99x - 16 = 0$	**(7 marks)**

4. The diagram below shows a quadrilateral *BCDF* whose vertices lie on the circumference of a circle. The tangent to the circle at the point *F* meets *CB* extended at *A* and *CD* extended at *E*. The lines *CF* and *BD* intersect at *G* and *BG* = *GF*.



(i) State an angle which is equal to angle *BFA*. [1]

(ii) Prove that angle *FDE* = angle *FAB* + angle *FCB*. [2]

(iii) Prove that quadrilateral *BCDF* is a trapezium. [3]

4. (i)	$\angle FCB = \angle BFA$ or $\angle BDF$ (Alternate Segment Theorem)	
(i)	$\angle CBF = \angle FAB + \angle BFA$ (Exterior angle of Δ) $= \angle FDE$ (ext ∠ of a cyclic quadrilateral) $\therefore \angle FDE = \angle FAB + \angle FCB$.	
(iii)	$\angle FBG = \angle BFG$ (Base ∠s of an isos Δ) $\angle FBG = \angle FCD$ (∠s in the same segment) $\angle FCD = \angle BFG$ Since the alternate angles, $\angle FCD$ and $\angle BFG$ are equal, BF is parallel to CD. Hence, quadrilateral *BCDF* is a trapezium.	**(6 marks)**

5. Solve the following equations.

(i) $(2^{3x+2})(3^{x-1}) = 32$ [2]

(ii) $7(3^{1-x}) = 3^{x+1} + 2$ [4]

(iii) $\log_3(x+5) - \log_{\sqrt{3}}(x-1) = \log_3 2$ [4]

5.	**(i)**	$2^{3x+2}3^{x-1} = 32$ $2^{3x}3^{x} = 32 \times \frac{3}{4}$ $24^x = 24$ $x = 1$	
	(ii)	$7(3^{1-x}) = 3^{x+1} + 2$ Let $t = 3^x$ $\frac{21}{t} = 3t + 2$ $3t^2 + 2t - 21 = 0$ $(3t-7)(t+3) = 0$ $t = 3^x = \frac{7}{3},\ -3\ (rej)$ $x = \frac{\lg\left(\frac{7}{3}\right)}{\lg 3}$ $= 0.771$	
	(iii)	$\log_3(x+5) - \frac{\log_3(x-1)}{\log_3\sqrt{3}} = \log_3 2$ $\log_3(x+5) - \frac{\log_3(x-1)}{\frac{1}{2}\log_3 3} = \log_3 2$ $\log_3 \frac{x+5}{(x-1)^2} = \log_3 2$ $x+5 = 2(x-1)^2$ $x+5 = 2x^2 - 4x + 2$ $2x^2 - 5x - 3 = 0$ $(2x+1)(x-3) = 0$ $x = -\frac{1}{2}\ \text{(rej)}, 3$	**(10 marks)**

6. A curve has the equation $y = \dfrac{x}{e^{2x}}$. The point (p, q) is a stationary point on the curve.

Determine

(i) the exact value of p and of q, [4]

(ii) the nature of the stationary point (p, q). [3]

Hence

(iii) write down the values of x for which y is increasing. [1]

<table>
<tr>
<td>6. (i)</td>
<td>$\dfrac{dy}{dx} = \dfrac{e^{2x} - x(2e^{2x})}{e^{4x}}$
$= \dfrac{1-2x}{e^{2x}}$
At (p, q), $\dfrac{1-2p}{e^{2p}} = 0$
$p = \dfrac{1}{2}$, $q = \dfrac{1}{2e}$</td>
<td></td>
</tr>
<tr>
<td>(ii)</td>
<td>$\dfrac{d^2y}{dx^2} = \dfrac{-2e^{2x} - (1-2x)(2e^{2x})}{(e^{2x})^2}$
$= \dfrac{-4+4x}{e^{2x}}$
At (p, q), $\dfrac{d^2y}{dx^2} = -\dfrac{2}{e} < 0$
Hence (p, q) is a maximum point.

<u>Alternative Method</u>
<table>
<tr><td>x</td><td>0.5^-</td><td>0.5</td><td>0.5^+</td></tr>
<tr><td>$\dfrac{dy}{dx}$</td><td>+</td><td>0</td><td>–</td></tr>
<tr><td>Slope</td><td>/</td><td>–</td><td>\</td></tr>
</table>
Hence (p, q) is a maximum point.</td>
<td></td>
</tr>
<tr>
<td>(iii)</td>
<td>y is increasing when $x < \dfrac{1}{2}$.</td>
<td>(8 marks)</td>
</tr>
</table>

7. The population, P, of a certain species of fish after t years is given by

$$P = 1750(1 + 2e^{-kt}).$$

When $t = 2$, the population is 4600.

