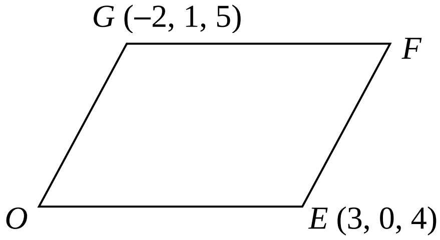
H2 Mathematics (9758)
Chapter 5 Vectors
Discussion Solutions

Level 1

1 **(a)** $OEFG$ is a parallelogram as shown in the diagram.

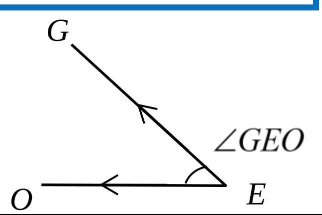


(i) Find $\overrightarrow{OF}$ and $\overrightarrow{EG}$.

(ii) Find the size of the angle GEO.

(b) The points P, Q and R have coordinates $(4,\ 1,\ 1)$, $(-8,\ 5, -15)$ and $(7, 0,\ 5)$ respectively. Show that P, Q, and R are collinear.

(c) Given $\mathbf{a}=\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{b}=\mathbf{j}-3\mathbf{k}$, find a vector perpendicular to both **a** and **b**.

Q1	Solution
(a)(i)	$\overrightarrow{OF}=\overrightarrow{OE}+\overrightarrow{EF}$ $=\overrightarrow{OE}+\overrightarrow{OG}$ $=\begin{pmatrix}3\\0\\4\end{pmatrix}+\begin{pmatrix}-2\\1\\5\end{pmatrix}$ $=\begin{pmatrix}1\\1\\9\end{pmatrix}$ Using the parallelogram, $\overrightarrow{OF}=\overrightarrow{OE}+\overrightarrow{EF}$ and $\overrightarrow{EF}=\overrightarrow{OG}$ $\overrightarrow{EG}=\overrightarrow{OG}-\overrightarrow{OE}$ $=\begin{pmatrix}-2\\1\\5\end{pmatrix}-\begin{pmatrix}3\\0\\4\end{pmatrix}$ $=\begin{pmatrix}-5\\1\\1\end{pmatrix}$ Recall: $\overrightarrow{EG}=\overrightarrow{OG}-\overrightarrow{OE}$ Use position vectors $\overrightarrow{OG}$ and $\overrightarrow{OE}$ to find $\overrightarrow{EG}$
(a)(ii)	$\overrightarrow{EG}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$ and $\overrightarrow{EO}=-\overrightarrow{OE}=\begin{pmatrix}-3\\0\\-4\end{pmatrix}$ To find $\angle GEO$, use vectors that are both either pointing 'outwards' from point E (vectors $\overrightarrow{EG}$ and $\overrightarrow{EO}$) or pointing 'inwards' from point E (vectors $\overrightarrow{GE}$ and $\overrightarrow{OE}$) $\cos\angle GEO=\dfrac{\overrightarrow{EG}\cdot\overrightarrow{EO}}{\lvert\overrightarrow{EG}\rvert\lvert\overrightarrow{EO}\rvert}=\dfrac{\begin{pmatrix}-5\\1\\1\end{pmatrix}.\begin{pmatrix}-3\\0\\-4\end{pmatrix}}{\sqrt{27}\sqrt{25}}=\dfrac{11}{15\sqrt{3}}$  $\therefore \angle GEO=65.0^\circ$ (1 d.p)

(b)	$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}-8\\5\\-15\end{pmatrix}-\begin{pmatrix}4\\1\\1\end{pmatrix}=\begin{pmatrix}-12\\4\\-16\end{pmatrix}$ $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}7\\0\\5\end{pmatrix}-\begin{pmatrix}4\\1\\1\end{pmatrix}=\begin{pmatrix}3\\-1\\4\end{pmatrix}$ $\overrightarrow{PQ}=\begin{pmatrix}-12\\4\\-16\end{pmatrix}=-4\begin{pmatrix}3\\-1\\4\end{pmatrix}=-4\overrightarrow{PR}$ Recommended to work out the constant like this to ensure that you have the correct relationship between the vectors Since $\overrightarrow{PQ}=-4\overrightarrow{PR}$, P, Q, R are collinear. (Other answers such as $\overrightarrow{PQ}=-\frac{4}{5}\overrightarrow{QR}$ is also accepted)
(c)	A vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{a}\times\mathbf{b}=\begin{pmatrix}1\\1\\-1\end{pmatrix}\times\begin{pmatrix}0\\1\\-3\end{pmatrix}=\begin{pmatrix}-2\\3\\1\end{pmatrix}$ $\left(\mathbf{b}\times\mathbf{a}=\begin{pmatrix}2\\-3\\-1\end{pmatrix}\text{ is also accepted}\right)$

2 2010(9740)/I/1

The position vectors **a** and **b** are given by

$$\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where $p > 0$. It is given that $|\mathbf{a}| = |\mathbf{b}|$.

(i) Find the exact value of *p*. [2]

(ii) Show that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$. [3]

