Notes for the Singapore-Cambridge 'A'-Level

Inequalities

9820 Mathematics III (2025 onward)

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Syllabus requirements

More inequalities outside the syllabus will be included, and some of the inequalities below will be detailed at greater depth.

- AM-GM inequality
- Cauchy-Schwarz inequality
- Triangle inequality

Read this first before continuing

This document is only meant to cover inequalities in the H3 syllabus. It may not be enough to cover inequalities expected at mathematical olympiads, although this can be a great starting point. Most non-trivial inequalities here are proven.

If you would like to see resources on mathematical olympiad inequalities, the author recommends Yufei Zhao's handout on inequalities and Evan Chen's brief notes.

Contents

1 The AM-GM inequality and other extensions

The *arithmetic mean-geometric mean* inequality is one of the most important inequalities in mathematics – whether it be in Olympiads or in general, it is used widely in proofs involving non-negative real numbers.

For starters, we have the following definitions:

Definition 1.1. The arithmetic mean of $a_1, a_2, \dots, a_n \in \mathbb{R}_0^+$ is defined as

$$\mathrm{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k$$

Definition 1.2. The geometric mean of $a_1, a_2, \dots, a_n \in \mathbb{R}_0^+$ is defined as

$$\mathrm{GM} = \sqrt[n]{a_1 a_2 \dots a_n} = \sqrt[n]{\prod_{k=1}^{n} a_k}$$

1.1 Proof of the AM-GM inequality

Now, we will prove the AM-GM inequality. Consider the simple case $n = 2$, where

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$$

This holds if and only if

$$\begin{aligned}
\frac{a_1^2 + 2a_1a_2 + a_2^2}{4} &\geq a_1a_2 \\
a_1^2 + 2a_1a_2 + a_2^2 - 4a_1a_2 &\geq 0 \\
a_1^2 - 2a_1a_2 + a_2^2 &\geq 0 \\
(a_1 - a_2)^2 &\geq 0
\end{aligned}$$

which is obviously true since $a_1, a_2 \in \mathbb{R}$.

Theorem 1.3 (AM-GM inequality)**.** *For any $a_1, a_2, \dots, a_n \in \mathbb{R}_0^+$,*

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}$$

with equality attained if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. The above inequality can be reduced to the inequality

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \geq a_1 a_2 \dots a_n$$

on which we proceed by induction.

The base case $n = 1$ results in equality. Hence, suppose that the inequality holds for some arbitrary n.

Then, consider the case $n + 1$, with $a_1, a_2, \ldots, a_n, a_{n+1} \in \mathbb{R}_0^+$. We know that

$$\alpha = \frac{a_1 + a_2 + \cdots + a_n + a_{n+1}}{n+1}$$

is their mean. If $a_1 = a_2 = \cdots = a_n = a_{n+1}$, then there is equality.

Otherwise, there exists $a_i < \alpha < a_j$. Since addition and multiplication are commutative, the list $a_1, \ldots, a_n, a_{n+1}$ can be reordered such that $a_i = a_n < \alpha$ and $a_j = a_{n+1} > \alpha$ without loss of generality. Since $a_{n+1} - \alpha > 0$ and $a_n - \alpha > 0$, it follows that $(a_{n+1} - \alpha)(a_n - \alpha) > 0$. We know that $\alpha(n + 1) = a_1 + \cdots + a_{n+1}$ so

$$\alpha = \frac{a_1 + \cdots + a_{n-1} + (a_n + a_{n+1} - \alpha)}{n}.$$

Due to the induction hypothesis, one has $\alpha^{n+1} = \alpha^n \alpha \geq a_1 a_2 \ldots a_{n-1}(a_n + a_{n+1} - \alpha)\alpha$.

We use the result $(a_{n+1} - \alpha)(a_n - \alpha) > 0$ to justify that $(a_{n+1} - \alpha)(a_n - \alpha) = \alpha(a_n + a_{n+1} - \alpha) - a_n a_{n+1} > 0$ implying $(a_n + a_{n+1} - \alpha)\alpha > a_n a_{n+1}$.