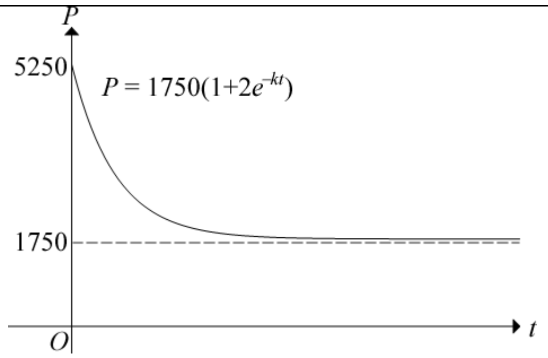
(i) Find the value of k. [3]

Any species is considered as an "Endangered Species" if its population falls below 2500.

(ii) Determine, with working, whether this species of fish will become an "Endangered Species" after 10 years. [2]

(iii) Hence sketch the population-time graph. [2]

7. (i)	When $t = 2$, $P = 4600$, $4600 = 1750(1 + 2e^{-2k})$ $2e^{-2k} = \frac{4600}{1750} - 1$ $e^{-2k} = \frac{57}{70}$ $-2k = \ln\left(\frac{57}{70}\right)$ $k = -\frac{1}{2}\ln\left(\frac{57}{70}\right)$ $= 0.10272$ ≈ 0.103	
(ii)	When $t = 10$, $P = 1750[1 + 2e^{\frac{1}{2}\ln\left(\frac{57}{70}\right)\times 10}]$ $= 3003 > 2500$ Since the population of the fish is greater than 2500, the species of fish will not be considered as an "Endangered Species".	
	<u>Alternative Method</u> When $P = 2500$, $2500 = 1750(1 + 2e^{-0.10272t})$ $\frac{2500}{1750} - 1 = 2e^{-0.10272t}$ $e^{-0.10272t} = \frac{3}{14}$ $-0.10272t = \ln\frac{3}{14}$ $t = \frac{\ln\frac{3}{14}}{-0.10272}$ $= 15.0 > 10$ Since the population of the fish takes 15 years to fall below 2500, the species of fish will not be considered as an "Endangered Species".	

(iii)		**(7 marks)**

8. The depth of water, d metres, at a pier, t hours after low tide, can be modelled by the formula

$$d = c - a\cos(bt),$$

where a, b and c are positive constants.

(i) If low tides occur every 12 hours, find the value of b. [1]

Given that the depth of water at the pier was 2 metres during low tide and 8 metres during high tide,

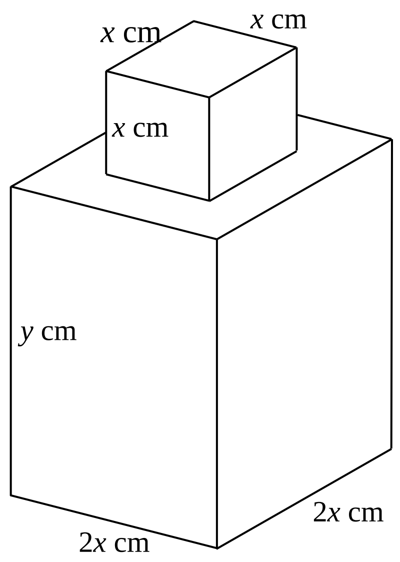
(ii) find the value of a and of c. [2]

The pier will be open when the depth of the water is more than 4 m.

(iii) For how long will the pier be open in a 12-hour period after low tide. [3]

8.	**(i)**	$b = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$	
	(ii)	$a = \dfrac{8-2}{2} = 3$ $c = 5$	
	(iii)	$d = 5 - 3\cos\left(\dfrac{\pi}{6}t\right)$ When $d = 4$, $5 - 3\cos\left(\dfrac{\pi}{6}t\right) = 4$ $\cos\left(\dfrac{\pi}{6}t\right) = \dfrac{1}{3}$ Basic angle = 1.2309 $\dfrac{\pi}{6}t = 1.2309, 5.0522$ $t = 2.3508, \ 9.6489$ Length of time for which the pier is open $= 9.6489 - 2.3508$ $= 7.30\text{ h}$	**(6 marks)**

9. The diagram shows a solid which consists of a cube fixed on top of a cuboid. The cube has sides x cm. The cuboid has a square base of side $2x$ cm and a height of y cm.



