1	Let $\mathrm{P}(N)$ be the statement $\sum_{n=1}^{N} \frac{n+3}{(n+1)(n+2)2^n} = \frac{1}{2} - \frac{1}{(N+2)2^N}$ for integers $N \geq 1$. When $N=1$: LHS = $\frac{1+3}{(1+1)(1+2)(2)} = \frac{1}{3}$ RHS = $\frac{1}{2} - \frac{1}{(1+2)(2)} = 1 - \frac{1}{6} = \frac{1}{3}$ LHS = RHS, hence $\mathrm{P}(1)$ is true. Assume $\mathrm{P}(k)$ is true for some $k \geq 1$, i.e. $\sum_{n=1}^{k} \frac{n+3}{(n+1)(n+2)2^n} = \frac{1}{2} - \frac{1}{(k+2)2^k}$. Claim $\mathrm{P}(k+1)$ is true, i.e. $\sum_{n=1}^{k+1} \frac{n+3}{(n+1)(n+2)2^n} = \frac{1}{2} - \frac{1}{(k+3)2^{k+1}}$. *Proof:* LHS $= \sum_{n=1}^{k+1} \frac{n+3}{(n+1)(n+2)2^n}$ $= \sum_{n=1}^{k} \frac{n+3}{(n+1)(n+2)2^n} + \frac{k+4}{(k+2)(k+3)2^{k+1}}$ $= \frac{1}{2} - \frac{1}{(k+2)2^k} + \frac{k+4}{(k+2)(k+3)2^{k+1}}$ $= \frac{1}{2} - \frac{(k+3)(2)-(k+4)}{(k+2)(k+3)2^{k+1}}$ $= \frac{1}{2} - \frac{2k+6-k-4}{(k+2)(k+3)2^{k+1}}$ $= \frac{1}{2} - \frac{k+2}{(k+2)(k+3)2^{k+1}}$ $= \frac{1}{2} - \frac{1}{(k+3)2^{k+1}}$ = RHS Hence $\mathrm{P}(k)$ is true $\Rightarrow$ $\mathrm{P}(k+1)$ is true. Since $\mathrm{P}(1)$ is true, and if $\mathrm{P}(k)$ is true then $\mathrm{P}(k+1)$ is also true, then by mathematical induction, $\mathrm{P}(N)$ is true for all positive integers $N \geq 1$. $\sum_{n=1}^{\infty} \frac{n+3}{(n+1)(n+2)2^n} = \lim_{N\to\infty} \frac{1}{2} - \frac{1}{(N+2)2^N} = \underline{\frac{1}{2}}$	
2(i)	The distributive axiom $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ is violated. $c(\mathbf{u}+\mathbf{v}) = c\begin{pmatrix} u_1v_1 \\ u_2v_2 \end{pmatrix} = \begin{pmatrix} cu_1v_1 \\ cu_2v_2 \end{pmatrix}$	

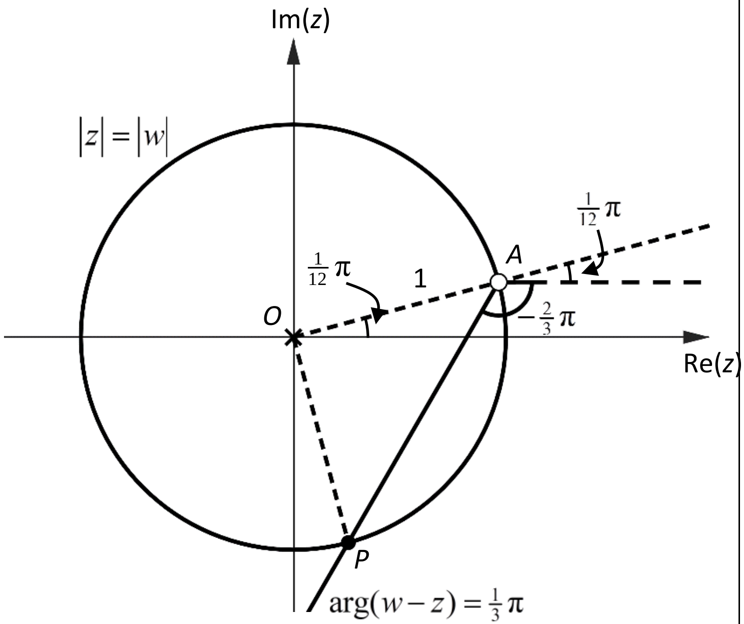
<table>
<tr><td></td><td>$c\mathbf{u}+c\mathbf{v}=\begin{pmatrix}cu_1\\cu_2\end{pmatrix}+\begin{pmatrix}cv_1\\cv_2\end{pmatrix}=\begin{pmatrix}c^2u_1v_1\\c^2u_2v_2\end{pmatrix}$
So $c(\mathbf{u}+\mathbf{v})\neq c\mathbf{u}+c\mathbf{v}$ in general