Hence

$$\alpha^{n+1} = \alpha^n \alpha \geq a_1 a_2 \ldots a_{n-1}(a_n + a_{n+1} - \alpha)\alpha \geq a_1 a_2 \ldots a_n a_{n+1}$$

and the proof is complete. □

Example 1.4 (IMO 2020 Shortlisted Problem)**.** Suppose $a, b, c, d \in \mathbb{R}^+$ satisfy $ac+bd = (a+c)(b+d)$. Find the smallest possible value of $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$.

Solution. We know that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = \left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right)$$

and

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \left(\frac{b}{c} + \frac{d}{a}\right) \geq 2\left(\sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}}\right)$$

Clearly,

$$\sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}} = \frac{\sqrt{ac}}{\sqrt{bd}} + \frac{\sqrt{bd}}{\sqrt{ac}} = \frac{\sqrt{ac}}{\sqrt{bd}} + \frac{\sqrt{bd}}{\sqrt{ac}} = \frac{ac + bd}{\sqrt{abcd}} = \frac{(a + c)(b + d)}{\sqrt{abcd}}$$

Using again the AM-GM inequality, $a + c \geq 2\sqrt{ac}$ and $b + d \geq 2\sqrt{bd}$ so

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2\left(\frac{(a + c)(b + d)}{\sqrt{abcd}}\right) \geq 2\left(\frac{(2\sqrt{ac})(2\sqrt{bd})}{\sqrt{abcd}}\right) = 8$$

and so the minimum value is 8. □

The AM-GM inequality appears in the formula booklet, unlike the more powerful inequalities like the weighted AM-GM inequality and the QM-AM-GM-HM inequality chain.

1.2 Weighted AM-GM inequality

Theorem 1.3 implies the *weighted* AM-GM inequality, which states:

Corollary 1.5 (Weighted AM-GM inequality)**.** *For any* $w_1, w_2, \dots, w_n \geq 0$ *corresponding to* $a_1, a_2, \dots, a_n \geq 0$*,*

$$\frac{w_1 a_1 + w_2 a_2 + \cdots + w_n a_n}{\Sigma w} \geq \sqrt[\Sigma w]{a_1^{w_1} a_2^{w_2} \dots a_n^{w_n}}$$

where $\Sigma w = w_1 + \cdots + w_n$*, with equality attained only when* $a_1 = a_2 = \cdots = a_n$*.*

A proof will not be provided here.

1.3 QM-AM-GM-HM inequality chain

The *quadratic mean-arithmetic mean-geometric mean-harmonic mean* inequality chain, also known as the *mean inequality chain*, state the relationship between the harmonic mean, geometric mean, arithmetic mean, and quadratic mean of positive real numbers. This time, no number can be zero, or the harmonic mean would be undefined.

Definition 1.6. The quadratic mean (root mean square) of $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ is defined as

$$\text{QM} = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} = \sqrt{\frac{1}{n} \sum_{k=1}^{n} a_k^2}$$

Definition 1.7. The harmonic mean of $a_1, a_2, \dots, a_n \in \mathbb{R}^+$ is defined as

$$\text{HM} = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

In order to prove that QM $\geq$ AM $\geq$ GM $\geq$ HM, one must prove separately that QM $\geq$ AM, then AM $\geq$ GM implies QM $\geq$ AM $\geq$ GM. Finally, proving GM $\geq$ HM completes the inequality chain.

Lemma 1.8 (QM-AM)**.** *For any* $a_1, a_2, \dots, a_n \in \mathbb{R}^+$*,*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Proof. This reduces to the inequality

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \geq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n^2} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2$$

which holds due to $n \geq 1, n \in \mathbb{Z}^+$ and Theorem 2.1, which asserts

$$\left(\sum_{k=0}^{n} u_k v_k\right)^2 \leq \left(\sum_{k=0}^{n} u_k^2\right)\left(\sum_{k=0}^{n} v_k^2\right)$$

for $u_k, v_k \in \mathbb{R}$. □

Lemma 1.9 (GM-HM)**.** *For any* $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$,

$$\sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Proof. This inequality reduces to

$$\frac{1}{\sqrt[n]{a_1 a_2 \ldots a_n}} \leq \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n}$$