Q2	**Suggested Solutions**
(i)	Since $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert$, $\sqrt{(2p)^2+(3p)^2+(6p)^2} = \sqrt{1^2+2^2+2^2}$ $4p^2+9p^2+36p^2 = 9$ $p^2 = \frac{9}{49}$ $p = \pm\frac{3}{7}$ Note that there are 2 possible values of *p* when you solve $p^2 = \frac{9}{49}$. Since $p > 0$, $p = \frac{3}{7}$ Always reject with reason
(ii)	**<u>Method 1:</u>** $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$ Make sure the dot for the dot product is visible in your working. Properties to note here: 1. $\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2$ 2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ $= \lvert\mathbf{a}\rvert^2 - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a} - \lvert\mathbf{b}\rvert^2$ $= \lvert\mathbf{a}\rvert^2 - \lvert\mathbf{a}\rvert^2 \quad (\because \lvert\mathbf{b}\rvert = \lvert\mathbf{a}\rvert)$ $= 0$ **<u>Method 2:</u>** $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \left[\frac{3}{7}\begin{pmatrix}2\\3\\6\end{pmatrix} + \begin{pmatrix}1\\-2\\2\end{pmatrix}\right] \bullet \left[\frac{3}{7}\begin{pmatrix}2\\3\\6\end{pmatrix} - \begin{pmatrix}1\\-2\\2\end{pmatrix}\right]$ $= \begin{pmatrix}13/7\\-5/7\\32/7\end{pmatrix} \bullet \begin{pmatrix}-1/7\\23/7\\4/7\end{pmatrix}$ $= -\frac{13}{49} - \frac{115}{49} + \frac{128}{49} = 0$

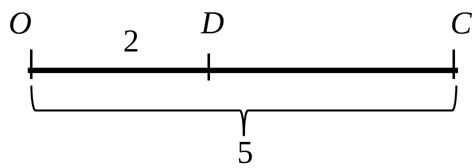
3 **2009 MJC Promo**

The points A, B and C have position vectors $\begin{pmatrix}2\\1\\5\end{pmatrix}$, $\begin{pmatrix}3\\y\\7\end{pmatrix}$ and $\begin{pmatrix}5\\10\\11\end{pmatrix}$ with respect to the origin O respectively.

(i) Find the value of y such that A, B and C are collinear. [2]

(ii) Find the exact area of triangle OAC. [3]

(iii) The point D divides OC internally such that $OD : OC = 2 : 5$. Find the vector $\overrightarrow{AD}$. [3]

Q3	Solution
(i)	$\overrightarrow{AB} = \lambda\overrightarrow{AC}$ for some $\lambda \in \mathbb{R}$. (or other valid combinations) $\begin{pmatrix}1\\y-1\\2\end{pmatrix} = \lambda\begin{pmatrix}3\\9\\6\end{pmatrix} \Rightarrow \lambda = \frac{1}{3}$ $\therefore y = 4$
(ii)	Area of triangle $OAC = \frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}11-50\\25-22\\20-5\end{pmatrix}\right\|$ $= \frac{1}{2}\sqrt{39^2+3^2+15^2}$ $= \frac{1}{2}\sqrt{1755} = \frac{3}{2}\sqrt{195}$ units^2
(iii)	 $OD : OC = 2 : 5$ $\overrightarrow{OD} = \frac{2}{5}\overrightarrow{OC}$ $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$ $= \frac{2}{5}\overrightarrow{OC} - \overrightarrow{OA}$ $= \begin{pmatrix}0\\3\\-3/5\end{pmatrix}$

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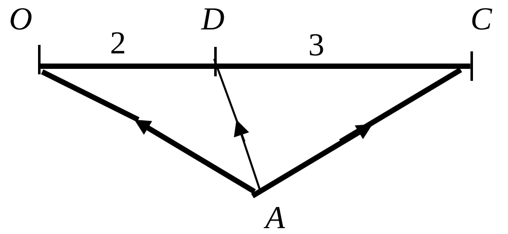
Alternative Method:



OD : *DC* = 2 : 3

By ratio theorem, $\overrightarrow{AD} = \dfrac{2\overrightarrow{AC} + 3\overrightarrow{AO}}{5}$

$$= \frac{1}{5}\left[\begin{pmatrix} 6 \\ 18 \\ 12 \end{pmatrix} + \begin{pmatrix} -6 \\ -3 \\ -15 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 \\ 3 \\ -3/5 \end{pmatrix}$$

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Level 2

4 The position vectors of the points A, B and C are $\overrightarrow{OA} = -8\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 5\mathbf{k}$ and $\overrightarrow{OC} = p\mathbf{i} + (2p - 11)\mathbf{j} - 4\mathbf{k}$ respectively. Find the

(i) unit vector(s) parallel to the vector $\overrightarrow{AB}$,

(ii) position vector of the midpoint of AB,

(iii) value of p such that A, B and C are collinear,

(iv) position vector of D such that $ABCD$ is a parallelogram and $p = 6$.

(v) position vector of E such that point E is on AB produced and $AB : AE = 2 : 5$.

Q4	Solution
(i)	$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 8 \\ 6 \end{pmatrix}$ Recall that: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ Good practice to express $\overrightarrow{AB}$ with reference to position vectors Unit vectors required $= \pm \dfrac{1}{\sqrt{101}} \begin{pmatrix} 1 \\ 8 \\ 6 \end{pmatrix} = \dfrac{\sqrt{101}}{101} \begin{pmatrix} 1 \\ 8 \\ 6 \end{pmatrix}$ or $-\dfrac{\sqrt{101}}{101} \begin{pmatrix} 1 \\ 8 \\ 6 \end{pmatrix}$
(ii)	Position vector of midpoint of AB $= \dfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right) = \dfrac{1}{2} \begin{pmatrix} 1 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -4 \\ 2 \end{pmatrix}$
(iii)	Since A, B and C are collinear, there exists $\lambda \in \mathbb{R}$ such that $\overrightarrow{AC} = \lambda \overrightarrow{AB}$ Good practice to express $\overrightarrow{AC}$ and $\overrightarrow{AB}$ with reference to position vectors Learning Point: Points A, B, and C are collinear $\Leftrightarrow \overrightarrow{AB} = \lambda \overrightarrow{AC}$, for some $\lambda \in \mathbb{R}, \lambda \neq 0$. $\Rightarrow \overrightarrow{OC} - \overrightarrow{OA} = \lambda\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$ $\Rightarrow \begin{pmatrix} p \\ 2p - 3 \\ -3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 8 \\ 6 \end{pmatrix}$

$$\Rightarrow \begin{cases} p = \lambda & \text{- (1)} \\ 2p-3=8\lambda & \text{- (2)} \\ -3=6\lambda & \text{- (3)} \end{cases}$$

Using (1) and (3), $\lambda=-\dfrac{1}{2}$ and $p=-\dfrac{1}{2}$

Verify whether the value of p and λ satisfy **<u>all three equations</u>**.