<u>Alternative</u>
The axiom $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ is violated.
$(c+d)\mathbf{u}=\begin{pmatrix}(c+d)u_1\\(c+d)u_2\end{pmatrix}$
$c\mathbf{u}+d\mathbf{u}=\begin{pmatrix}cu_1\\cu_2\end{pmatrix}+\begin{pmatrix}du_1\\du_2\end{pmatrix}=\begin{pmatrix}cdu_1{}^2\\cdu_2{}^2\end{pmatrix}$
So $(c+d)\mathbf{u}\neq c\mathbf{u}+d\mathbf{u}$ in general</td><td></td></tr>
<tr><td>2(ii)(a)</td><td>Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be matrices such that $\mathbf{A}_1\mathbf{B}=\mathbf{B}\mathbf{A}_1$ and $\mathbf{A}_2\mathbf{B}=\mathbf{B}\mathbf{A}_2$
$(\mathbf{A}_1+\mathbf{A}_2)\mathbf{B}=\mathbf{A}_1\mathbf{B}+\mathbf{A}_2\mathbf{B}=\mathbf{B}\mathbf{A}_1+\mathbf{B}\mathbf{A}_2=\mathbf{B}(\mathbf{A}_1+\mathbf{A}_2)$
$(k\mathbf{A}_1)\mathbf{B}=k(\mathbf{A}_1\mathbf{B})=k(\mathbf{B}\mathbf{A}_1)=\mathbf{B}(k\mathbf{A}_1)$
The set is closed under addition and scalar multiplication. Also, the set is non-empty since $\mathbf{0B}=\mathbf{B0}=\mathbf{0}$.
Hence it is a subspace.</td><td></td></tr>
<tr><td>2(ii)(b)</td><td>$\mathbf{I}^{\mathrm{T}}\mathbf{I}=\mathbf{I}$ but $(2\mathbf{I})^{\mathrm{T}}(2\mathbf{I})=4(\mathbf{I}^{\mathrm{T}}\mathbf{I})=4\mathbf{I}\neq\mathbf{I}$
The set is not closed under scalar multiplication.
Hence it is not a subspace.</td><td></td></tr>
<tr><td>3(i)</td><td>$I=\int_{-2}^{2}3^x\,\mathrm{d}x$ Let $\mathrm{f}(x)=3^x$ and $h=\frac{2-(-2)}{4}=1$
<table><tr><td>n</td><td>t_n</td><td>$y_n=\mathrm{f}(t_n)$</td></tr><tr><td>0</td><td>–2</td><td>$\frac{1}{9}$</td></tr><tr><td>1</td><td>–1</td><td>$\frac{1}{3}$</td></tr><tr><td>2</td><td>0</td><td>1</td></tr><tr><td>3</td><td>1</td><td>3</td></tr><tr><td>4</td><td>2</td><td>9</td></tr></table>
Let T denotes the approximation to $I=\int_{-2}^{2}3^x\,\mathrm{d}x$, found using trapezium rule with 5 ordinates.
$T=\frac{h}{2}\left[y_0+2y_1+2y_2+2y_3+y_4\right]$ --- (1)
$T=8\frac{8}{9}$</td><td></td></tr>
<tr><td>3(ii)</td><td>$\mathrm{f}(x)=3^x$
$\mathrm{f}'(x)=(\ln 3)(3^x)$</td><td></td></tr>
</table>

