

H2 MATHEMATICS 9758 16 Sept 2024 JC2 Prelim Paper 2 (100 marks) 3 hours Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME	
CLASS	

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.

Please write clearly and use capital letters.

Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.

Do not use staples, paper clips, glue or correction fluid.

/

Answer all the questions and write your answers in this booklet. Do not tear out any part of this booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.

The number of marks is given in brackets [] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 13 printed pages and 3 blank pages.

Section A: Pure Mathematics [40 marks]

1 (a)
Let
$$x = \cot \theta$$

 $\frac{dx}{d\theta} = -\csc^2 \theta$
 $\int \frac{1}{x^2 \sqrt{1+x^2}} dx$
 $= \int \frac{1}{\cot^2 \theta \sqrt{1+\cot^2 \theta}} \times (-\csc e^2 \theta) d\theta$
 $= \int \frac{1}{\cot^2 \theta \sqrt{\cos e^2 \theta}} \times (-\csc e^2 \theta) d\theta$
 $= -\int \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{1}{\sin \theta} d\theta$
 $= \int (-\sin \theta)(\cos \theta)^{-2} d\theta$ OR $= -\int \tan \theta \sec \theta d\theta$
 $= -\frac{1}{\cos \theta} + C$
 $= -\frac{\sqrt{x^2+1}}{x} + C$

2 (ai)

$$\int \frac{9u - 8}{4 + 9u^2} du$$

= $\int \frac{\frac{1}{2}(18u)}{4 + 9u^2} - \frac{8}{2^2 + (3u)^2} du$
= $\left[\frac{1}{2}\ln(4 + 9u^2) - \frac{8}{(2)(3)}\tan^{-1}\left(\frac{3u}{2}\right)\right] + C$
= $\frac{1}{2}\ln(4 + 9u^2) - \frac{4}{3}\tan^{-1}\left(\frac{3u}{2}\right) + C$
(ii) Area required = $\int_1^3 y \, dx$
= $\int_0^1 \frac{9u}{4 + 9u^2} \times (2u + 1) \, du$
= $\int_0^1 \frac{18u^2 + 9u}{4 + 9u^2} \, du$
= $2\int_0^1 1 \, du + \int_0^1 \frac{(9u - 8)}{4 + 9u^2} \, du$
= $\left[2u + \frac{1}{2}\ln(4 + 9u^2) - \frac{4}{3}\tan^{-1}\left(\frac{3u}{2}\right)\right]_0^1$ (from part (i))

$$= \left(2 + \frac{1}{2}\ln 13 - \frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right)\right) - \left(0 + \frac{1}{2}\ln 4 - \frac{4}{3}\tan^{-1}(0)\right)$$

$$= 2 + \frac{1}{2}\ln \frac{13}{4} - \frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right)\text{units}^{2}$$

(b) Given $(x+2)^{2} + 4(y-1)^{2} = 4$
 $(x+2)^{2} = 4[1-(y-1)^{2}] \Longrightarrow x + 2 = \pm 2\sqrt{1-(y-1)^{2}}$

The shaded region is bounded by the section of the ellipse where $x \le -2$. Hence $x = -2 - 2\sqrt{1 - (y-1)^2}$.



Volume of solid formed r^2

$$= \pi 4^{2} (2-1) - \pi \int_{1}^{2} x^{2} dy$$

= $16\pi - \pi \int_{1}^{2} \left(-2 - 2\sqrt{1 - (y-1)^{2}}\right)^{2} dy$
= 9.6 units³ (From GC)

3 (a)
$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$
 and $\overrightarrow{BE} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix} - \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$
Consider $\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -15 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$
Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = 2 - 15 + 5 = -5$
 $\Rightarrow x(1) + y(-5) + z(2) = -5$

Thus equation of surface *ABE* is x-5y+2z = -5

(b)
$$\overrightarrow{OM} = \frac{1}{2} \begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix} + \begin{bmatrix} 8 \\ 9 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 13 \\ 15 \\ 16 \end{bmatrix}$$

 $\overrightarrow{DM} = \frac{1}{2} \begin{bmatrix} 13 \\ 15 \\ 16 \end{bmatrix} - \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix}$