Checking (2),

$\text{LHS}=2\left(-\dfrac{1}{2}\right)-3=-4$

$\text{RHS}=8\left(-\dfrac{1}{2}\right)=-4$

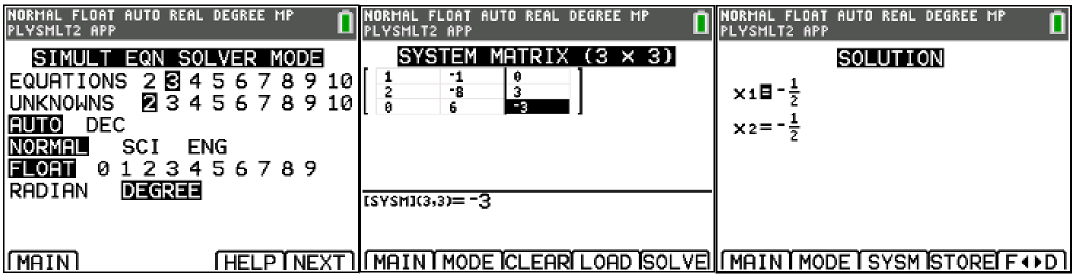
LHS = RHS

$\therefore \lambda=-\dfrac{1}{2}$ and $p=-\dfrac{1}{2}$.

Alternatively,

$$\Rightarrow \begin{cases} p = \lambda \\ 2p-3=8\lambda \\ -3=6\lambda \end{cases} \Rightarrow \begin{cases} p-\lambda=0 & \text{- (1)} \\ 2p-8\lambda=3 & \text{- (2)} \\ 6\lambda=-3 & \text{- (3)} \end{cases}$$

Using GC, $\lambda=-\dfrac{1}{2}$ and $p=-\dfrac{1}{2}$.

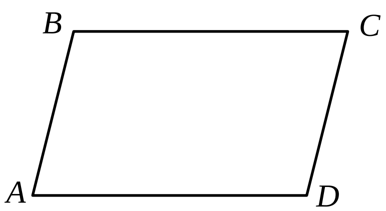


(iv) Since $ABCD$ is a parallelogram,

$\overrightarrow{AD}=\overrightarrow{BC}$

Label the vertices in clockwise or anti-clockwise direction

Good practice to express $\overrightarrow{AD}$ and $\overrightarrow{BC}$ with reference to position vectors



$\Rightarrow \overrightarrow{OD}-\overrightarrow{OA}=\overrightarrow{OC}-\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{OC}-\overrightarrow{OB}$

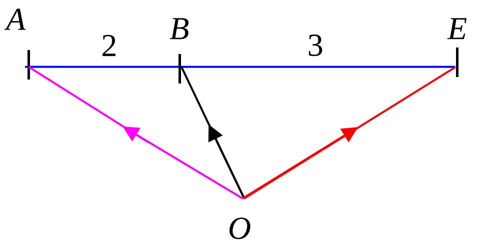
$$=\begin{pmatrix}0\\-8\\-1\end{pmatrix}+\begin{pmatrix}6\\1\\-4\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}=\begin{pmatrix}5\\-7\\-10\end{pmatrix}$$

Alternatively, we can also use $\overrightarrow{AB}=\overrightarrow{DC}$

(v) Given $AB : AE = 2 : 5$

Learning Point:
Draw diagram to clearly show the ratio that divides AE internally before applying ratio theorem.

By ratio theorem,

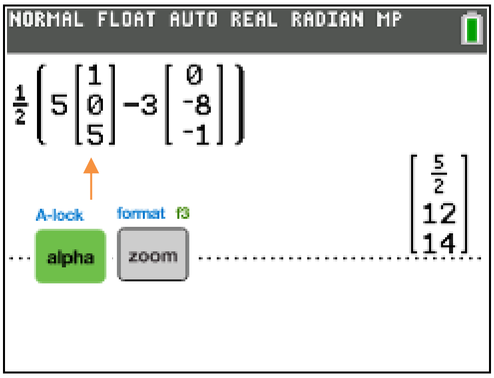


$$\overrightarrow{OB} = \frac{3\overrightarrow{OA} + 2\overrightarrow{OE}}{5}$$

$$\overrightarrow{OE} = \frac{5\overrightarrow{OB} - 3\overrightarrow{OA}}{2}$$

$$= \frac{1}{2}\left[5\begin{pmatrix}1\\0\\5\end{pmatrix} - 3\begin{pmatrix}0\\-8\\-1\end{pmatrix}\right]$$

$$= \begin{pmatrix}2.5\\12\\14\end{pmatrix}$$



Useful GC Tip:
Use F3 to create 3 rows and 1 column to represent column vector to compute tedious operations.

5 **2009(9740)/II/2**

Relative to the origin *O*, two points *A* and *B* have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(i) The point *P* divides the line *AB* in the ratio $2:1$. Find the coordinates of *P*. [2]

(ii) Show that *AB* and *OP* are perpendicular. [2]

(iii) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector, and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$. [2]

(iv) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of triangle *OAP*. [4]

Q5	Suggested Solutions

(i)

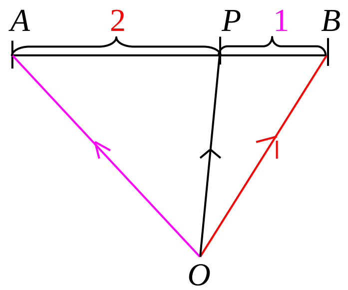
Learning Point:
Draw diagram to clearly show the ratio that divides *AB* internally before applying ratio theorem.

