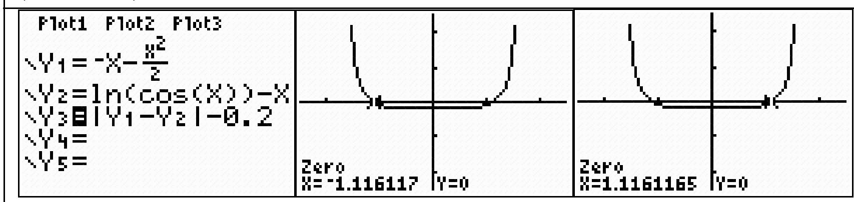
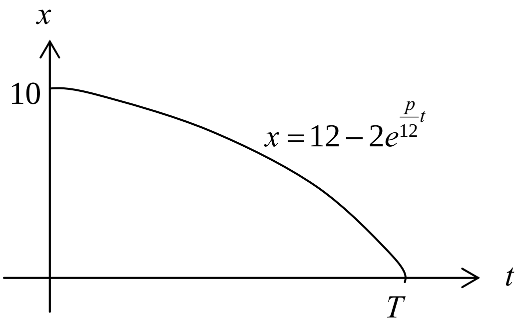
2014 H2 Mathematics Prelim P2 Worked Solutions

Qn	Solution	Notes
1(a) (i)	$\mathrm{e}^{y+x} = \cos x$ $y = \ln(\cos x) - x$ Differentiate wrt x $\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan x - 1$	
	Differentiate wrt x $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\sec^2 x$	
	$x = 0$ $y = 0$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -1$ $y = -x - \frac{x^2}{2} + \ldots$	
(a) (ii)	$\lvert \mathrm{h}(x) - y \rvert < 0.2$ $\lvert \mathrm{h}(x) - y \rvert - 0.2 < 0$	
	 From GC, $-1.12 < x < 1.12$ (to 3 s.f)	

1(b)	$\left(a-\frac{x}{3}\right)^n = a^n\left(1-\frac{x}{3a}\right)^n$	
	$= a^n\left(1-\frac{xn}{3a}+\frac{n(n-1)}{2}\left(\frac{x}{3a}\right)^2+..\right)$ $= a^n\left(1-\frac{n}{3a}x+\frac{n(n-1)}{18a^2}x^2+..\right)$	
	$-\frac{n}{3a}=\frac{4n(n-1)}{18a^2}$ $n=1-\frac{3a}{2}$	
	$a^n=\frac{1}{4}$ Sub $n=1-\frac{3a}{2}$ $a^{1-\frac{3a}{2}}=\frac{1}{4}$	
	From GC, $a=2$ or 0.16086 (to 5 s.f) when $a=2$, $n=-2$ when $a=0.161$ (to 3 s.f), $n=0.759$ (to 3 s.f)	

2(a)	$\frac{d^2y}{dx^2} = ae^{-2x}$ $\frac{dy}{dx} = \frac{ae^{-2x}}{-2} + c$ $y = \frac{ae^{-2x}}{4} + cx + d$	
2(b)	$\frac{dx}{dt} = kx - p$ Given that $\frac{dx}{dt} = 0$ when $x = 12$. $\Rightarrow 12k - p = 0$ $\Rightarrow k = \frac{p}{12}$ $\therefore \quad \frac{dx}{dt} = \frac{px}{12} - p$ $\Rightarrow \quad \frac{dx}{dt} = \frac{p}{12}(x-12)$ (Shown)	
(i)	$\int \frac{1}{x-12}\, dx = \int \frac{p}{12}\, dt$ $\ln\lvert x-12\rvert = \frac{p}{12}t + c$ $x - 12 = \pm\, e^{\frac{p}{12}t + c}$ $x = 12 + Ae^{\frac{p}{12}t}$, where $A = \pm e^{c}$ When $t = 0$, $x = 10 \Rightarrow A = -2$ $\therefore \quad x = 12 - 2e^{\frac{p}{12}t}$	
(ii)	When $x = 0$, $t = T$ $\Rightarrow 12 - 2e^{\frac{p}{12}T} = 0$ $\Rightarrow \frac{p}{12}T = \ln 6$ $\Rightarrow T = \frac{12}{p}\ln 6$	
(c)	 	