Line *DM* has equation: $\mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 1 \\ 10 \end{pmatrix}, \lambda \in \Box$ (c) Let foot of perpendicular from *M* to surface be *N*. Thus $\overrightarrow{ON} = \frac{1}{2} \begin{pmatrix} 13\\15\\16 \end{pmatrix} + t \begin{pmatrix} 1\\-5\\2 \end{pmatrix}$ for some $t \in \Box$ $\Rightarrow \begin{pmatrix} \frac{13}{2} + t \\ \frac{15}{2} - 5t \\ 8 + 2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} = -5$ $\Rightarrow \frac{13}{2} + t - \frac{75}{2} + 25t + 16 + 4t = -5$ $\Rightarrow t = \frac{1}{2}$ Thus $\vec{ON} = \begin{pmatrix} \frac{13}{2} + \frac{1}{3} \\ \frac{15}{2} - 5\left(\frac{1}{3}\right) \\ 8 + 2\left(\frac{1}{3}\right) \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 41 \\ 35 \\ 52 \end{pmatrix}$ Thus we have $N\left(\frac{41}{6}, \frac{35}{6}, \frac{26}{3}\right)$. (**d**) $\begin{vmatrix} \rightarrow \\ MN \end{vmatrix} = \begin{vmatrix} \frac{1}{6} \begin{pmatrix} 41 \\ 35 \\ 52 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 13 \\ 15 \\ 16 \end{pmatrix} = \begin{vmatrix} \frac{1}{3} \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$ $=\frac{1}{3}\sqrt{1+25+4}=\frac{\sqrt{30}}{3}$ (e) Consider $\begin{pmatrix} 4+5\lambda\\7+\lambda\\3+10\lambda \end{pmatrix} = \begin{pmatrix} 9\\8\\13 \end{pmatrix} \Rightarrow \begin{cases} \lambda=1\\\lambda=1\\\lambda=1 \end{cases}$

Since the value of λ is consistent, *P* lines on the line *DM*. Also, 9-5(8)+2(13)=-5

Hence P also lies on the plane ABE. Thus P is the point of intersection between the line DM and the plane ABE. Let the reflection of point M about the surface ABE be the point M'. By ratio theorem (mid-point theorem),

$$\vec{ON} = \frac{\vec{OM} + \vec{OM'}}{2}$$

$$\vec{OM}' = 2\vec{ON} - \vec{OM} = \frac{1}{3} \begin{pmatrix} 41\\35\\52 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 13\\15\\16 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 43\\25\\56 \end{pmatrix}$$
$$\vec{PM}' = \frac{1}{6} \begin{pmatrix} 43\\25\\56 \end{pmatrix} - \begin{pmatrix} 9\\8\\13 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 11\\23\\22 \end{pmatrix}$$

Thus equation of the reflection of DM about the surface ABE is

$$\mathbf{r} = \begin{pmatrix} 9\\8\\13 \end{pmatrix} + \mu \begin{pmatrix} 11\\23\\22 \end{pmatrix}, \mu \in \Box$$

4 (a)



$$\frac{dV}{dr} = \frac{\pi(60) \left[8(50^2) - 3(60^2) \right]}{\sqrt{4(50^2) - 60^2}} = 6900\pi$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$100\pi = 6900\pi \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{69} \text{ cm s}^{-1}$$
(c) $\frac{dV}{dr} = \frac{\pi r \left(2\sqrt{2}a + \sqrt{3}r \right) \left(2\sqrt{2}a - \sqrt{3}r \right)}{\sqrt{4a^2 - r^2}} = 0$

$$\frac{\pi r \left(2\sqrt{2}a + \sqrt{3}r \right) \left(2\sqrt{2}a - \sqrt{3}r \right)}{\sqrt{4a^2 - r^2}} = 0$$

$$r = 0 \text{ (rej $\thereforms r > 0$) or $r = -\frac{2\sqrt{2}a}{\sqrt{3}}$ (rej $\thereform r > 0$) or $r = \frac{2\sqrt{2}a}{\sqrt{3}}$$$
When $0 < r < \frac{2\sqrt{2}}{\sqrt{3}}a$,
$$\frac{\pi r \left(2\sqrt{2}a + \sqrt{3}r \right)}{\sqrt{4a^2 - r^2}} > 0 \quad \& \ 2\sqrt{2}a - \sqrt{3}r > 0 \quad \Rightarrow \ \frac{dV}{dr} > 0.$$
When $r > \frac{2\sqrt{2}}{\sqrt{3}}a$,
$$\frac{\pi r \left(2\sqrt{2}a + \sqrt{3}r \right)}{\sqrt{4a^2 - r^2}} > 0 \quad \& \ 2\sqrt{2}a - \sqrt{3}r < 0 \quad \Rightarrow \ \frac{dV}{dr} < 0.$$

$$\frac{\left(\frac{2\sqrt{2}a}{3} \right)^{-1} \qquad \left(\frac{2\sqrt{2}a}{3} \right) \qquad \left(\frac{2\sqrt{2}a}{3} \right)^{+} \qquad \left(\frac{2\sqrt{2}a}{3} \right)^{-} \qquad \left(\frac{2\sqrt{2}a}{3} \right)^{+} \qquad \left($$

