ANDERSON SERANGOON JUNIOR COLLEGE
JC2 Preliminary Examinations 2021
Higher 3

MATHEMATICS **9820/01**

Paper 1 **22 September 2021**

3 hours

Additional Materials: Answer Booklet

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Answer **all** the questions.

1 Find the set of integer values of x that satisfy

$$x^4 + x^3 + x - 1 \equiv 0 \pmod{5}$$

Express $x^4 + x^3 + x - 1$ as a product (modulo 5) of four linear factors with integer coefficients. [6]

2 It is given that g is a function with domain $[-1, \pi]$. Using the substitution $t = \pi - x$ or otherwise, show that $\int_0^{\pi} x \cdot \mathrm{g}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \mathrm{g}(\sin x)\,\mathrm{d}x$. Hence find the exact values of the following definite integrals, simplifying your answers.

(a) $\displaystyle\int_0^{\pi} \frac{x \sin x}{\sqrt{8+\sin^2 x}}\,\mathrm{d}x$

(b) $\displaystyle\int_0^{\pi} x\sqrt{1+\sin x}\,\mathrm{d}x$ [9]

3 There are n identical balls to be distributed into k distinct boxes such that every box must be filled.

(i) Find the number of ways if there are no restrictions. [1]

(ii) It is given that n and k are both even and $n \geq 2k$.

(a) Find the number of ways if every box contains an even number of balls. [2]

(b) Find the number of ways if every box contains an odd number of balls. [2]

(iii) Use principle of inclusion and exclusion to find the number of ways to distribute the n objects if no box can have more than m objects. [4]

4 It is given that a, b and c are the lengths of the sides of a triangle.

(i) Prove that it is always possible to form a triangle with sides of lengths $\sqrt[3]{a}$, $\sqrt[3]{b}$ and $\sqrt[3]{c}$. [2]

(ii) Prove that it is never possible to form a triangle with sides of lengths $\left|\sqrt{a}-\sqrt[3]{b}\right|$, $\left|\sqrt[3]{b}-\sqrt[4]{c}\right|$ and $\left|\sqrt[4]{c}-\sqrt{a}\right|$. [2]

(iii) Determine with a proof whether it is always possible to form a triangle with sides of lengths a^2+2bc, b^2+2ca and c^2+2ab. [3]

(iv) Prove that if the triangle with sides of lengths a, b and c is a right-angled triangle, the triangle in part **(i)** cannot be a right-angled triangle. [3]

5 It is given that $\mathrm{f}(x)=\sum_{r=0}^{n} c_r x^r$, where $c_0=c_n=1$, $|c_r|\le k$ for all $r=1, 2, \ldots, n-1$, and k is a positive constant greater than or equal to 1.

(i) By using triangle inequality, show that $|\mathrm{f}(x)-1|\le \dfrac{k|x|}{1-|x|}$ if $-1<x<1$. [3]

(ii) Prove that if r is a root of f and $-1<r<1$, then r must satisfy the inequality $\dfrac{1}{k+1}\le |r|\le k+1$. [2]

(iii) Show that the inequality in **(ii)** also holds if r is a root of f and $|r|\ge 1$. [3]

(iv) Consider the polynomial equation

$$(2021+2n)x^5-2020x^4-2019x^3-2018x^2-2017x+2021+2n=0.$$

where n is a non-negative integer.

Given that the above equation has integer solution(s), determine all possible values of n, justifying your answer clearly using the result in **(iii)**. [3]

6 **(a)** Show, by induction or otherwise, that if n is a positive integer, then any odd number a satisfies $a^{2^n} \equiv 1 \left(\text{mod } 2^{n+2}\right)$. [7]

(b) Show that if a and k are positive integers then $a+1$ divides $a^{2k+1}+1$. [1]

By considering the case $a = 2^{2^r}$, or otherwise, show that if 2^n+1 is a prime then n is a power of 2. [4]

7 Show that, if n and r are positive integers, then

$$(n, n+r) = (n, r),$$

where (n, r) denotes the greatest common divisor of n and r. [2]

Show further that , if n is odd,

$$(2n, n+r) = \begin{cases} 2(n,r) & \text{for } r \text{ odd,} \\ (n,r) & \text{for } r \text{ even.} \end{cases}$$ [5]

For any positive integer n, $\phi(n)$ is defined to be the number of positive integers not exceeding n which are coprime to n. For example, $\phi(4) = 2$ since positive integers not exceeding 4 and coprime to 4 are 1 and 3.