3	**(i)**	Since A lies in the plane π_1, $\begin{pmatrix} -10 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = -30$. $\Rightarrow -10\alpha = -30$ $\Rightarrow \alpha = 3$	
		Perpendicular distance from O to $\pi_1 = 6$ $\Rightarrow \dfrac{\left\|\overrightarrow{OA} \cdot \mathbf{n}_1\right\|}{\left\|\mathbf{n}_1\right\|} = 6$ $\Rightarrow \dfrac{\left\|-30\right\|}{\left\|\mathbf{n}_1\right\|} = 6$ $\Rightarrow \left\|\mathbf{n}_1\right\| = 5$ $\Rightarrow \alpha^2 + \beta^2 = 5^2$ $\Rightarrow \beta = 4$	
	(ii)	Acute angle between OA and π_2 $= \sin^{-1} \dfrac{\left\|\overrightarrow{OA} \cdot \mathbf{n}_2\right\|}{\left\|\overrightarrow{OA}\right\|\left\|\mathbf{n}_2\right\|}$ $= \sin^{-1} \dfrac{\left\|\begin{pmatrix} -10 \\ 0 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right\|}{\left\|\begin{pmatrix} -10 \\ 0 \\ 5 \end{pmatrix}\right\|\left\|\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}_2\right\|}$ $= \sin^{-1} \dfrac{20}{\sqrt{125}\sqrt{6}}$ $= 46.9°$	
	(iii)	$\begin{pmatrix} -10 \\ 0 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ -10 \end{pmatrix}$ A normal to plane π_3 is $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$. A cartesian equation is $x - 3y + 2z = 0$.	

4(i)	$p=1-\mathrm{i}, \quad q=1+\sqrt{3}\,\mathrm{i}, \quad r=k+\mathrm{i}$ $\Rightarrow \lvert p\rvert=\sqrt{1^2+(-1)^2}=\sqrt{2}, \quad \lvert q\rvert=\sqrt{1^2+(\sqrt{3})^2}=2, \quad \lvert r\rvert=\sqrt{k^2+1}$ $\arg(p)=-\frac{1}{4}\pi, \quad \arg(q)=\frac{1}{3}\pi$ $\lvert s\rvert=\frac{1}{2}$ $\Rightarrow \frac{\lvert p\rvert^2\lvert r\rvert}{\lvert q\rvert^3}=\frac{1}{2}$ $\Rightarrow \frac{\left(\sqrt{2}\right)^2\sqrt{k^2+1}}{(2)^3}=\frac{1}{2}$	
	$\Rightarrow \sqrt{k^2+1}=2$ $\Rightarrow k^2+1=4$ $\Rightarrow k=\pm\sqrt{3}$	
	$\arg(s)=-\frac{2}{3}\pi$ $\Rightarrow 2\arg(p)+\arg(r)-3\arg(q)=-\frac{2}{3}\pi$ $\Rightarrow \arg(r)=-\frac{2}{3}\pi-2\left(-\frac{1}{4}\pi\right)+3\left(\frac{1}{3}\pi\right)=\frac{5}{6}\pi$	
	If $k=\sqrt{3}$, then $\arg(r)=\frac{1}{6}\pi$ If $k=-\sqrt{3}$, then $\arg(r)=\frac{5}{6}\pi$ $\therefore \quad k=-\sqrt{3}$	

4(ii)	$z^3 = 4(1+\sqrt{3}\,\mathrm{i})$ $z^3 = 8\mathrm{e}^{\mathrm{i}\left(\frac{1}{3}\pi+2k\pi\right)}$ $= 8\mathrm{e}^{\mathrm{i}\left(\frac{1+6k}{3}\right)\pi}$	
	$z = 2\mathrm{e}^{\mathrm{i}\left(\frac{1+6k}{9}\right)\pi}, \quad k = -1, 0, 1$ $= 2\mathrm{e}^{\mathrm{i}\left(-\frac{5}{9}\pi\right)}, \quad 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)}, \quad 2\mathrm{e}^{\mathrm{i}\left(\frac{7}{9}\pi\right)}$	
	$w^3 = 4\left(\sqrt{3} - \mathrm{i}\right)$ $w^3 = 4\mathrm{i}\left(-\sqrt{3}\mathrm{i} - 1\right)$ $w^3 = -4\mathrm{i}\left(1+\sqrt{3}\mathrm{i}\right)$ $w^3 = (\mathrm{i})^3\, 4\left(1+\sqrt{3}\mathrm{i}\right)$ since $\mathrm{i}^2 = -1$ $\left(\frac{w}{\mathrm{i}}\right)^3 = 4\left(1+\sqrt{3}\mathrm{i}\right)$	
	Comparing with $z^3 = 4(1+\sqrt{3}\,\mathrm{i})$ $\frac{w}{\mathrm{i}} = z$ $w = \mathrm{i}z$	
	$w = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)} z$ $= \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)} \times 2\mathrm{e}^{\mathrm{i}\left(-\frac{5}{9}\pi\right)}, \quad \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)} \times 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)}, \quad \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)} \times 2\mathrm{e}^{\mathrm{i}\left(\frac{7}{9}\pi\right)}$ $= 2\mathrm{e}^{\mathrm{i}\left(-\frac{1}{18}\right)\pi}, 2\mathrm{e}^{\mathrm{i}\left(\frac{11}{18}\right)\pi}, 2\mathrm{e}^{\mathrm{i}\left(-\frac{13}{18}\right)\pi}$	

5(i)	This is quota sampling. A disadvantage of quota sampling is that the sample is not a good representation of the population (residents of the town).	
(ii)	Select the following number of students randomly from each stratum below.	