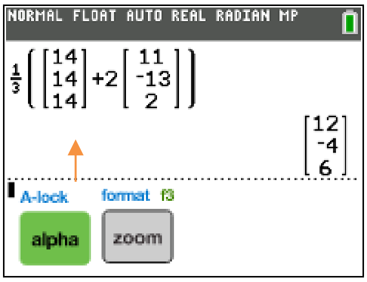
By Ratio Theorem,

$$\overrightarrow{OP} = \frac{\overrightarrow{OA} + 2\overrightarrow{OB}}{3}$$

$$= \frac{\begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} + 2\begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix}}{3}$$

$$= \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$$





Useful GC Tip:
Use F3 to create 3 rows and 1 column to represent column vector to compute tedious operations.

Coordinates of *P* are (12, −4, 6).

Remember to leave answer as coordinate form as stated in question.

(ii)

Good practice to express $\overrightarrow{AB}$ with reference to position vectors

$$\overrightarrow{OP} \bullet \overrightarrow{AB} = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \bullet \left[\begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} - \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} \right]$$

Since this is a showing question, working must be shown clearly. Do not skip steps.

$$= \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ -27 \\ -12 \end{pmatrix}$$

$$= -36 + 108 - 72 = 0$$

Thus AB and OP are perpendicular.

Learning Point:
$\mathbf{a}$ is perpendicular to $\mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$

(iii)

$$\mathbf{c} = \frac{1}{\sqrt{12^2 + (-4)^2 + 6^2}} \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$$

$$= \frac{1}{14} \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$$

Recall:
Unit vector $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$

Note:
$\mathbf{c}$ is a unit vector.

$|\mathbf{a.c}|$ is the length of projection of **a** onto **c**.

(iv)

$$\mathbf{a} \times \mathbf{p} = \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} \times \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 140 \\ 84 \\ -224 \end{pmatrix}$$

Recall:
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$$ **[in MF27]**

$|\mathbf{a} \times \mathbf{p}|$ is the area of the parallelogram in which OA and OP are a pair of adjacent sides.

Area of triangle $OAP = \frac{1}{2}|\mathbf{a} \times \mathbf{p}|$

$$= \frac{1}{2} \left| \begin{pmatrix} 140 \\ 84 \\ -224 \end{pmatrix} \right|$$

$$= \frac{1}{2}\sqrt{76832}$$

$$= 98\sqrt{2} \text{ square units}$$

6 **(a)** Find the unit vector in the direction of $-\mathbf{i}-3\mathbf{j}$.

(b) The vector $\mathbf{v}$ has a magnitude of 5 and is parallel to $\begin{pmatrix}2\\3\\-1\end{pmatrix}$.

Find the possible vector(s) $\mathbf{v}$.

(c) Find the vector $\mathbf{r}$ given that $r+j-2k$ is parallel to the x-axis and $\mathbf{r}-2\mathbf{i}$ is parallel to $-2\mathrm{j}+4\mathrm{k}$.

Q6	**Solution**
(a)	Unit vector in the direction of $-\mathbf{i}-3\mathbf{j}$ $=\frac{1}{\sqrt{1^2+3^2}}\begin{pmatrix}-1\\-3\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}-1\\-3\end{pmatrix}=\frac{\sqrt{10}}{10}\begin{pmatrix}-1\\-3\end{pmatrix}$ or $=\frac{1}{\sqrt{1^2+3^2}}\begin{pmatrix}-1\\-3\\0\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}-1\\-3\\0\end{pmatrix}=\frac{\sqrt{10}}{10}\begin{pmatrix}-1\\-3\\0\end{pmatrix}$
(b)	Unit vector parallel to $\begin{pmatrix}2\\3\\-1\end{pmatrix}=\frac{\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$ or $-\frac{\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$ $\therefore \mathbf{v}=\frac{5\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$ or $-\frac{5\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$
(c)	**<u>Method 1</u>** $\mathbf{r}+\mathbf{j}-2\mathbf{k}$ parallel to x-axis $\Rightarrow \mathbf{r}+\begin{pmatrix}0\\1\\-2\end{pmatrix}=\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ for some $\lambda\in\mathbb{R}\setminus\{0\}$ $\mathbf{r}=\begin{pmatrix}\lambda\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\-2\end{pmatrix}=\begin{pmatrix}\lambda\\-1\\2\end{pmatrix}$ ---- (1) $\mathbf{r}-2\mathbf{i}$ parallel to $-2\mathbf{j}+4\mathbf{k}$ $\Rightarrow \mathbf{r}+\begin{pmatrix}-2\\0\\0\end{pmatrix}=\mu\begin{pmatrix}0\\-2\\4\end{pmatrix}$ for some $\mu\in\mathbb{R}\setminus\{0\}$ $\mathbf{r}=\begin{pmatrix}0\\-2\mu\\4\mu\end{pmatrix}+\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}2\\-2\mu\\4\mu\end{pmatrix}$ ---- (2) Equating (1) and (2). $\begin{pmatrix}\lambda\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\-2\mu\\4\mu\end{pmatrix}$

$\therefore \lambda = 2$ and $\mu = \dfrac{1}{2}$

$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

<u>Method 2</u>

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\mathbf{r} + \mathbf{j} - 2\mathbf{k}$ parallel to x-axis $\Rightarrow \begin{pmatrix} x \\ y+1 \\ z-2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$

$\Rightarrow y = -1,\ z = 2$ and $x = \lambda$

$\mathbf{r} - 2\mathbf{i}$ parallel to $-2\mathbf{j} + 4\mathbf{k}$ $\Rightarrow \begin{pmatrix} x-2 \\ y \\ z \end{pmatrix} = \mu \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ for some $\mu \in \mathbb{R} \setminus \{0\}$

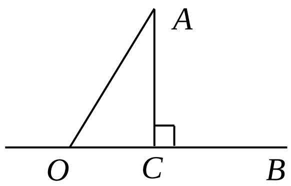
$\Rightarrow x = 2$

$\therefore \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

7 Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ respectively.