V is maximum when $r = \frac{2\sqrt{2}a}{\sqrt{3}} = \frac{2\sqrt{6}a}{3}$ When $r = \frac{2\sqrt{6}a}{3}$, $V = \pi \left(\frac{2\sqrt{6}a}{3}\right)^2 \sqrt{4a^2 - \left(\frac{2\sqrt{6}a}{3}\right)^2}$ $V = \frac{16\pi a^3}{3\sqrt{3}} = \frac{16\sqrt{3}\pi a^3}{9}$ cm³



Section B: Probability and Statistics [60 marks]



=0.501 (3 s.f)

6 (i) Table of outcomes:

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

The probability distribution of *T* is given by:

Т	0	1	2	3	4	6	9
$\mathbf{P}(T=t)$	7	1	2	2	1	2	1
	16	16	16	16	16	16	16
			_ 1	_ 1		_ 1	
			$=\frac{-}{8}$	$=\frac{-}{8}$		$=\frac{-}{8}$	

(ii)

$$E(T) = 0\left(\frac{7}{16}\right) + 1\left(\frac{1}{16}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + 6\left(\frac{1}{8}\right) + 9\left(\frac{1}{16}\right) = \frac{9}{4}$$

$$E(T^{2}) = 0^{2} \left(\frac{7}{16}\right) + 1^{2} \left(\frac{1}{16}\right) + 2^{2} \left(\frac{1}{8}\right) + 3^{2} \left(\frac{1}{8}\right) + 4^{2} \left(\frac{1}{16}\right) + 6^{2} \left(\frac{1}{8}\right) + 9^{2} \left(\frac{1}{16}\right) = \frac{49}{4}$$

$$Var(T) = E(T^{2}) - \left[E(T)\right]^{2} = \frac{49}{4} - \left(\frac{9}{4}\right)^{2} = \frac{115}{16}$$

$$(iii) P(|T - 2\mu| > \sigma) = P(T - 2\mu > \sigma) + P(T - 2\mu < -\sigma)$$

$$= P(T > 2\mu + \sigma) + P(T < 2\mu - \sigma)$$

$$= P(T > 7.18095) + P(T < 1.81905)$$

$$= P(T = 9) + P(T = 0) + P(T = 1)$$

$$= \frac{1}{16} + \frac{7}{16} + \frac{1}{16} = \frac{9}{16} \text{ or } 0.5625$$

7 (a) Let X be the mass, in grams, of a randomly chosen packet of semolina. $X \sim N(225, 25^2)$

$$4X \square N(4 \times 225, 4^{2} \times 25^{2})$$

$$4X \square N(900, 100^{2})$$

$$P(850 \le 4X \le 1050) = 0.62466 (5 \text{ sf})$$

$$= 0.624 (3 \text{ sf})$$

(b) Let *Y* be the mass, in grams, of a randomly chosen packet of millet flour. *Y* ~ N(μ , σ^2)

Let
$$M = \frac{X_1 + X_2 + X_3 + Y_2 + Y_2}{5}$$

 $M \square N\left(\frac{675 + 2\mu}{5}, \frac{3(25^2) + 2(\sigma^2)}{25}\right)$
 $P(M < 125) = P(M > 265) = 0.02$
 $\Rightarrow \frac{675 + 2\mu}{5} = \frac{125 + 265}{2}$
 $\Rightarrow \frac{675 + 2\mu}{5} = 195$
 $\Rightarrow \mu = 150$
 $P(M < 125) = 0.02$
 $P\left(Z < \frac{125 - 195}{\sqrt{\frac{1875 + 2\sigma^2}{25}}}\right) = 0.02$ where $Z \sim N(0, 1)$
 $-\frac{70}{\sqrt{\frac{1875 + 2\sigma^2}{25}}} = -2.0537$
 $1875 + 2\sigma^2 = 29042.99478$
 $\sigma = 116.55$ (5 s.f.) = 116 (3 s.f.)

9

(i) Number of ways = $\binom{10}{5} \times 5! = 30240$

8

(ii) Number of ways = $\begin{bmatrix} 7\\5\\0 \end{bmatrix} + \begin{bmatrix} 7\\4\\1 \end{bmatrix} = 5!$

 $=(21\times1+35\times3)5!=15120$

(iii) Number of ways = (8-1)!2! = 10080

(iv) Case 1 – Beth in 1 row while Anne and Cathie are in another row

$$\binom{7}{4} \times 5 \times 5 \times 2 = 1008000$$

Case 2 – Beth and one of them in 1 row

$$\binom{7}{3} \times 3 \times \binom{2}{1} \times \binom{4}{2} \times 2 \times 5 \times 2 = 1209600$$

Case 3 – Anne, Beth and Cathie are in the same row
$$\binom{7}{2} \times 2 \times \binom{3}{2} \times 2 \times 2 \times 5 \times 2 + \binom{7}{2} \times 2 \times 3 \times 5 \times 2 = 181440$$