	18 - 25 year old	**26 - 35 year old**
Male	$\frac{75}{250}\times 100 \approx 30$	$\frac{36}{250}\times 100 \approx 14$
Female	$\frac{99}{250}\times 100 \approx 40$	$\frac{40}{250}\times 100 \approx 16$

6

Given $X \sim N(\mu, \sigma^2)$,

$$\mathrm{P}(2X_1 < X_2) = 0.8$$

$$\mathrm{P}(2X_1 - X_2 < 0) = 0.8$$

Now, $2X_1 - X_2 \sim N(\mu, 5\sigma^2)$

$$\mathrm{P}\left(Z < -\frac{\mu}{\sqrt{5}\sigma}\right) = 0.8$$

$$-\frac{\mu}{\sqrt{5}\sigma} = 0.84162$$

$$\mu = -1.8819\sigma$$

$$\sigma = -0.53138\mu$$

$$\mathrm{P}(X_1 + X_2 > 2a) = 0.8$$

$$\mathrm{P}(X_1 + X_2 < 2a) = 0.2$$

Now, $X_1 + X_2 \sim N(2\mu, 2\sigma^2)$

$$\mathrm{P}\left(Z < \frac{2a - 2\mu}{\sqrt{2}\sigma}\right) = 0.2$$

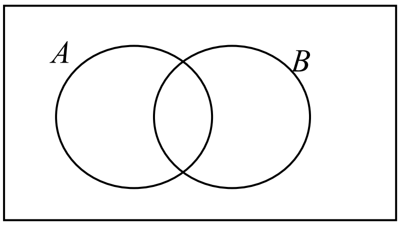
$$\frac{2a - 2\mu}{\sqrt{2}\sigma} = -0.84162$$

$$a - \mu = -0.59512\sigma$$

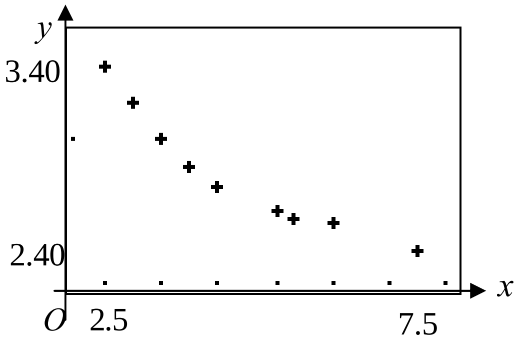
$$a - \mu = -0.59512(-0.53138\mu)$$

$$\mu = 0.75974a$$

$$= 0.760a$$

7(i)	$\mathrm{P}\left(B \mid A'\right) = 0.75$ $\dfrac{\mathrm{P}\left(B \cap A'\right)}{\mathrm{P}(A')} = 0.75$	
	$\mathrm{P}\left(B \cap A'\right) = 0.75\left(1-0.6\right)$ $= 0.3$  	
	$\mathrm{P}\left(A' \cap B'\right) = 1 - \mathrm{P}(A) - \mathrm{P}(B \cap A')$ $= 1 - 0.6 - 0.3$ $= 0.1$	
(ii)	$\mathrm{P}\left(A \mid B\right) = 0.4$ $\dfrac{\mathrm{P}\left(A \cap B\right)}{\mathrm{P}(B)} = 0.4$ $\mathrm{P}\left(A \cap B\right) = 0.4P(B)$ $\mathrm{P}(B) = \mathrm{P}\left(B \cap A'\right) + \mathrm{P}\left(A \cap B\right)$ $= 0.3 + 0.4\mathrm{P}(B)$ $\therefore 0.6\mathrm{P}(B) = 0.3$ $\mathrm{P}(B) = 0.5$	
	$\mathrm{P}(A) = 0.6$ $\mathrm{P}\left(A \mid B\right) = 0.4$ $\therefore \mathrm{P}\left(A \mid B\right) \neq \mathrm{P}(A)$ $\therefore$ A and B are **not** independent.	