(i) Find the length of the projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$.

(ii) Hence, or otherwise, find the position vector of the point C on OB such that AC is perpendicular to OB.

Q7	Solutions
(i)	Length of the projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ $= \left\| \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \cdot \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} \right\| = \frac{20}{\sqrt{50}} = 2\sqrt{2}$
(ii)	**Method 1:**  From **(i)**, $\left\|\overrightarrow{OC}\right\| = 2\sqrt{2}$ $\overrightarrow{OC} = 2\sqrt{2}\left[\frac{1}{\sqrt{50}}\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}\right] = \frac{2}{5}\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ **Method 2:** Since OC is parallel to OB: $\overrightarrow{OC} = \lambda\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \lambda\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5\lambda - 3 \\ -4\lambda - 1 \\ 3\lambda - 3 \end{pmatrix}$ Since $\overrightarrow{AC} \perp \overrightarrow{OB}$, $\begin{pmatrix} 5\lambda - 3 \\ -4\lambda - 1 \\ 3\lambda - 3 \end{pmatrix} . \begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} = 0$ $(5\lambda - 3)(5) + (-4\lambda - 1)(-4) + (3\lambda - 3)(3) = 0$ $50\lambda - 20 = 0$ $\lambda = \frac{2}{5}$ $\overrightarrow{OC} = \frac{2}{5}\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix}$

8 $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 1$, and the angle between them is $\frac{5\pi}{6}$.

(a) By considering $(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$, find the exact value of $|2\mathbf{a}+\mathbf{b}|$.

(b) Find

(i) the length of projection of **a** onto **b**,

(ii) the projection vector of **a** onto **b**.

Q8	Solution

(a)

Useful technique of applying the concept $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

Properties of scalar product:

$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$

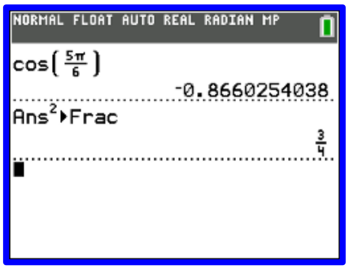
$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$

$\lambda\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\lambda\mathbf{b} = \lambda(\mathbf{a}\cdot\mathbf{b})$

$$(2\mathbf{a}+\mathbf{b})\bullet(2\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\bullet 2\mathbf{a} + 2\mathbf{a}\bullet\mathbf{b} + \mathbf{b}\bullet 2\mathbf{a} + \mathbf{b}\bullet\mathbf{b}$$

$$= 4\mathbf{a}\bullet\mathbf{a} + 4\mathbf{a}\bullet\mathbf{b} + |\mathbf{b}|^2$$

$$= 4|\mathbf{a}|^2 + 4|\mathbf{a}||\mathbf{b}|\cos\left(\frac{5\pi}{6}\right) + |\mathbf{b}|^2$$



$$\left[\cos\left(\frac{5\pi}{6}\right)\right]^2 = \frac{3}{4}$$

$$\cos\left(\frac{5\pi}{6}\right) = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$$

$$= 4\left(\sqrt{3}\right)^2 + 4\left(\sqrt{3}\right)(1)\left(-\frac{\sqrt{3}}{2}\right) + 1^2$$

$$= 7$$

Since $(2\mathbf{a}+\mathbf{b})\bullet(2\mathbf{a}+\mathbf{b}) = |2\mathbf{a}+\mathbf{b}|^2$

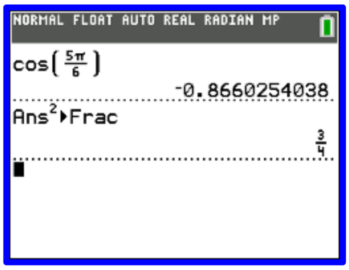
$\therefore\ |2\mathbf{a}+\mathbf{b}|^2 = 7 \quad\Rightarrow\quad |2\mathbf{a}+\mathbf{b}| = \sqrt{7}$

(b) (i)

Length of projection of **a** onto **b**

$$= \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right| = \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right|$$

$$= \left|\frac{|\mathbf{a}||\mathbf{b}|\cos\frac{5\pi}{6}}{|\mathbf{b}|}\right|$$



$$\left[\cos\left(\frac{5\pi}{6}\right)\right]^2 = \frac{3}{4}$$

$$\cos\left(\frac{5\pi}{6}\right) = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$$

$$= \left|\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)\right|$$

$$= \left|-\frac{3}{2}\right| = \frac{3}{2} \text{ units}$$

(b) **(ii)**	Projection vector of **a** onto **b** $=\left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}=\left(-\frac{3}{2}\right)\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$ $=-\frac{3}{2}\mathbf{b}\quad(\text{since }\lvert\mathbf{b}\rvert=1)$

Take note that there is no modulus sign when computing the projection vector

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(i) Given that $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, what can be deduced about the vectors **a** and **b**? [2]

(ii) Find a unit vector **n** such that $\mathbf{n}\times\left(\mathbf{i}+2\mathbf{j}-2\mathbf{k}\right)=\mathbf{0}$. [2]

(iii) Find the cosine of the acute angle between $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and the *z*-axis. [1]