However,

$$\frac{1}{\sqrt[n]{a_1 a_2 \ldots a_n}} = \left(\sqrt[n]{a_1 a_2 \ldots a_n}\right)^{-1} = (a_1 a_2 \ldots a_n)^{-\frac{1}{n}} = \sqrt{\frac{1}{a_1}\frac{1}{a_2}\ldots\frac{1}{a_n}}$$

and thus the inequality above holds, due to the AM-GM inequality. □

Theorem 1.10 (QM-AM-GM-HM)**.** *For any* $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$,

$$\begin{aligned}\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} &\geq \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &\geq \sqrt[n]{a_1 a_2 \ldots a_n} \\ &\geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}\end{aligned}$$

where no number is zero.

Proof. This follows by Lemma 1.8 and Lemma 1.9, as well as Theorem 1.3. □

2 The Cauchy-Schwarz inequality

The *Cauchy-Schwarz inequality* (or *Cauchy-Bunyakovsky-Schwarz inequality*) provides an upper bound on the inner product between two vectors in an inner product space in terms of the product of the vector norms (or magnitudes).

Formally, an *inner product space* is a real or complex vector space (permitting vector arithmetic) bundled with an operation called the *inner product*. An inner product is usually denoted $\langle \vec{u}, \vec{v} \rangle$. The inner product of $\mathbb{R}^n$ satisfies the following conditions:

- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$. In other words, the inner product is commutative for real vectors $\vec{u}, \vec{v}$.
- $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$ for scalars $a, b \in \mathbb{R}$.
- $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x}$ is not the zero vector.

We will operate in the real vector space $\mathbb{R}^n$ where vectors are of length n, and we use the dot product as the inner product; all the properties above are satisfied.

2.1 Proof of the inequality

Theorem 2.1 (Cauchy-Schwarz in $\mathbb{R}^n$)**.** *For any $u_1, \ldots, u_n$ and $v_1, \ldots, v_n \in \mathbb{R}$,*

$$\left(\sum_{k=0}^{n} u_k v_k \right)^2 \leq \left(\sum_{k=0}^{n} u_k^2 \right) \left(\sum_{k=0}^{n} v_k^2 \right)$$

with equality if and only if there exists $c \neq 0$ such that $u_k = cv_k$ for all $i = 1, 2, \ldots, n$.

Proof. Consider the real vectors

$$\vec{a} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

of length n, where $u_i, v_i \in \mathbb{R}$ for any $i = 1, 2, \ldots, n$. Then, if $|\vec{x}|$ denotes the Euclidean norm $|\vec{x}| = \sqrt{x_1^2 + \cdots + x_n^2}$,

$$|\vec{a}||\vec{b}| \cos(\theta) = \vec{a} \cdot \vec{b} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

and so

$$\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \cos(\theta) = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Since $\cos^2(\theta) \in [0,1]$ for any θ, it is obvious that after squaring both sides one has

$$\left(u_1^2 + u_2^2 + \cdots + u_n^2\right)\left(v_1^2 + v_2^2 + \cdots + v_n^2\right) \geq \left(u_1 v_1 + u_2 v_2 + \cdots + u_n v_n\right)^2$$

and in summation notation,

$$\left(\sum_{k=0}^{n} u_k v_k\right)^2 \leq \left(\sum_{k=0}^{n} u_k^2\right)\left(\sum_{k=0}^{n} v_k^2\right)$$

Thus the proof is complete. The condition of equality is obvious and shall not be proven here. □

Example 2.2. Prove that

$$x + y + z \leq 2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right)$$

for all positive $x, y, z \in \mathbb{R}$.