	$f''(x) = (\ln 3)^2 (3^x) > 0$ for $-2 \le x \le 2$ $f(x) = 3^x$ is concave upwards over the interval $[-2, 2]$ Trapezium rule produces an overestimate T to $I = \int_{-2}^{2} 3^x \, dx$.	
3(iii)	Let S denotes the approximation to $I = \int_{-2}^{2} 3^x \, dx$, found using Simpson rule with 5 ordinates. $S = \frac{1}{3} h [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$ --- (2) $S = 8\frac{4}{27}$	
3(iv)	$I = \int_{-2}^{2} 3^x \, dx$ $= \frac{1}{\ln 3} \left[3^x \right]_{-2}^{2}$ $= \frac{1}{\ln 3} \left[3^2 - 3^{-2} \right]$ $= \frac{80}{9} \left(\frac{1}{\ln 3} \right)$	
3(v)	Numerical integration using the Simpson rule produces a more accurate approximation compared to the Trapezium rule, with the same number of ordinates. The Simpson rule makes use of a quadratic approximation as opposes to the Trapezium rule which makes use of a linear approximation. Hence Simpson rule uses a better approximation to the curve $y = 3^x$.	
3(vi)	Absolute percentage error $= \frac{\lvert I - S \rvert}{I} \times 100\% \approx 0.706\%$	
4(a)	Differentiate (1) with respect to *x*: $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} - 2\frac{dz}{dx} = 0$ From (2), $\frac{dz}{dx} = y - 5z + 16x$ $\Rightarrow \frac{d^2 y}{dx^2} + 4\frac{dy}{dx} - 2(y - 5z + 16x) = 0$ $\Rightarrow \frac{d^2 y}{dx^2} + 4\frac{dy}{dx} - 2y + 10z - 32x = 0$ From (1), $2z = \frac{dy}{dx} + 4y - 8$ $\Rightarrow \frac{d^2 y}{dx^2} + 4\frac{dy}{dx} - 2y + 5\left(\frac{dy}{dx} + 4y - 8 \right) - 32x = 0$	

	$\Rightarrow \dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}-2y+5\dfrac{dy}{dx}+20y-40-32x=0$ $\Rightarrow \dfrac{d^2y}{dx^2}+9\dfrac{dy}{dx}+18y=32x+40$ (shown)	
4(b)	Auxiliary equation: $m^2+9m+18=0$ $\Rightarrow (m+3)(m+6)=0$ $\Rightarrow m=-6$ or -3 Complementary function: $y=Ae^{-6x}+Be^{-3x}$ for arbitrary constants A, B For particular integral, let $y=cx+d \Rightarrow \dfrac{dy}{dx}=c$, $\dfrac{d^2y}{dx^2}=0$ Substitute into DE: $0+9c+18(cx+d)=32x+40$ Comparing coefficients: x: $18c=32 \Rightarrow c=\dfrac{16}{9}$ x^0: $9c+18d=40 \Rightarrow d=\dfrac{40-16}{18}=\dfrac{4}{3}$ $\therefore$ General solution for y is $y=Ae^{-6x}+Be^{-3x}+\dfrac{16}{9}x+\dfrac{4}{3}$ $\dfrac{dy}{dx}=-6Ae^{-6x}-3Be^{-3x}+\dfrac{16}{9}$ Substitute into (1): $-6Ae^{-6x}-3Be^{-3x}+\dfrac{16}{9}+4\left(Ae^{-6x}+Be^{-3x}+\dfrac{16}{9}x+\dfrac{4}{3}\right)-2z=8$ $\Rightarrow -2Ae^{-6x}+Be^{-3x}+\dfrac{64}{9}x-\dfrac{8}{9}=2z$ $\Rightarrow z=-Ae^{-6x}+\dfrac{B}{2}e^{-3x}+\dfrac{32}{9}x-\dfrac{4}{9}$ Sub. $x=0$, $y=0$: $A+B+\dfrac{4}{3}=0 \Rightarrow A+B=-\dfrac{4}{3}$ .... (3) Sub. $x=0$, $z=0$: $-A+\dfrac{B}{2}-\dfrac{4}{9}=0 \Rightarrow -2A+B=\dfrac{8}{9}$ .... (4) Using GC to solve (3) and (4), $A=-\dfrac{20}{27}$, $B=-\dfrac{16}{27}$ $\therefore$ Solutions for y and z are: $\underline{y=-\dfrac{20}{27}e^{-6x}-\dfrac{16}{27}e^{-3x}+\dfrac{16}{9}x+\dfrac{4}{3}}$ $\underline{z=\dfrac{20}{27}e^{-6x}-\dfrac{8}{27}e^{-3x}+\dfrac{32}{9}x-\dfrac{4}{9}}$	
5(a)	$2v^4=1+\sqrt{3}i$	