A and C together
Thus number of ways = 1008000 + 1209600 + 181440 = 2399

1 hus number of ways = 1008000 + 1209600 + 181440 = 2399040

9 (a) The probability of a cookie is flawed is constant at *p* for each cookie. OR

The event that a cookie is flawed is independent of another cookie being flawed. (b) $C \sim B(20, p)$

$$P(C=0) + P(C=1) = 0.15$$

$$(1-p)^{20} + \binom{20}{1} p^{1} (1-p)^{19} = 0.15$$

$$(1-p)^{19} (1+19p) = 0.15$$
Using G.C, $p = 0.15891 = 0.159$
(c) Let X denote the number of flawed cookies in a box of 20 cookies.
 $X \sim B(20, 0.08)$
 $P(X < 4) = P(X \le 3) = 0.92938$
Let Y be the number of rejected boxes out of 10 boxes.
 $Y \sim B(10, 1-0.92938)$
 $Y \sim B(10, 0.070615)$
 $P(2 \le Y \le 5) = P(Y \le 5) - P(Y \le 1)$
 $= 0.15388$
 $= 0.154$ (to 3 sig fig)
(d) Let W be the number of rejected boxes out of 15 boxes.
 $W \sim B(14, 0.070615)$
Let V be the number of rejected box is the 15th box | V = 3)
 $= \frac{P(W = 2) \times 0.070615}{P(V = 3)}$
 $= 0.2$

10 (a) Using G.C, $\overline{g} = 3.075$ $\overline{u} = \frac{84+a}{8}$ Since $(\overline{u}, \overline{g})$ lies on the regression line, $3.075 = -0.0765 \left(\frac{84+a}{8}\right) + 3.99$ $a = 11.686 \approx 11.7$ (correct to 1 decimal place)

(bi) and (ii)



(iii) From the scatter diagram, as *x* increases, *y* decreases at an increasing rate. Hence a linear model is not a suitable model.

(iv) Using G.C,
$$r_A = -0.9981$$

 $r_B = -0.8970$

Since r_A is closer to -1 than r_B , so model (A) is a better model than model (B). From the G.C.

$$y = -0.00511019x^{2} + 49.2444$$

$$y = -0.00511x^{2} + 49.2(3.s.f)$$

(v) When $x = 80$,

$$y = -0.00511019(80)^{2} + 49.2444 = 16.53916$$

$$y = 16.5 (3.s.f)$$

(vi) The estimate is unreliable because the data substituted is outside the data range $(10 \le x \le 70)$ and so the linear relationship between y and x^2 may not hold true.

11 (a) Let X denote the length of a randomly chosen green leaf, in centimetres.

Let *L* be the total lengths of 100 green leaves.

$$L = X_1 + X_2 + X_3 \dots + X_{100}$$

Since 100 > 30 (*n* is considered large), by Central Limit Theorem,

 $L \square$ N(12×100, 3.5²×100) approx.

 $P(L \ge 1138) = 0.96175 \approx 0.962 \ (3 \ sf)$

(bi) Unbiased estimate of the population variance

$$s^{2} = \frac{64}{63} (16.4^{2}) = 273.229 \approx 273.23 (2 \text{ dp}) (2 \text{ decimal places})$$

Let *Y* denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location *A* and μ denote the population mean time spent in minutes using the one-seater pod facilities at location *A*.

To test H₀: $\mu = 131$ Against H₁: $\mu < 131$ (Workspace operator overstating the claim)

Conduct a one-tail test at 3% level of significance, i.e., $\alpha = 0.03$ Under H₀,

Since n = 64 (> 30) is large, by Central Limit Theorem,

$$\overline{Y} \sim N\left(131, \frac{273.229}{64}\right)$$
 approximately.
 $\overline{t} = 127$

Using GC, p-value = $0.026438 \approx 0.0264$ (3 sf)

Since p-value = 0.0264 < 0.03, we reject H₀. There is sufficient evidence at 3% level of significance to conclude that that the centre manager was overstating his claim.

(ii) There is a probability of 0.03 of concluding that the population average time spent using the one-seater pod facilities at location A is less than 131 minutes when in fact the population average time spent using the one-seater pod facilities in location A is 131 minutes.

(iii) Assume that the time spent by the users of the one-seater pods facilities in location B follows a Normal Distribution.

Assume also that the time spent, on the one-seater pod facilities in location B by users, are independent of each other.

Let W denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location B.

To test H₀: $\mu = 140$

Against H₁: $\mu \neq 140$

at 5% level of significance

Under H₀,
$$\overline{W} \sim N\left(140, \frac{20.1^2}{15}\right)$$

Since H₀ is not rejected,

$$-1.95996 < \frac{\overline{w} - 140}{\left(\frac{20.1}{\sqrt{15}}\right)} < 1.95996$$
$$129.828 < \overline{w} < 150.1718$$

$$129.8 < \overline{w} < 150.2$$
 (1 d.p)