Given that the volume of the solid is 270 cm^3,

(i) show that the total surface area, A cm^2, of the solid is given by

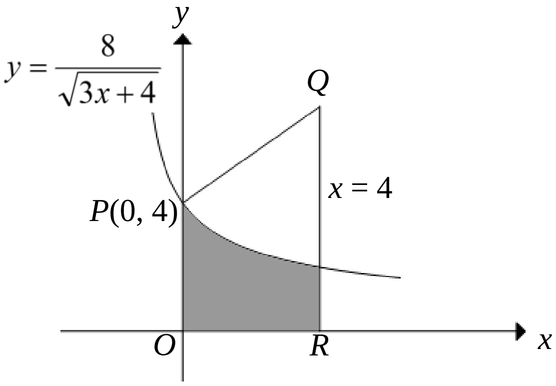
$$A = 10x^2 + \frac{540}{x}.$$ [5]

Given that x can vary,

(ii) find the value of x for which A has a stationary value and determine whether this value of A is a maximum or a minimum. [5]

9. (i)	$x^3 + 4x^2y = 270$ $4x^2y = 270 - x^3$ $y = \frac{270 - x^3}{4x^2}$ $A = 4x^2 + 2(2x)^2 + 4(2xy)$ $= 12x^2 + 8x\left(\frac{270 - x^3}{4x^2}\right)$ $= 12x^2 + \frac{540}{x} - 2x^2$ $= 10x^2 + \frac{540}{x}$	
(ii)	$\frac{\mathrm{d}A}{\mathrm{d}x} = 20x - \frac{540}{x^2}$ When $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$, $20x - \frac{540}{x^2} = 0$ $20x^3 = 540$ $x^3 = 27$ $x = 3$ $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 20 + \frac{1080}{x^3}$ When $x = 3$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 60 > 0$ Hence, A has a minimum value when $x = 3$.	**(10 marks)**

10. The diagram shows part of the curve $y=\dfrac{8}{\sqrt{3x+4}}$. The curve intersects the y-axis at $P(0, 4)$. The normal to the curve at P intersects the line $x = 4$ at the point Q and the line segment QR is parallel to the y-axis.



(i) Find the coordinates of Q. [5]

(ii) Find the ratio of the area of the shaded region to the area of the trapezium $OPQR$ in the form $1: n$. [5]

10. (i)	$\dfrac{dy}{dx}=8\left(-\dfrac{1}{2}\right)(3x+4)^{-\frac{3}{2}}(3)$ $=-\dfrac{12}{(3x+4)^{\frac{3}{2}}}$ At P, $\dfrac{dy}{dx}=-\dfrac{12}{4^{\frac{3}{2}}}$ $=-\dfrac{3}{2}$ Equation of PQ, $y=\dfrac{2x}{3}+4$ At Q, $x=4$, $y=\dfrac{20}{3}$ $\therefore Q\left(4, \dfrac{20}{3}\right)$	
(ii)	Area of A $=\displaystyle\int_0^4 8(3x+4)^{-\frac{1}{2}}\,dx$ $=\left[\dfrac{8(3x+4)^{\frac{1}{2}}}{\frac{1}{2}\times 3}\right]_0^4$ $=\left[\dfrac{16}{3}\sqrt{3x+4}\right]_0^4$ $=\dfrac{16}{3}(4-2)$ $=\dfrac{32}{3}$ unit^2 Area of $OPQR$ $=\dfrac{1}{2}\times\left(4+\dfrac{20}{3}\right)\times 4$ $=\dfrac{64}{3}$ unit^2	

	Area of shaded region : Area of $OPQR$ $=\dfrac{32}{3}:\dfrac{64}{3}$ $=1:2$	**(10 marks)**

11. (a) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [3]