	$2v^4 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ $v^4 = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}+2k\pi\right)}$, where $k \in \mathbb{Z}$ $v = \mathrm{e}^{\mathrm{i}\left[\frac{1}{4}\left(\frac{\pi}{3}+2k\pi\right)\right]}$ $v = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{12}+\frac{k\pi}{2}\right)}$ For arguments in the principal range, choose $k = 0, \pm 1, -2$ $\therefore\ z = \underline{\mathrm{e}^{-\mathrm{i}\left(\frac{11\pi}{12}\right)},\ \mathrm{e}^{-\mathrm{i}\left(\frac{5\pi}{12}\right)},\ \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{12}\right)},\ \mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{12}\right)}}$	
5(b)	Let $w = \mathrm{e}^{\mathrm{i}p} = \cos p + \mathrm{i}\sin p$ where $p = \frac{\pi}{12}$. By De Moivre's Theorem, for any positive integer *n*, $w^n + \frac{1}{w^n} = w^n + w^{-n}$ $= \cos np + \mathrm{i}\sin np + \cos(-np) + \mathrm{i}\sin(-np)$ $= \cos np + \mathrm{i}\sin np + \cos np - \mathrm{i}\sin np$ $= \underline{2\cos np}$ (shown) $2\cos p = w + \frac{1}{w}$ $\Rightarrow (2\cos p)^4 = \left(w + \frac{1}{w}\right)^4$ $\Rightarrow 16\cos^4 p = \left(w^4 + \frac{1}{w^4}\right) + 4\left(w^2 + \frac{1}{w^2}\right) + 6$ $= 2\cos 4p + 4(2\cos 2p) + 6$ $= 2\cos\frac{\pi}{3} + 8\cos\frac{\pi}{6} + 6$ $= 2\left(\frac{1}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right) + 6$ $= 7 + 4\sqrt{3}$ $\Rightarrow \underline{\cos^4 p = \frac{7+4\sqrt{3}}{16}}$ (shown)	
5(c)	$\lvert z\rvert = \lvert w\rvert = 1$ Locus is a circle centred at the origin *O* with radius 1 unit. $\arg(w-z) = \frac{\pi}{3} \Rightarrow \arg(-(z-w)) = \frac{\pi}{3}$ $\Rightarrow \arg(-1) + \arg(z-w) = \frac{\pi}{3}$ $\Rightarrow \arg(z-w) = \frac{\pi}{3} - \pi$ $\Rightarrow \arg(z-w) = -\frac{2\pi}{3}$	

	Locus is a half-line starting from (and excluding the point *A* representing *w*), at an argument of $-\dfrac{2\pi}{3}$ rad.  	
5(d)	Triangle *OAP* is isosceles triangle. $\angle OAP = \angle OPA$ $= \pi - \dfrac{2\pi}{3} - \dfrac{\pi}{12}$ $= \dfrac{\pi}{4}$ $\angle AOP = \pi - 2\left(\dfrac{\pi}{4}\right)$ $= \dfrac{\pi}{2}$ The argument of the complex number represented by *P* is $\dfrac{\pi}{12} - \dfrac{\pi}{2} = -\dfrac{5\pi}{12}$ Since *P* lies on the circle centred at origin with radius 1, the complex number represented by *P* is $\underline{e^{-i\left(\frac{5\pi}{12}\right)}}$, which is one of the roots of the equation in **(a)**.	
6(a)	$p_s = \dfrac{7}{9},\ z_{0.95} = 1.645,\ n = 900$ Confidence interval $= \left(P_s - z_{1-\alpha/2}\sqrt{\dfrac{P_s\left(1-P_s\right)}{n}},\ P_s + z_{1-\alpha/2}\sqrt{\dfrac{P_s\left(1-P_s\right)}{n}}\right)$ $= \left(\dfrac{7}{9} - 1.645\sqrt{\dfrac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{900}},\ \dfrac{7}{9} + 1.645\sqrt{\dfrac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{900}}\right)$	

$= (0.7550, 0.8006)$

Answer is 4 dp as interval width is 0.0456 to 3 sf

6(b)

$$z_{1-\alpha/2}\sqrt{\frac{P_s(1-P_s)}{n}} = 0.02$$

$$1.645\sqrt{\frac{\frac{7}{9}\left(1-\frac{7}{9}\right)}{n}} = 0.02$$

$$1.645\frac{\sqrt{\frac{7}{9}\left(1-\frac{7}{9}\right)}}{\sqrt{n}} = 0.02$$

$$1.645\frac{\sqrt{\frac{7}{9}\left(1-\frac{7}{9}\right)}}{0.02} = \sqrt{n}$$

$n = 1169.27 \approx 1169$

7

H_0: the number of heads obtained follows a Binomial distribution with $p = 0.6$.