Q9	**Suggested Solutions**
(i)	Since $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, **a** or **b** are zero vectors or **a** and **b** are parallel to each other. **<u>Alternative method</u>** $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ $\lvert\mathbf{a}\times\mathbf{b}\rvert=0$ $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta=0$ $\therefore\lvert\mathbf{a}\rvert=0,\ \lvert\mathbf{b}\rvert=0\ \text{ or }\ \sin\theta=0\Rightarrow\theta=0^\circ\ \text{ or }180^\circ$ Hence, either **a** or **b** or both are zero vectors or **a** and **b** are parallel to each other.
(ii)	Since **n** is not a zero vector, **n** must be parallel to $\begin{pmatrix}1\\2\\-2\end{pmatrix}$. Learning point: $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ $\Rightarrow$ **a** is parallel to **b** **OR** $\mathbf{a}=\mathbf{0}$ **OR** $\mathbf{b}=\mathbf{0}$ $\left\lvert\begin{pmatrix}1\\2\\-2\end{pmatrix}\right\rvert=\sqrt{1^2+2^2+(-2)^2}=3$ Recall: Unit vector $\hat{\mathbf{r}}=\frac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$ $\therefore\mathbf{n}=\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}\ \text{or}\ -\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$
(iii)	Let the required acute angle be θ. Since the angle required is acute, we can include a modulus sign Vector parallel to *z*-axis $\cos\theta=\frac{\left\lvert\begin{pmatrix}1\\2\\-2\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}1\\2\\-2\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}$ $=\frac{\lvert-2\rvert}{3}=\frac{2}{3}$ Recall: $\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ where θ is the angle between the two non-zero vectors **a** and **b**

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Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}.$$

The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ where λ and μ are positive constants.

(i) Given that the area of triangle OAC is $\sqrt{126}$, find μ. [4]

(ii) Given instead that $\mu = 4$ and that $OC = 5\sqrt{3}$, find the possible coordinates of C. [4]

Q10	**Solutions**
(i)	$\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\overrightarrow{OC} = \lambda\mathbf{a} + \boldsymbol{\mu}b = \begin{pmatrix}\lambda+\mu\\-\lambda+2\mu\\\lambda\end{pmatrix}$ Area of triangle OAC = $\sqrt{126}$ $\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OC}\right\rvert = \sqrt{126}$ $\left\lvert\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}\lambda+\mu\\-\lambda+2\mu\\\lambda\end{pmatrix}\right\rvert = 2\sqrt{126}$ Recall: $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\-(a_1b_3 - a_3b_1)\\a_1b_2 - a_2b_1\end{pmatrix}$ **[in MF27]** $\left\lvert\begin{pmatrix}-2\mu\\\mu\\3\mu\end{pmatrix}\right\rvert = 2\sqrt{126}$ $\mu\left\lvert\begin{pmatrix}-2\\1\\3\end{pmatrix}\right\rvert = 2\sqrt{126}$ $\mu\sqrt{4+1+9} = 2\sqrt{126}$ $14\mu^2 = 504$ $\mu^2 = 36$ $\mu = 6$ since $\mu > 0$

Alternatively (Applying properties of vector product)

$$\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OC}\right| = \sqrt{126}$$

$$|\mathbf{a}\times\mathbf{c}| = 2\sqrt{126}$$

$$|\mathbf{a}\times(\lambda\mathbf{a}+\mu\mathbf{b})| = 2\sqrt{126}$$

$$|\mathbf{a}\times\lambda\mathbf{a}+\mathbf{a}\times\mu\mathbf{b}| = 2\sqrt{126}$$

$$|\mathbf{a}\times\mu\mathbf{b}| = 2\sqrt{126}$$

Properties of vector product:
1. $\mathbf{a}\times(\mathbf{c}+\mathbf{b}) = \mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}$
2. $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ (zero vector)
3. $k\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times k\mathbf{a}$

$$\mu\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}\right| = 2\sqrt{126}$$

Recall:
$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$$ **[in MF27]**

$$\mu\left|\begin{pmatrix}-2\\1\\3\end{pmatrix}\right| = 2\sqrt{126}$$

$$\mu\sqrt{4+1+9} = 2\sqrt{126}$$

$$14\mu^2 = 504$$

$$\mu^2 = 36$$

$$\mu = 6 \text{ since } \mu > 0$$

(ii)

Given $\mu = 4$,

$$OC = 5\sqrt{3}$$

$$\left|\overrightarrow{OC}\right| = \lambda\mathbf{a}+4\mathbf{b} = \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}+4\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}\lambda+4\\-\lambda+8\\\lambda\end{pmatrix}$$

$$\left|\begin{pmatrix}\lambda+4\\-\lambda+8)\\\lambda\end{pmatrix}\right| = 5\sqrt{3}$$

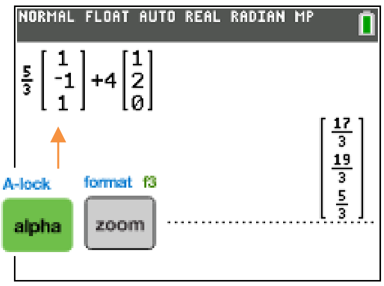
$$\sqrt{(\lambda+4)^2+(8-\lambda)^2+\lambda^2} = 5\sqrt{3}$$

$$3\lambda^2-8\lambda+80 = 75$$

$$3\lambda^2-8\lambda+5 = 0$$

$$(3\lambda-5)(\lambda-1) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = \frac{5}{3}$$



Useful GC Tip:
Use F3 to create 3 rows and 1 column to represent column vector to compute tedious operations.

$$\therefore C(5,7,1) \quad \text{or} \quad C\left(\frac{17}{3},\frac{19}{3},\frac{5}{3}\right)$$

Level 3

11 The points *A*, *B* and *C* have position vectors **a**, **b** and $\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ respectively. The point *P* on *AB* is such that $AP:PB=\lambda:1-\lambda$ and the point *P* on *OC* is such that $OP:PC=\mu:1-\mu$.

(i) Express $\overrightarrow{OP}$ in terms of λ, **a** and **b**.

(ii) By expressing $\overrightarrow{OP}$ in terms of μ, **a** and **b**, find the values of λ and μ. Hence show that P is the midpoint of OC.

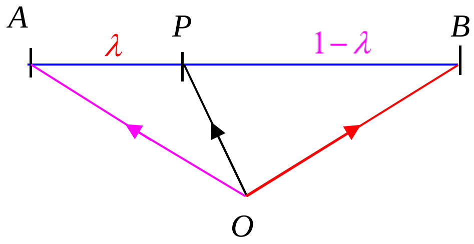
(iii) It is given that the position vectors of the points *A* and *B* are $2\mathbf{j}+\mathbf{k}$ and $12\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ respectively. The point *Q* lies on *OA* such that *PQ* is perpendicular to *OA*. Find the position vector of the point *Q*.

