FUNCTIONS - Additional Practice Questions' Solutions

1. **[EJC/2019/JC1 Promo/Q7]**

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$$

where a and b are non-zero constants.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Hence show that $\mathrm{f}^2(x) = x$, and write down $\mathrm{f}^n(x)$ where n is an odd number. [2]

The function g is defined by $\mathrm{g}: x \mapsto 2+\mathrm{e}^{-x}, \quad \text{for } x \in \mathbb{R}, x \geq 0$. If $a=2$ and $b=1$ for function f,

(iii) Explain why the composite function gf does not exist. [1]

(iv) Find an expression for $\mathrm{fg}(x)$ and state the domain and exact range of fg. [3]

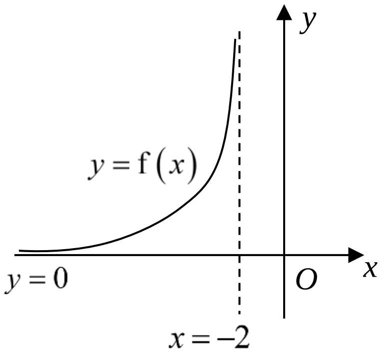
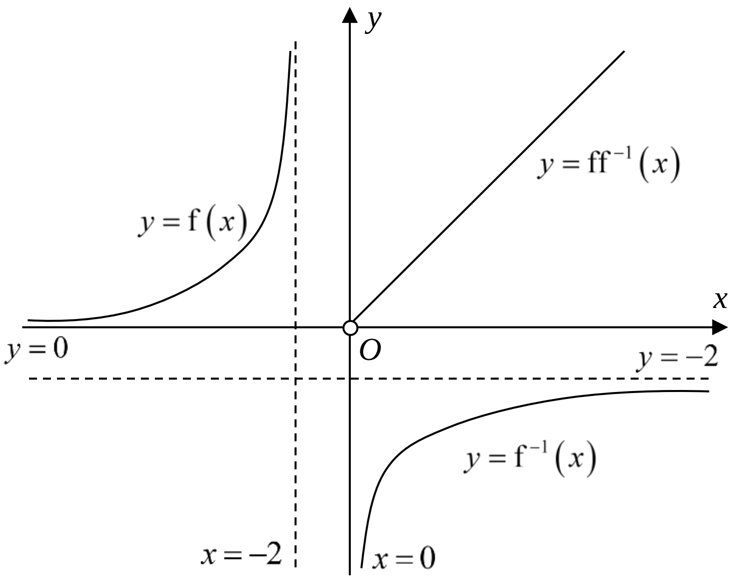
EJC JC1 Promo 9758/2019/01/Q7 (Solutions)	
	Suggested solution
(i)	$\mathrm{f}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$ Let $y = \frac{bx}{ax-b} \Rightarrow y(ax-b) = bx$ $\Rightarrow axy - bx = by$ $\Rightarrow x = \frac{by}{ay-b}$ $\mathrm{f}^{-1}: x \mapsto \frac{bx}{ax-b}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{b}{a},$
(ii)	Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}^2(x) = \mathrm{f}\left[\mathrm{f}^{-1}(x)\right] = x$ $\mathrm{f}^3(x) = \mathrm{f}\left[\mathrm{f}^2(x)\right] = \mathrm{f}(x)$, therefore for $\mathrm{f}^n(x) = \mathrm{f}(x) = \frac{bx}{ax-b}$ for n odd,
(iii)	If $a=2$ and $b=1$ for function f, $\mathrm{f}: x \mapsto \frac{x}{2x-1}, \quad \text{for } x \in \mathbb{R}, x \neq \frac{1}{2},$ $\mathrm{R_f} = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$ $\mathrm{D}_g = [0, \infty)$ As $\mathrm{R_f} \not\subset \mathrm{D}_g$, gf does not exist.
(iv)	$\mathrm{fg}(x) = \mathrm{f}\left(2+\mathrm{e}^{-x}\right)$ $= \frac{2+\mathrm{e}^{-x}}{2\left(2+\mathrm{e}^{-x}\right)-1}$ $= \frac{2+\mathrm{e}^{-x}}{3+2\mathrm{e}^{-x}}, \quad x \geq 0$ $\mathrm{D_{fg}} = \mathrm{D_g} = [0, \infty)$ Range of $\mathrm{R_{fg}} = \left[\frac{3}{5}, \frac{2}{3}\right)$

2. **[MI/2019/J1 Promo/01/Q5]**

The function f is defined by $\mathrm{f}: x \mapsto \dfrac{1}{x^2-4}$ for $x \in \mathbb{R},\ x < -2$.