Solution. We recognize that the Cauchy-Schwarz inequality uses squares, and express the inequality as follows:

$$x + y + z \leq 2\left(\overbrace{\left(\frac{x}{\sqrt{y+z}}\right)^2}^{u_1^2} + \overbrace{\left(\frac{y}{\sqrt{z+x}}\right)^2}^{u_2^2} + \overbrace{\left(\frac{z}{\sqrt{x+y}}\right)^2}^{u_3^3}\right)$$

We can get rid of the denominator, multiplying both sides by $v_1^2 + v_2^2 + v_3^2 = \sqrt{(y+z)^2} + \sqrt{(z+x)^2} + \sqrt{(x+y)^2} = 2(x+y+z)$. Hence one obtains

$$(x+y+z)^2 \leq 2\left(\left(\frac{x}{\sqrt{y+z}}\right)^2 + \left(\frac{y}{\sqrt{z+x}}\right)^2 + \left(\frac{z}{\sqrt{x+y}}\right)^2\right)(x+y+z)$$

which is true due to the Cauchy-Schwarz inequality. Hence our inequality is proven. □

Exercise 2.3 (APMO 1991)**.** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Prove that

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + b_k} \geq \frac{1}{2}\sum_{k=1}^{n} a_k$$

2.2 Titu's lemma and Nesbitt's inequality

Titu's lemma is often used in mathematical competitions to provide a wide range of inequalities, like Nesbitt's inequality.

Proposition 2.4 (Titu's lemma)**.** *The following inequality*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

holds for all $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}$ where $b_i > 0$ for any i.

Proof. Consider $u_k = a_k$ and $v_k = \frac{1}{\sqrt{b_k}}$. The inequality follows after Theorem 2.1. □

Proposition 2.5 (Nesbitt's inequality)**.** *The following inequality*

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

holds for any $a, b, c \in \mathbb{R}^+$, and equality is attained only when $a = b = c$.

Nesbitt's inequality can be proved using various means, like the AM-HM inequality which is part of the QM-AM-GM-HM chain covered in the previous chapter, as well as the Cauchy-Schwarz inequality.

Exercise 2.6. Prove Theorem 2.5 using

1. the AM-HM inequality
2. the Cauchy-Schwarz inequality

3 The triangle inequality

The triangle inequality states that the sum of the lengths of any two sides of a triangle must be greater than or equal to the length of the remaining side.

3.1 Two-variable triangle inequality

If we consider a triangle whose sides are vectors $\vec{a}, \vec{b}, (\vec{a}+\vec{b})$, we have the following:

Theorem 3.1 (Triangle inequality)**.** *For any vectors $\vec{a}, \vec{b}$,*

$$|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$

with equality if and only if $\vec{a}$ and $\vec{b}$ point in the same direction and are collinear.

Proof. We know that $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$ for any vector $\vec{v}$. Hence, we square both sides of the above inequality and obtain

$$|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2 \iff (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) \leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2.$$

By the associativity and distributive property of the dot product,

$$(\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \vec{a} \cdot (\vec{a}+\vec{b}) + \vec{b} \cdot (\vec{a}+\vec{b}) = |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos(\theta) + |\vec{b}|^2$$

since $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$ where θ is the angle between $\vec{a}$ and $\vec{b}$. Therefore one has

$$|\vec{a}+\vec{b}|^2 \leq (|\vec{a}| + |\vec{b}|)^2 \iff |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos(\theta) + |\vec{b}|^2 \leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2.$$

This finally reduces to the inequality

$$|\vec{a}||\vec{b}|\cos(\theta) \leq |\vec{a}||\vec{b}|$$

which is true because $-1 \leq \cos(\theta) \leq 1$. If $\vec{a}$ and $\vec{b}$ point in the same direction, $\cos(\theta) = \cos(0) = 1$ and equality is achieved. Hence the proof is complete. □

Example 3.2. The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are defined such that $|\vec{u}| = 1$, $|\vec{v}| = 2$ and $|\vec{w}| = 3$. If $\vec{w} = \vec{u} + \vec{v}$, explain why $\vec{u}$ is a unit vector parallel to $\vec{w}$.