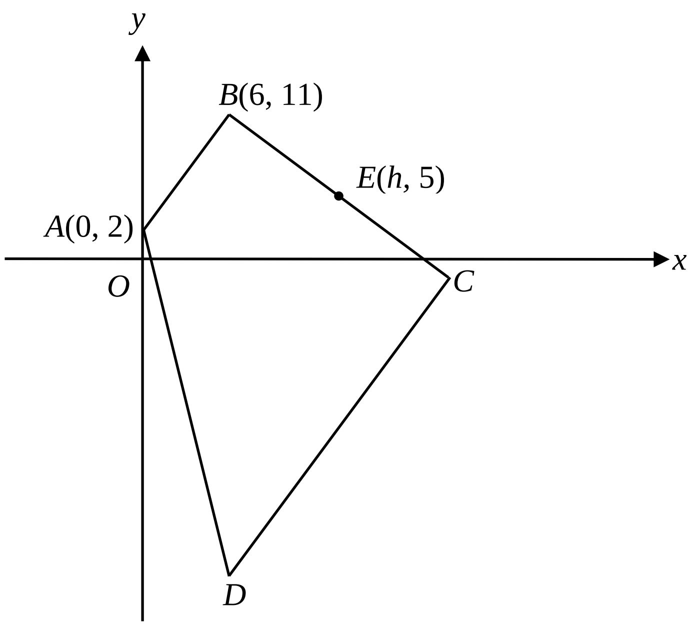
(ii) Find all the angles between $-\frac{\pi}{2}$ and π which satisfy the equation

$$\frac{4\cos^3\theta - 3\cos\theta}{\cos\frac{3\theta}{2}} = 4\sin\frac{3\theta}{2}.$$ [4]

(b) Solve the equation $6\cos x - 3\sec x = 7$ for $-180^{\circ} \le x \le 180^{\circ}$. [4]

11. (a)(i)	$\cos 3\theta = \cos(2\theta+\theta)$ $= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$ $= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$ $= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$ $= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ $= 4\cos^3\theta - 3\cos\theta$	
(ii)	$\dfrac{4\cos^3\theta - 3\cos\theta}{\cos\frac{3\theta}{2}} = 4\sin\dfrac{3\theta}{2}$ $\cos 3\theta = 2\left(2\sin\dfrac{3\theta}{2}\cos\dfrac{3\theta}{2}\right)$ $\cos 3\theta = 2\sin 3\theta$ $\tan 3\theta = \dfrac{1}{2}$ Basic $\angle = 0.46364$ $3\theta = -2.6779,\ 0.46364,\ 3.6052,\ 6.74682$ $\theta = -0.893,\ 0.155,\ 1.20,\ 2.25$	
(b)	$6\cos x - 3\sec x = 7$ $6\cos x - \dfrac{3}{\cos x} = 7$ $6\cos^2 x - 7\cos x - 3 = 0$ $(3\cos x + 1)(2\cos x - 3) = 0$ $\cos x = -\dfrac{1}{3},\ \dfrac{3}{2}$ (NA) Basic $\angle = 70.528°$ $x = -109.5°,\ 109.5$	**(11 marks)**

12. Solutions to this question by accurate drawing will not be accepted.



The diagram, which is not drawn to scale, shows a trapezium *ABCD* in which *AB* is parallel to *DC* and *BD* is parallel to the y–axis. The coordinates of *A* and of *B* are (0, 2) and (6, 11) respectively.
$E(h, 5)$ is the midpoint of *BC* such that $AB = BE$ and $h > 0$.

(i) Show that the value of h is 15. [2]

Find

(ii) the coordinates of *C*, [2]

(iii) the equation of *DC*, [3]

(iv) the coordinates of *D*. [1]

Given that the area of triangle $ABE = 58\frac{1}{2}$ units^2,

(v) find the perpendicular distance from *B* to the line segment *AE*. [2]

12.	**(i)**	$AB = BE$ $\sqrt{6^2+9^2} = \sqrt{(6-h)^2+6^2}$ $9^2 = (6-h)^2$ $6-h = \pm 9$ $h = -3,\ 15$ $h > 0,\ \therefore h = 15$	
	(ii)	Let $C(x, y)$. $\left(\dfrac{6+x}{2}, \dfrac{11+y}{2}\right) = (15, 5)$ $x = 24, y = -1$ $C(24,\ -1)$	
	(iii)	Gradient of $DC = \dfrac{9}{6} = \dfrac{3}{2}$ Equation of DC, $y+1 = \dfrac{3}{2}(x-24)$ $2y - 3x + 74 = 0$ or $y = \dfrac{3}{2}x - 37$	
	(iv)	$D\ (6, -28)$	
	(v)	$AE = \sqrt{15^2+3^2}$ $= \sqrt{234}$ or $3\sqrt{26}$ units Let the perpendicular distance from B to AE = p units. $\dfrac{1}{2} \times p \times \sqrt{234} = 58\dfrac{1}{2}$ $p = \dfrac{2\left(58\dfrac{1}{2}\right)}{\sqrt{234}}$ $= \dfrac{3\sqrt{26}}{2}$ or 7.65 units	**(10 marks)**