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [3]

(ii) On the same diagram, sketch the graphs of $y=\mathrm{f}(x)$, $y=\mathrm{f}^{-1}(x)$ and $y=\mathrm{ff}^{-1}(x)$, stating the equations of any asymptotes and showing the relationships between the graphs clearly. [4]

M1 JC1 Promo 9758/2019/01/Q5 (Solutions)	
5(i)	 $y=\dfrac{1}{x^2-4}$ $x^2-4=\dfrac{1}{y}$ $x^2=4+\dfrac{1}{y}$ $x=\sqrt{4+\dfrac{1}{y}}$ $\left(\text{rejected } \because x<-2\right)$ or $x=-\sqrt{4+\dfrac{1}{y}}$ $\therefore \mathrm{f}^{-1}(x)=-\sqrt{4+\dfrac{1}{x}}$ Domain of $\mathrm{f}^{-1}=$ Range of $\mathrm{f}=(0,\infty)$
5(ii)	

3. **[TMJC/2019/JC2 Prelim/01/Q7]**

The function f is defined by

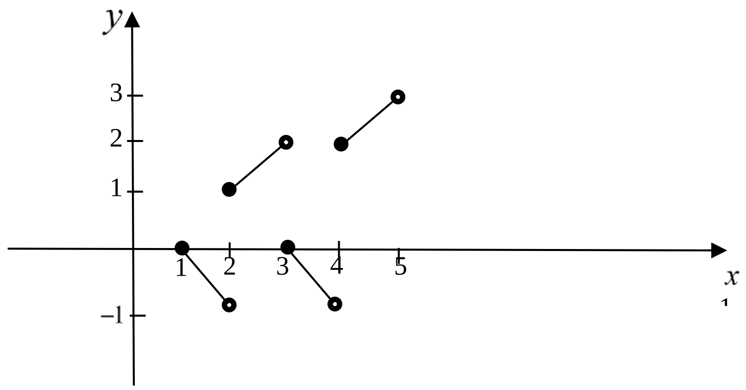
$$\mathrm{f}(x)=\begin{cases} n-x & \text{for } n \le x < n+1, \text{ where } n \text{ is any positive odd integer,} \\ x-\dfrac{n}{2} & \text{for } n \le x < n+1, \text{ where } n \text{ is any positive even integer.} \end{cases}$$

(i) Show that $\mathrm{f}(1.5)=-0.5$ and find $\mathrm{f}(2.5)$. [2]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $1 \le x < 5$. [2]

(iii) Does f have an inverse for $1 \le x < 5$? Justify your answer. [2]

(iv) The function g is defined by $\mathrm{g}: x \mapsto \dfrac{2x-1}{x+1}, x \in \mathbb{R}, x \neq -1$. For $2 \le x < 3$, find an expression for $\mathrm{gf}(x)$ and hence, or otherwise, find $(\mathrm{gf})^{-1}\left(\dfrac{2}{3}\right)$. [4]