8	1A, 1B, 1S, 2E, 1N, 1C No. of code words formed with 2'E's = $^5C_2 \times \frac{4!}{2!}$ $= 120$	
	No. of code words formed with 1 or no 'E' = $^6C_4 \times 4!$ $= 360$	
	Total no. of code words $= 120 + 360$ $= 480$ (shown)	
(i)	P(four-letter code words contain distinct letters) $= \frac{\text{No. of code words formed with 1 or no 'E'}}{\text{No. of code words formed without restrictions}}$ $= \frac{360}{480}$ =0.75	
(ii)	$\text{P}\left(\begin{matrix}\text{code words do not contain} \\ \text{any vowels}\end{matrix} \middle\vert \begin{matrix}\text{code words contain} \\ \text{distinct letters}\end{matrix}\right)$ $= \frac{\left(\frac{^4C_4 \times 4!}{480}\right)}{0.75}$ $= \frac{1}{15}$	

9(i)	$y=-0.175x+3.57$ $\bar{y}=-0.175\bar{x}+3.57$	
	$\frac{22.94+k}{9}=-0.175\left(\frac{38.8}{9}\right)+3.57$	
	$\therefore k=2.40\ \ (\text{shown})$	
(ii)		
	Using GC: $r=-0.943\ (3\text{ s.f.})$	
	Even though r is close to – 1, from the scatter diagram, the data points do not follow a straight line. Therefore, a linear model may not be suitable.	
(iii)	C is the appropriate model as the data points in the scatter diagram follow the graph of $y=e+\frac{f}{x}$.	
(iv)	Equation of regression line $y=\frac{2.7480}{x}+2.0651$ $y=\frac{2.75}{x}+2.07$ (to 3 s.f.)	
	When x = 8.0, $y=\frac{2.7480}{8.0}+2.0651=2.41$ (to 2 d.p.)	
	Since x = 8.0 falls outside the data range of x, the estimation of y is unreliable.	

10(i)	$\sum x = 1620$ and $\sum x^2 = 221175$ Unbiased estimate of population mean $= \frac{1620}{12}$ $= 135$	
	Unbiased estimate of population variance $= \frac{1}{11}\left[221175 - \frac{1620^2}{12}\right]$ $= 225$	
(ii)	$H_0 : \mu = 124.5$ $H_1 : \mu \neq 124.5$	
	Let X = number of packets of cereal sold daily. Since n = 12 is small, Under H_0, $T = \frac{\overline{X} - \mu_0}{\sqrt{\frac{S^2}{n}}} \sim t(n-1)$	
	$\alpha = 0.01$ From GC, p-value = 0.0337	
	Since p-value = 0.0337 > α = 0.01, we do not reject H_0 at 1% level of significance and conclude there is insufficient evidence that the mean number of packets of cereal sold per day has changed.	
(iii)	$H_0 : \mu = 124.5$ $H_1 : \mu > 124.5$	
	Under H_0, $\overline{X} \sim N\left(124.5, \frac{13^2}{12}\right)$ Test statistic: $z = \frac{\bar{x} - 124.5}{\sqrt{\frac{169}{12}}}$	
	$\alpha = 0.01$ Reject H_0 if $z \geq 2.3263$	
	Since H_0 is rejected, $\frac{\bar{x} - 124.5}{\sqrt{\frac{13^2}{12}}} \geq 2.3263$ $\bar{x} \geq 133.23$ $\bar{x} \geq 133$	
	It means that there is a probability of 0.01 of concluding that the mean number of packets of cereal sold per day is more than 124.5, when in fact the mean number is 124.5.	

11 (i)	Let X be the number of emergency admissions per day $X \sim \text{Po}(3)$ $X_1 + X_2 \sim \text{Po}(6)$ $\text{P}(X_1 + X_2 = 3) = 0.0892$ (to 3 s.f.)	
(ii)	Let length of time = t hr Let T be the number of emergency admissions in t hours. $T \sim \text{Po}\left(\frac{3t}{24}\right)$	
	$\text{P}(T = 0) = e^{-\frac{3t}{24}} = 0.2$ $-\frac{3t}{24} = \ln 0.2$ $t = -8\ln 0.2$ $\approx 13\text{hr}$	
(iii)	$\text{P}(X > 4) = 1 - \text{P}(X \le 4)$ $= 0.18474 \approx 0.185$ (to 3 s.f.)	
(iv)	Let Y be the number of days with at most 4 admissions out of 50 days. $Y \sim \text{B}(50, 0.81526)$ Since $n = 50$ is large, $np = 40.763 > 5$, $nq = 9.237 > 5$, $Y \sim \text{N}(40.763, 7.5306)$ approximately $\text{P}(Y < 40) \xrightarrow{\text{C.C.}} \text{P}(Y < 39.5) \approx 0.323$	
(v)	Let W be the number of patients who require surgery out of 15. $W \sim \text{B}(15, 0.4)$ $\text{E}(W) = 15(0.4) = 6$ $\text{Var}(W) = 15(0.4)(1-0.4) = 3.6$	
	Let $\overline{W}$ be the mean number of patients who require surgery. Since n is large, so by Central Limit Theorem, $\overline{W} \sim \text{N}(6, \frac{3.6}{50})$ approximately. $\text{P}(\overline{W} > 5.5) = 0.96880$ ≈ 0.969 (to 3 s.f)	