Q11	Solution

(i) Given $AP:PB=\lambda:1-\lambda$

> Learning Point:
> Draw diagram to clearly show the ratio that divides *AB* internally before applying ratio theorem.

By ratio theorem,



$$\overrightarrow{OP}=\frac{(1-\lambda)\overrightarrow{OA}+\lambda\overrightarrow{OB}}{1-\lambda+\lambda}$$

$$\overrightarrow{OP}=(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$$

(ii) Given $OP:PC=\mu:1-\mu$

> Learning Point:
> Draw diagram to clearly show the ratio that divides *OC* internally

$$\overrightarrow{OP}=\mu\overrightarrow{OC}=\mu\left(\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$$

> Answer from part **(i)**

$$\therefore(1-\lambda)\mathbf{a}+\lambda\mathbf{b}=\mu\left(\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$$

> **Useful technique**: Solving 2 unknowns in a given equation involving vectors.

Since **a** and **b** are non-zero and non-parallel vectors,

$$1-\lambda=\frac{3}{2}\mu \quad \text{------ (1)}$$

$$\lambda=\frac{1}{2}\mu \quad \text{------ (2)}$$

> Learning Point:
> If **a** and **b** are non-zero and non-parallel vectors and $\lambda\,a=\mu\,b$ for some λ, $\mu\in\mathbb{R}$, then $\lambda=\mu=0$.

Solving (1) & (2)

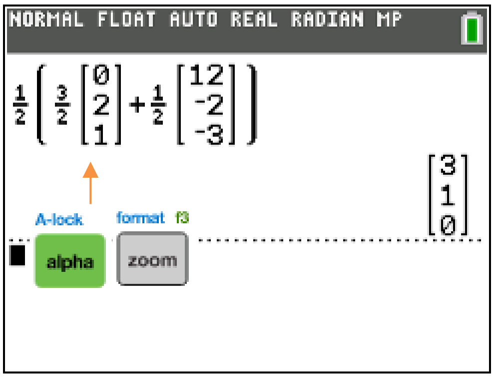
$\lambda=\frac{1}{4}$ and $\mu=\frac{1}{2}$

Since $\overrightarrow{OP}=\mu\overrightarrow{OC}=\frac{1}{2}\overrightarrow{OC}$, hence *P* is the midpoint of *OC*.

(iii)

$\overrightarrow{OA}=\begin{pmatrix}0\\2\\1\end{pmatrix}$ $\overrightarrow{OB}=\begin{pmatrix}12\\-2\\-3\end{pmatrix}$

$$\overrightarrow{OP}=\frac{1}{2}\overrightarrow{OC}=\frac{1}{2}\left(\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)=\frac{1}{2}\left[\frac{3}{2}\begin{pmatrix}0\\2\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}12\\-2\\-3\end{pmatrix}\right]=\begin{pmatrix}3\\1\\0\end{pmatrix}$$



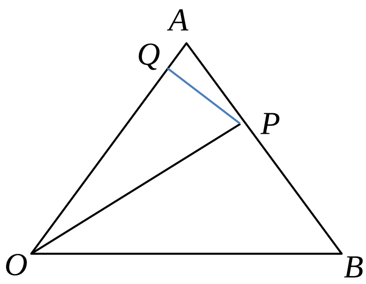
Useful GC Tip:
Use F3 to create 3 rows and 1 column to represent column vector to compute tedious operations.

Learning Point:
If a point lies on a line, the position vector of that point will satisfy the equation of the line (Chapter 6).

Since Q lies on OA, $\overrightarrow{OQ}=\alpha\begin{pmatrix}0\\2\\1\end{pmatrix}$ for some $\alpha\in\mathbb{R}$

$\overrightarrow{PQ}=\alpha\begin{pmatrix}0\\2\\1\end{pmatrix}-\begin{pmatrix}3\\1\\0\end{pmatrix}$ for some $\alpha\in\mathbb{R}$

Good practice to express $\overrightarrow{PQ}$ with reference to position vectors



Learning Point:
Draw simple diagram to clearly show the relationship.

$\overrightarrow{PQ}\bullet\overrightarrow{OA}=0$

Learning Point:
$\mathbf{a}$ is perpendicular to $\mathbf{b}\Leftrightarrow\mathbf{a}\cdot\mathbf{b}=0$

	$\left[\alpha\begin{pmatrix}0\\2\\1\end{pmatrix}-\begin{pmatrix}3\\1\\0\end{pmatrix}\right]\bullet\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $\begin{pmatrix}-3\\2\alpha-1\\\alpha\end{pmatrix}\bullet\begin{pmatrix}0\\2\\1\end{pmatrix}=0$ $\boxed{\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3}$ $4\alpha-2+\alpha=0$ $\alpha=\frac{2}{5}$ $\overrightarrow{OQ}=\frac{2}{5}\begin{pmatrix}0\\2\\1\end{pmatrix}$

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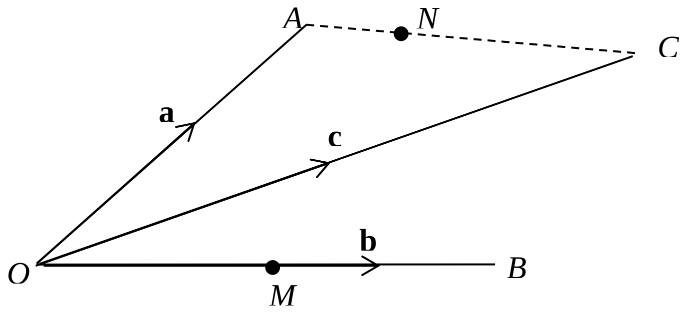


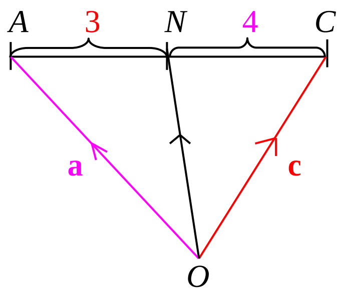
The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ (see diagram).