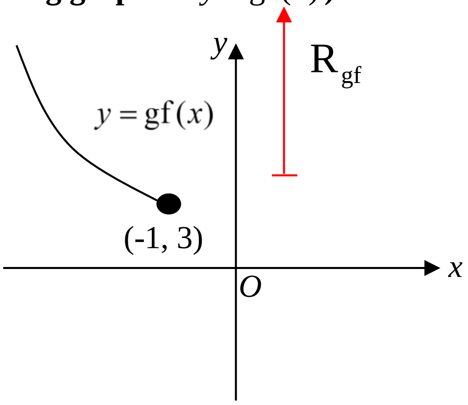
TMJC JC2 Prelim 9758/2019/01/Q7 (Solutions)	
For $1 \le x < 2$, $\mathrm{f}(x)=1-x$ $\therefore \mathrm{f}(1.5)=1-1.5=-0.5$ (shown) For $2 \le x < 3$, $\mathrm{f}(x)=x-1$ $\therefore \mathrm{f}(2.5)=2.5-1=1.5$	
For $1 \le x < 2$, $\mathrm{f}(x)=1-x$ For $2 \le x < 3$, $\mathrm{f}(x)=x-1$ For $3 \le x < 4$, $\mathrm{f}(x)=3-x$ For $4 \le x < 5$, $\mathrm{f}(x)=x-2$ 	
Since $\mathrm{f}(1)=0=\mathrm{f}(3)$, f is not a one-to-one function. Hence, f does not have an inverse. **<u>Alternative</u>** Since $y=0$ cuts the graph of f at 2 distinct points, f is not a one-to-one function. Hence, f does not have an inverse.	
For $2 \le x < 3$, $\mathrm{f}(x)=x-1$	

$\text{gf}(x)=\dfrac{2(x-1)-1}{(x-1)+1} = \dfrac{2x-3}{x}$ Let $x=(\text{gf})^{-1}\left(\dfrac{2}{3}\right)$ $\text{gf}(x)=\dfrac{2}{3}$ $\dfrac{2x-3}{x}=\dfrac{2}{3}$ $x=\dfrac{9}{4}$ **Alternative:** For $2\le x<3$, $\text{f}(x)=x-1$, $\text{gf}(x)=\dfrac{2(x-1)-1}{(x-1)+1} = \dfrac{2x-3}{x}$ Let $y=\dfrac{2x-3}{x}$ $xy=2x-3$ $2x-xy=3$ $x(2-y)=3$ $x=\dfrac{3}{2-y}$ $(\text{gf})^{-1}(x)=\dfrac{3}{2-x}$ $(\text{gf})^{-1}\left(\dfrac{2}{3}\right)=\dfrac{3}{2-\left(\dfrac{2}{3}\right)}=\dfrac{9}{4}$	

4(i) $R_f = (-\infty, 16] \subseteq (-\infty, 25] = D_g$

$\therefore$ gf exists.

Method 1 (Using graph of $y = \text{gf}(x)$):



From graph, $R_{gf} = [3, \infty)$

Method 2 (Using graph of $y = \text{g}(x)$:

$(-\infty, -1] \xrightarrow{f} (-\infty, 16] \xrightarrow{g} [3, \infty)$

$\therefore R_{gf} = [3, \infty)$

(ii) Let $y = (3-x)(x+5)$

$y = (3-x)(x+5)$

$\quad = 15 - 2x - x^2$

$\quad = 16 - (x+1)^2$

$(x+1)^2 = 16 - y$

$x = -1 \pm \sqrt{16-y}$

Since $x \le -1$, $\therefore x = -1 - \sqrt{16-y}$

$\therefore f^{-1}(x) = -1 - \sqrt{16-x}$

$D_{f^{-1}} = R_f = (-\infty, 16]$

(iii) $f(x) = f^{-1}(x)$

$\Rightarrow f(x) = x$

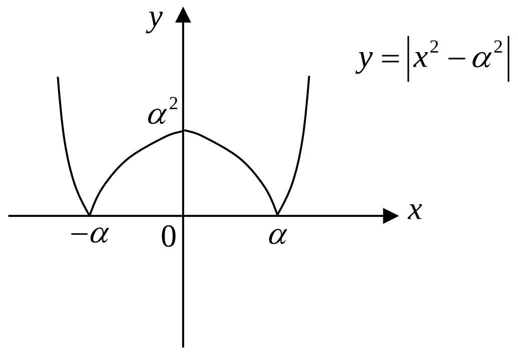
$15 - 2x - x^2 = x$

$x^2 + 3x - 15 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-15)}}{2(1)} = \frac{-3 \pm \sqrt{69}}{2}$$

$$\therefore x = \frac{-3 - \sqrt{69}}{2} \quad (\because x \le -1)$$

5(i)



Greatest $k = \alpha$.