Using the above results or otherwise, show that

(i) if n is odd, $\phi(2n) = \phi(n)$,

(ii) if n is even, $\phi(2n) = 2\phi(n)$,

(iii) $\phi\left(2^n\right) = 2^{n-1}$, where n is a positive integer. [7]

[Note that any result involving $\phi(n)$ should be proven before it can be used.]

8 **(i)** Verify that $\dfrac{1}{1+t^2}=1-t^2+t^4-t^6+\ldots+t^{4n}-\dfrac{t^{4n+2}}{1+t^2}$. [2]

Let $P_n(x)=\displaystyle\int_0^x \frac{t^{4n+2}}{1+t^2}\,\mathrm{d}t$ for $n\in\mathbb{Z}^+$, $0\le x\le 1$.

(ii) Prove that $0\le P_n(x)\le \dfrac{1}{4n+3}$. Hence or otherwise, show that

$$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$ [4]

(iii) D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty}a_r$ converges if $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|<1$, diverges if $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|>1$, and is inconclusive if $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=1$.

Given that the series expansion for $\tan^{-1}x$ is convergent for $x=\pm 1$, prove that the series expansion in **(ii)** is valid for $-1\le x\le 1$. [3]

(iv) Justify that $\tan^{-1}t+\tan^{-1}\dfrac{1}{t}=\dfrac{\pi}{2}$ if t is a positive real number. Demonstrate how a numerical approximation of $\tan^{-1}2021$ can be obtained using the first two terms of the series expansion in **(ii)**. [2]

One way to obtain an approximation of π is to sum sufficient initial terms of the series

$$4\left(\frac{1}{1(2^1)}+\frac{1}{1(3^1)}-\frac{1}{3(2^3)}-\frac{1}{3(3^3)}+\frac{1}{5(2^5)}+\frac{1}{5(3^5)}-\frac{1}{7(2^7)}-\frac{1}{7(3^7)}+\ldots\right)$$

(iii) By considering $\tan^{-1}\dfrac{1}{2}$ and $\tan^{-1}\dfrac{1}{3}$, explain how the above result is obtained.

[2]

9 The Bell number B_n gives the number of ways to distribute n distinct objects into no more than n identical boxes such that none of the boxes is empty. It is given that $B_0 = 1$ and $B_1 = 1$.

(i) Find B_2 and B_3. [2]

(ii) Show that $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. [4]

Stirling number of the second kind $S_{n,k}$ gives the number of ways to distribute n distinct objects into k identical boxes such that none of the boxes is empty.

(iii) Write down an expression for $S_{n,n-1}$. [1]

(iv) Find an expression for B_n in terms of Stirling numbers of the second kind. [1]

The set A is defined as $\{1, 2, 3, \cdots, n, a, b\}$. We are now going to distribute the elements in A into identical boxes such that no box is empty.

(v) Explain why there are B_n ways to distribute the elements in A into identical boxes, given that a and b are both alone in each of their respective boxes. [1]

(vi) Explain why there are B_{n+1} ways to distribute the elements in A into identical boxes, given that a and b are both in the same box. [1]

(vii) Suppose that a and b are in different boxes and that both a and b are not alone in their respective boxes, show that the number of ways to distribute the elements in A is given by $\sum_{k=1}^{n} k(k-1)S_{n,k}$. [2]

(viii) Hence show that $B_{n+2} \equiv \left(B_{n+1} + B_n\right)\left(\mathrm{mod}\, 2\right)$. [4]