H_1: the number of heads obtained does not follow a Binomial distribution with $p = 0.6$.

Level of significance: 5%

No of heads	0	1	2	3	4
Frequency	5	35	64	66	30
Expected frequency	5.12	30.72	69.12	69.12	25.92

Degree of freedom is 4.

$$\chi^2 = \sum_{i=0}^{4}\frac{(O_i - E_i)^2}{E_i} \sim \chi^2(4)$$

By GC, the *p*-value is 0.780 > 0.05.

Hence, we do not reject H_0 and conclude at 5% significance level that a Binomial distribution with $p = 0.6$ is a good fit.

If the experiment is repeated 1000 times, the new χ^2 value will be $1.7614 \times 5 = 8.807 < 9.488$ and so there is no change to the result of the test.

8(a)

We may not be able to assume that the difference in the depths of tread on the front and rear tyres is normally distributed.

8(b)

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	2.4	1.5	2.3	2.4	2.6	2.5	2.1	2.6
	2.3	1.9	2.1	1.8	1.8	2.8	1.4	2.1

Diff	–0.1	0.4	–0.2	–0.6	–0.8	0.3	–0.7	–0.5
Rank	–1	4	–2	–6	–8	3	–7	–5

H_0: $m_d = 0$ H_1: $m_d \neq 0$

$P = 7$, $Q = 29$, so $T = P = 7$, the 10% two-tail critical region for $n = 8$ is $T \le 5$.
Therefore, we do not reject H_0 at 10% significance level and conclude there is insufficient evidence that there is a difference between the average wear for the front and rear tyres.

8(c)

After correcting the mistakes, we have only one positive difference.
For the change in conclusion, $P = T \le 5$. This means that $0.3 < 1.9 - b < 0.5$. Hence $1.4 < b < 1.6$.

9(a)

$0 < x < 1$

$-5 < -5x^2 < 0$

$-4 < 1 - 5x^2 < 1$

$-4 < Y < 1$

$\mathrm{P}(Y \le y)$

$= \mathrm{P}(1 - 5X^2 \le y)$

$= \mathrm{P}\left(\frac{1-y}{5} \le X^2\right)$

$= \mathrm{P}\left(X \ge \sqrt{\frac{1-y}{5}}\right) + \mathrm{P}\left(-\sqrt{\frac{1-y}{5}} \ge X\right)$

$= \mathrm{P}\left(X \ge \sqrt{\frac{1-y}{5}}\right)$ $(\text{since } 0 < x < 1)$

$= \int_{\sqrt{\frac{1-y}{5}}}^{1} 4x^3 \, \mathrm{d}x$

$= \left[x^4\right]_{\sqrt{\frac{1-y}{5}}}^{1}$

$= 1 - \left(\frac{1-y}{5}\right)^2$

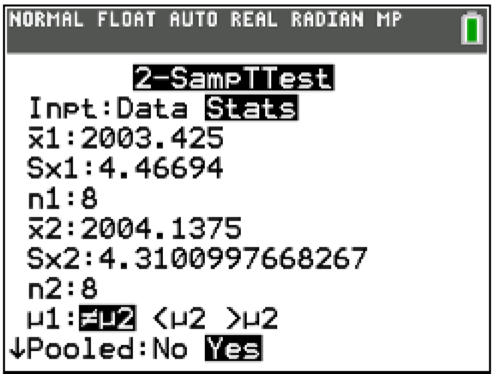
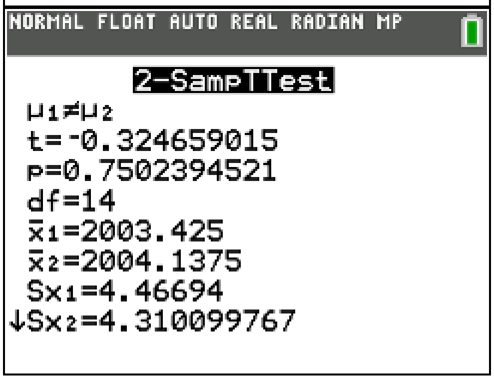
p.d.f of Y