5(ii) Let $y = -\left(x^2 - \alpha^2\right)$.

$x^2 = \alpha^2 - y$

$\Rightarrow x = \sqrt{\alpha^2 - y}$ ($\because x \geq 0$)

$\therefore \mathrm{f}^{-1}(x) = \sqrt{\alpha^2 - x}, \quad x \in \left(0, \alpha^2\right]$

5(iii) $\mathrm{f}^{-1}\mathrm{g}(x) = 1$

$\Rightarrow \sqrt{25 - \frac{1}{x^2}} = 1$

$\Rightarrow x^2 = \frac{1}{24}$

$\Rightarrow x = -\frac{1}{2\sqrt{6}} \quad \left(\because x \in \mathrm{D}_{\mathrm{f}^{-1}\mathrm{g}} = \left(-\infty, -\frac{1}{5}\right)\right)$

OR

$\mathrm{f}^{-1}\mathrm{g}(x) = 1$

$\Rightarrow \mathrm{g}(x) = \mathrm{f}(1)$

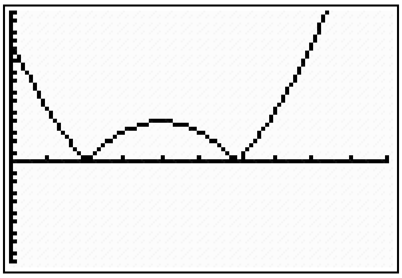
$\Rightarrow \frac{1}{x^2} = 24$

$\Rightarrow x^2 = \frac{1}{24}$

$\Rightarrow x = -\frac{1}{2\sqrt{6}} \quad \left(\because x \in \mathrm{D}_{\mathrm{f}^{-1}\mathrm{g}} = \left(-\infty, -\frac{1}{5}\right)\right)$

6(i)

Least $a = 6$.



Let $y = x^2 - 8x + 12$

$x^2 - 8x + 12 - y = 0$

$x = \dfrac{8 \pm \sqrt{8^2 - 4(1)(12 - y)}}{2}$

$x = 4 \pm \sqrt{4 + y}$

As $x \geq 6$, $x = 4 + \sqrt{4 + y}$

Hence $\mathrm{f}^{-1} : x \mapsto 4 + \sqrt{4 + x}$, $\quad x \geq 0$

Alternatively,

Let $y = x^2 - 8x + 12$

$x^2 - 8x + 12 - y = 0$

$(x - 4)^2 - 4^2 + 12 - y = 0$

$(x - 4)^2 = 4 + y$

$x - 4 = \pm\sqrt{4 + y}$

$x = 4 \pm \sqrt{4 + y}$

As $x \geq 6$, $x = 4 + \sqrt{4 + y}$. Hence $\mathrm{f}^{-1} : x \mapsto 4 + \sqrt{4 + x}$, $\quad x \geq 0$

(ii)

Reflect $y = \mathrm{f}(x)$ about the line $y = x$ to get $y = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = x$

$x^2 - 8x + 12 = x$

$x^2 - 9x + 12 = 0$

$x = \dfrac{9 \pm \sqrt{9^2 - 4(1)(12)}}{2}$

$x = \dfrac{9 \pm \sqrt{33}}{2}$

$x = \dfrac{9 + \sqrt{33}}{2}$ as $x \geq 6$

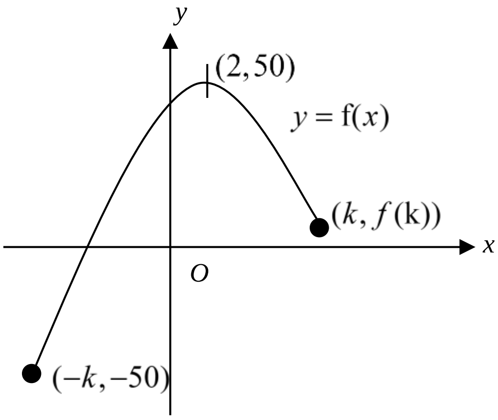
7(a)

$$\begin{aligned} & g(x) > 2 \\ \Rightarrow\ & \frac{3x-3}{x^2-8x+18} > 2 \\ \Rightarrow\ & \frac{3x-3-2\left(x^2-8x+18\right)}{x^2-8x+18} > 0 \\ \Rightarrow\ & \frac{2x^2-19x+39}{x^2-8x+18} < 0. \end{aligned}$$