Solution. By the triangle inequality, $|\vec{u}| + |\vec{v}| \geq |\vec{w}|$, with equality attained if and only if $\vec{u}$ and $\vec{v}$ (and $\vec{w} = \vec{u} + \vec{v}$) point in the same direction and are collinear. Since $|1| + |2| = |3|$, $\vec{u}$ is collinear with $\vec{w}$ and hence they are parallel. Also, $|\vec{u}| = 1$ so $\vec{u}$ is a unit vector. □

This also applies to the real numbers:

Corollary 3.3. *For any real a, b,*

$$|a+b| \leq |a| + |b|$$

with equality if and only if $ab \geq 0$.

Proof. Consider the one-dimensional vectors $\vec{u} = \langle a \rangle$ and $\vec{u} = \langle b \rangle$. By Theorem 3.1, $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$, where equality is attained when $ab \geq 0$, as it implies that $\vec{u}$ points in the same direction as $\vec{v}$ (because a, b have the same sign). Obviously since the vectors are one-dimensional and span a line, they are always collinear. □

3.2 Generalized triangle inequality

Theorem 3.4. *For any vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$,*

$$\left| \sum_{k=1}^{n} \vec{v_k} \right| \leq \sum_{k=1}^{n} |\vec{v_k}|$$

with equality if and only if all the vectors point in the same direction and are collinear.

Proof. By induction. We have already proved the base case for $n = 2$. Now, for our induction hypothesis, suppose that

$$\left| \sum_{k=1}^{n} \vec{v_k} \right| \leq \sum_{k=1}^{n} |\vec{v_k}|.$$

Our induction step is to prove that

$$\begin{aligned} \left| \sum_{k=1}^{n+1} \vec{v_k} \right| \leq \sum_{k=1}^{n+1} |\vec{v_k}| \iff \left| \vec{v}_{n+1} + \sum_{k=1}^{n} \vec{v_k} \right| &\leq |\vec{v}_{n+1}| + \left| \sum_{k=1}^{n} \vec{v_k} \right| \\ &\leq |\vec{v}_{n+1}| + \sum_{k=1}^{n} |\vec{v_k}| \end{aligned}$$

and hence we are done. □

Again, we must always find some way to apply the induction hypothesis by some kind of substitution or algebraic manipulation. Above we treated $\sum_{k=1}^{n} \vec{v_k}$ as another vector, before applying Theorem 3.1, and then the induction hypothesis.

Example 3.5 (BMO 2010)**.** Let a, b and c be the lengths of the sides of a triangle. Suppose that $ab + bc + ca = 1$. Show that $(a+1)(b+1)(c+1) < 4$.

Solution. Expanding, one has

$$\begin{aligned}(a+1)(b+1)(c+1) &= (ab+1+a+b)(c+1)\\ &= abc+ac+bc+c+ab+1+a+b\\ &= abc+(ab+bc+ca)+c+1+a+b\\ &= abc+(a+b+c)+2\end{aligned}$$

If we were to obtain another expression of this form, we could consider

$$\begin{aligned}(a-1)(b-1)(c-1) &= (ab+1-a-b)(c-1)\\ &= abc+(a+b+c)-2\\ &= abc+(a+b+c)+2-4\\ &= (a+1)(b+1)(c+1)-4.\end{aligned}$$

We now claim $0 < a, b, c < 1$. Clearly, $a, b, c > 0$ because they are lengths. Without loss of generality, we have $c < a + b$ by the **triangle inequality**. Hence $c^2 < ac + bc$ and $c^2 < 1 - ab < 1$. Hence $a, b, c < 1$.

Now, we can confidently state $(a-1)(b-1)(c-1) < 0$, so we have $(a+1)(b+1)(c+1) - 4 < 0$. The inequality thus follows. □

In the above problem, one may be wondering how we managed to get $(a-1)(b-1)(c-1)$. It is generally useful to play around with the expression $(a+1)(b+1)(c+1)$ to see if we can try to substitute it into another expression, as we have done above. We could then prove that $0 < a, b, c < 1$, which allowed us to show that $(a-1)(b-1)(c-1)$ was negative, and we obtained an inequality from that.