$= \frac{\mathrm{d}}{\mathrm{d}y}\left[1 - \left(\frac{1-y}{5}\right)^2\right]$

$= \frac{2}{25}(1-y)$

$\mathrm{f}(y) = \frac{2}{25}(1-y), -4 < y < 1.$

9(b)	$\mathrm{E}(Y)=\int_{-4}^{1}\frac{2}{25}(1-y)y\,\mathrm{d}y=-\frac{7}{3}$	
10(i)	Visitors answering the surveys on the website must occur randomly and independently of each other. The average number of surveys is constant and proportional to the length of time.	
10(ii)	Let S be the number of surveys received in an hour, i.e. $S\sim\mathrm{Po}\left(\frac{5}{6}\right)$. $\mathrm{P}(S_1=0, S_2=0 \text{ and } S_3\geq 1)$ $=\mathrm{P}(S_1=0)\mathrm{P}(S_2=0)\mathrm{P}(S_3\geq 1)$ $=\mathrm{P}(S_1=0)\mathrm{P}(S_2=0)\left[1-\mathrm{P}(S_3=0)\right]$ $=0.10679$ $\approx 0.107\,(3\text{ s.f.})$	
10(iii)	Let X be the number of surveys received in a day, i.e. $X\sim\mathrm{Po}(20)$ Let W be the total number of surveys received in 2 days, i.e. $W\sim\mathrm{Po}(40)$. Required probability $=\frac{2\left(\begin{array}{l}[\mathrm{P}(X=16)][\mathrm{P}(X=14)]+[\mathrm{P}(X=17)][\mathrm{P}(X=13)]\\+[\mathrm{P}(X=18)][\mathrm{P}(X=12)]\end{array}\right)}{\mathrm{P}(W=30)}$ $=\frac{2(0.0060479)}{0.018465}$ $=0.65505$ $=0.655\ (3\text{s.f.})$	
10(iv)	$\mathrm{f}(t)=\frac{1}{72}\mathrm{e}^{-\frac{1}{72}t}, t>0$	
10(v)	$\mathrm{P}(T>n)\geq 0.3$ $\mathrm{e}^{-\frac{1}{72}n}\geq 0.3$ $n\leq \ln 0.3(-72)=86.7$ $\therefore$ greatest $n=86$	
11(i)	1-sample t test	
11(ii)	By GC, $\bar{x}=2003.425$ and $s_X{}^2=4.46694^2$ Let μ_X be the mean mass of rice in a packet in g. H_0: $\mu_X=2000$ H_1: $\mu_X>2000$ $T=\frac{\bar{X}-\mu_x}{s/\sqrt{8}}\sim t_7$	

	Test statistic $t=\dfrac{2003.425-2000}{4.46694/\sqrt{8}}=2.16868$ By GC, p-value = 0.0334. Since the null hypothesis is rejected, $\dfrac{\alpha}{100}\geq 0.0334 \Rightarrow \alpha \geq 3.34$. Thus the minimum value of α is 3.34.	
11(iii)	Appropriate hypothesis test is the 2-sample t test. We need to assume that the variance of the masses of packets of rice of both batches are the same. We have $\bar{y}=2004.1375$ and $s_Y{}^2=\dfrac{1}{7}\left(266.99-\dfrac{33.1^2}{8}\right)=18.57696$. Let μ_Y be the mean mass of rice in a packet in g for the second batch. H_0: $\mu_X-\mu_Y=0$ H_1: $\mu_X-\mu_Y\neq 0$ $s_p{}^2=\dfrac{7s_X{}^2+7s_Y{}^2}{14}$ $T=\dfrac{\overline{X}-\overline{Y}}{s_p\sqrt{\frac{1}{8}+\frac{1}{8}}}\sim t_{14}$   By GC, p-value = 0.750 > 0.05. Hence we do not reject the null hypothesis and conclude that there is no significant difference between the mean	

	masses of packets in the two batches.	
11(iv)	It could be possible that the second sample of rice consisted of packets that are heavier.	