Since $x^2-8x+18=\left(x-4\right)^2+2>0$ for all $x\in\mathbb{R},$ the inequality reduces to

$$\begin{aligned} & 2x^2-19x+39<0 \\ \Rightarrow\ & \left(2x-13\right)\left(x-3\right)<0 \\ \Rightarrow\ & 3<x<\frac{13}{2}. \end{aligned}$$

7(b)(i)



Given that the range of f is $[-50,50]$, we have

$$\begin{aligned} & 50-\left(2(-k)-4\right)^2=f(-k)=-50 \\ \Rightarrow\ & k=-\frac{4\pm 10}{2} \\ \therefore\ & k=3 \text{ (since } k>0\text{)} \end{aligned}$$

7(b)(ii) Consider the inequality

$$\begin{aligned} & g(x)>0 \\ \Rightarrow\ & 3x-3>0 \\ \Rightarrow\ & x>1. \end{aligned}$$

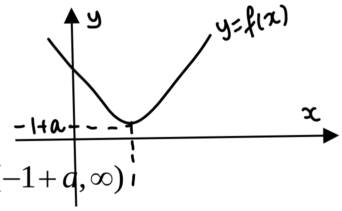
So to solve $gf(x)>0,$ we replace x with $f(x)$ to obtain

$$\begin{aligned} & f(x)>1 \\ \Rightarrow\ & 50-\left(2x-4\right)^2>1 \\ \Rightarrow\ & \left(2x-4\right)^2<49 \\ \Rightarrow\ & -7<2x-4<7 \\ \Rightarrow\ & -\frac{3}{2}<x<\frac{11}{2}. \end{aligned}$$

Since $D_{gf}=D_f=[-3,3],$ $\quad -\frac{3}{2}<x\le 3.$

8(i)

$$\mathrm{f}(x) = x^2 - 2x + a$$
$$= (x-1)^2 - 1 + a$$



From the graph, range of f $= [-1+a, \infty)$

$R_{g^{-1}} = D_g = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$R_{g^{-1}} \subseteq D_f = \mathbb{R}$

Therefore, composite function fg^{-1} exists

Finding range of fg^{-1}:

$R_{g^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$D_{g^{-1}} \to R_{g^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R_{fg^{-1}} = [a-1, \frac{\pi^2}{4} + \pi + a)$

Therefore, range of $R_{\mathrm{fg}^{-1}} = \left[a-1, \frac{\pi^2}{4} + \pi + a\right)$

(ii) when $a = -3$,

$y = x^2 - 2x - 3$

$y = (x-1)^2 - 4$

$x = \sqrt{y+4} + 1 \;\; or -\sqrt{y+4} + 1 (rej)$

$\mathrm{f}^{-1} : x \to \sqrt{x+4} + 1 \;\; \text{for } x \in \mathbb{R},\; x \geq -4$

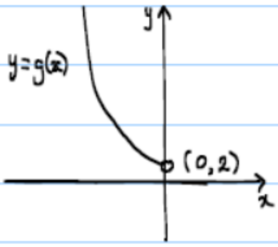
9. The functions f, g and h are defined by

$$\mathrm{f}: x \mapsto x(x-2), \quad x \in \mathbb{R}^{+},$$
$$\mathrm{g}: x \mapsto 3x^2+2, \quad x \in \mathbb{R}^{-}.$$
$$\mathrm{h}: x \mapsto \mathrm{e}^{-2x}, \quad x \geq 0.$$

(i) Determine, stating your reasons, whether each of the following functions exist:
(a) g^{-1}, (b) fg.
If the function exists, give its domain, rule and range.

(ii) Given that $\mathrm{f}(\alpha)+\mathrm{h}(\ln\alpha)=0$ and $\alpha \neq 0,1$, show that $\alpha^3-\alpha^2-\alpha-1=0$.

i/ (a)



By the horizontal line test,
g is one-one, $\therefore$ g^{-1} exists.

$$\mathrm{g}^{-1}: x \mapsto -\sqrt{\frac{x-2}{3}}, x \in (2,\infty), R_{\mathrm{g}^{-1}} = \mathbb{R}^{-}$$

(b) $R_g = (2,\infty) \subseteq \mathbb{R}^+ = D_f$
$\therefore$ fg exists.
$fg: x \mapsto 3x^2(3x^2+2), \ x \in \mathbb{R}^-$
$R_{fg} = \mathbb{R}^+$

ii/
$f(\alpha) + h(\ln\alpha) = 0$
$\alpha(\alpha-2) + e^{-2\ln\alpha} = 0$
$\alpha^2 - 2\alpha + e^{\ln\alpha^{-2}} = 0$
$\alpha^2 - 2\alpha + \frac{1}{\alpha^2} = 0$
$\alpha^4 - 2\alpha^3 + 1 = 0$
$\therefore (\alpha-1)(\alpha^3-\alpha^2-\alpha-1) = 0$
$\because \alpha \neq 1$
$\alpha^3-\alpha^2-\alpha-1 = 0$

We notice $\alpha = 1$ is a factor:

$$\begin{array}{r|l} & \alpha^3 - \alpha^2 - \alpha - 1 \\ \alpha - 1 & \alpha^4 - 2\alpha^3 + 0\alpha^2 + 0\alpha + 1 \\ & -(\alpha^4 - \alpha^3) \\ \hline & -\alpha^3 + 0\alpha^2 + 0\alpha + 1 \\ & -(-\alpha^3 + \alpha^2) \\ \hline & -\alpha^2 + 0\alpha + 1 \\ & -(-\alpha^2 + \alpha) \\ \hline & -\alpha + 1 \\ & -(-\alpha + 1) \\ \hline & 0 \end{array}$$

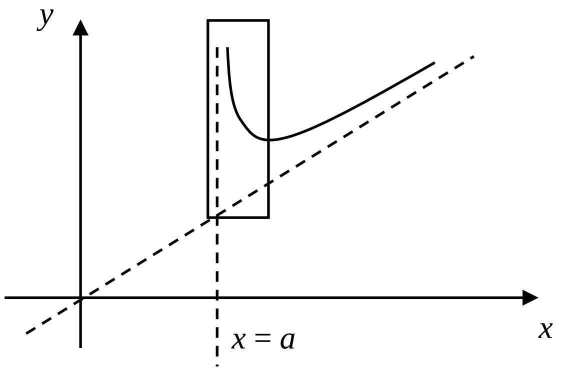
10. **[DHS /2019 Prelim/P1/Q4]**
The function f is defined as follows:

$$\mathrm{f}(x) = x + \frac{1}{x-a}, \quad a < x \le b$$

where a is a positive constant.

(i) Given that f^{-1} exist, show that $b \le a+1$. [2]

(ii) Given that $a = 1$ and $b = 2$, find $\mathrm{f}^{-1}(x)$ and the domain of f^{-1}. [5]

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>10
(i)</td><td>Since the shape of the curve is


For f to be 1-1, the largest b can take is the x-coordinate of the turning point.
$\mathrm{f}'(x) = 1 - \frac{1}{(x-a)^2}$
$1 - \frac{1}{(x-a)^2} = 0$
$x = a \pm 1$
x-coordinate of turning point is a+1, since $b > a$
For graph to be 1-1, $b \le a+1$,</td></tr>
<tr><td>(ii)</td><td>Let $y = \mathrm{f}(x)$
$y = x + \frac{1}{x-1}$
$(x-1)y = x(x-1)+1$
$xy - y = x^2 - x + 1$
$x^2 - (1+y)x + 1 + y = 0$
$\left(x - \frac{(1+y)}{2}\right)^2 - \frac{(1+y)^2}{4} + 1 + y = 0$
$\left(x - \frac{(1+y)}{2}\right)^2 = \frac{(y-1)^2}{4} - 1$
$x = \frac{(1+y)}{2} \pm \sqrt{\frac{(y-1)^2}{4} - 1}$</td></tr>
</table>

	Since $\left(\frac{3}{2},\frac{7}{2}\right)$ is a point on the curve of $y=\mathrm{f}(x)$, $x=\frac{(1+y)}{2}-\sqrt{\frac{(y-1)^2}{4}-1}$ $\mathrm{f}^{-1}(x)=\frac{(1+x)}{2}-\sqrt{\frac{(x-1)^2}{4}-1}$ The domain of f^{-1} is the range of f = $[3,\infty)$.

11. **[JPJC /2019 Prelim/P1/Q1]**

It is given that

$$\mathrm{f}(x)=\begin{cases}4a^2-x^2, & \text{for } 0<x\le 2a,\\ 2a(x-2a), & \text{for } 2a<x\le 4a,\end{cases}$$

and that $\mathrm{f}(x)=\mathrm{f}(x+4a)$ for all real values of x, where a is a positive real constant.

(i) Evaluate $\mathrm{f}(2019a)$ in terms of a. [1]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-3a\le x\le 5a$. [3]

The function g is defined by

$$\mathrm{g}: x\mapsto \sqrt{4a^2-(x-2a)^2}, \qquad 2a<x<4a.$$

(iii) Determine whether the composite function gf exists, justifying your answer. [1]

(iv) Give, in terms of a, a definition of fg. [2]

(v) Given that $(\mathrm{fg})^{-1}(27)=\frac{7}{2}a$, find the exact value of a. [2]

JPJC JC2 Prelim 9758/2019/01/Q7 (Solutions)

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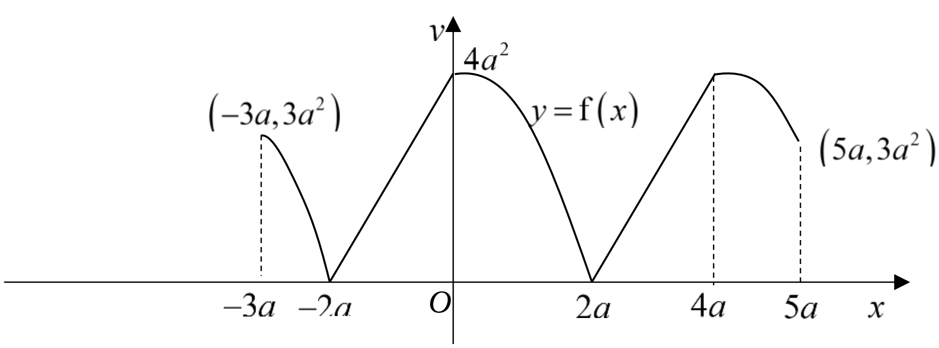
(i)

Since $\mathrm{f}(x)=\mathrm{f}(x+4a)$,

$\mathrm{f}(2019a)=\mathrm{f}(3a)=2a(3a-2a)=2a^2$

(ii)



(iii)

Since $\mathrm{R_f}=[0,4a^2]\not\subset \mathrm{D_g}=(2a,4a)$, gf does not exist.

(iv) Please note that g(2*a*) = 0, g(4*a*) = 2*a*, so f takes 0 to 2*a*

$$\text{fg}(x) = 4a^2 - \left(\sqrt{4a^2-(x-2a)^2}\right)^2 = (x-2a)^2$$

$$\text{fg}: x \mapsto (x-2a)^2, \qquad 2a < x < 4a,$$

(v)

$$(\text{fg})^{-1}(27) = \frac{7}{2}a$$

$$\text{fg}\left(\frac{7}{2}a\right) = 27$$

$$\left(\frac{7}{2}a - 2a\right)^2 = 27$$

$$\frac{9}{4}a^2 = 27$$

$$a^2 = 12$$

$$a = \pm 2\sqrt{3}$$

Since $a > 0$, $a = 2\sqrt{3}$