(i) Explain why c can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ. [1]

The point N is on AC such that $AN : NC = 3:4$.

(ii) Write down the position vector of N in terms of **a** and **c**. [1]

(iii) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms **a** and **b**, find λ in terms of μ in the case where λ and μ are both positive. [5]

Q12	**Suggested Solutions**
(i)	Since O,A,B,C lie on the same plane, $\overrightarrow{OC}$ can be expressed as a sum of a scalar multiple of **a** and a scalar multiple of **b**. **Alternatively,** The equation of plane OAB: $\mathbf{r} = \mathbf{0} + \lambda\mathbf{a} + \mu\mathbf{b}$, $\lambda, \mu \in \mathbb{R}$. Since C lies on plane OAB, $\overrightarrow{OC}$ satisfies the equation of the plane, i.e. $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for some $\lambda, \mu \in \mathbb{R}$
(ii)	Given $AN : NC = 3:4$. Learning Point: Draw diagram to clearly show the ratio that divides AC internally before applying ratio theorem. Using ratio theorem, $\overrightarrow{ON} = \dfrac{3\mathbf{c} + 4\mathbf{a}}{7}$ 

(iii)	Since M is the mid-point of OB, $\overrightarrow{OM} = \frac{1}{2}\mathbf{b}$ Area of ΔONC = Area of ΔOMC $\frac{1}{2}\left\|\overrightarrow{ON}\times\overrightarrow{OC}\right\| = \frac{1}{2}\left\|\overrightarrow{OC}\times\overrightarrow{OM}\right\|$ $\left\|\frac{3\mathbf{c}+4\mathbf{a}}{7}\times\mathbf{c}\right\| = \left\|\frac{1}{2}\mathbf{c}\times\mathbf{b}\right\|$ $\left\|\frac{3}{7}\mathbf{c}\times\mathbf{c}+\frac{4}{7}\mathbf{a}\times\mathbf{c}\right\| = \left\|\frac{1}{2}\mathbf{c}\times\mathbf{b}\right\|$ $\left\|\frac{4}{7}\mathbf{a}\times\mathbf{c}\right\| = \left\|\frac{1}{2}\mathbf{c}\times\mathbf{b}\right\|$ Recall: Area of $\Delta ABC = \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$ Properties of vector product: 1. $\mathbf{a}\times(\mathbf{c}+\mathbf{b}) = \mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}$ 2. $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ (zero vector) 3. $k\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times k\mathbf{a}$ Hint from question: "By finding the areas of these triangles in terms **a** and **b**" $\Rightarrow$ replace $\mathbf{c} = \lambda\mathbf{a}+\mu\mathbf{b}$ $\left\|\frac{4}{7}\mathbf{a}\times(\lambda\mathbf{a}+\mu\mathbf{b})\right\| = \left\|\frac{1}{2}(\lambda\mathbf{a}+\mu\mathbf{b})\times\mathbf{b}\right\|$ $\left\|\frac{4}{7}\lambda\mathbf{a}\times\mathbf{a}+\frac{4}{7}\mu\mathbf{a}\times\mathbf{b}\right\| = \left\|\frac{1}{2}\lambda\mathbf{a}\times\mathbf{b}+\frac{1}{2}\mu\mathbf{b}\times\mathbf{b}\right\|$ $\left\|\frac{4}{7}\mu\mathbf{a}\times\mathbf{b}\right\| = \left\|\frac{1}{2}\lambda\mathbf{a}\times\mathbf{b}\right\|$ Properties of vector product: 1. $\mathbf{a}\times(\mathbf{c}+\mathbf{b}) = \mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}$ 2. $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ (zero vector) 3. $k\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times k\mathbf{a}$ Since $\mu, \lambda > 0$, $\frac{4}{7}\mu = \frac{1}{2}\lambda \Rightarrow \lambda = \frac{8}{7}\mu$

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The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ in terms of a and b. [2]

(ii) Given that the **i**- and **k**-components of the answer to part **(i)** are equal, express $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

Q13	Suggested Solutions
(i)	$\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ $\quad \mathbf{v} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$ $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = \mathbf{u}\times\mathbf{u} - \mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{u} - \mathbf{v}\times\mathbf{v}$ $= 2(\mathbf{v}\times\mathbf{u})$ $= 2\left(\begin{pmatrix} a \\ 0 \\ b \end{pmatrix}\times\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}\right)$ $= 2\begin{pmatrix} b \\ 2b-2a \\ -a \end{pmatrix}$ Properties of vector product: 1. $\mathbf{a}\times(\mathbf{c}+\mathbf{b}) = \mathbf{a}\times\mathbf{c} + \mathbf{a}\times\mathbf{b}$ 2. $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ (zero vector) 3. $k\mathbf{a}\times\mathbf{c} = -\mathbf{c}\times k\mathbf{a}$ Recall: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\times\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$ **[in MF27]**
(ii)	Given $b=-a$, $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = 2\begin{pmatrix} -a \\ 2(-a)-2a \\ -a \end{pmatrix} = -2a\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ $\left\| -2a\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \right\| = 1$ Unit vector: Vector of magnitude 1. $\|-2a\|\sqrt{18} = 1$ $2a\sqrt{18} = \pm 1$ $a = \pm\frac{1}{6\sqrt{2}}$ Rationalise the denominator. $\therefore a = \frac{\sqrt{2}}{12}$ or $a = -\frac{\sqrt{2}}{12}$ Hint in question: '*possible values of a*', hence there should be more than 1 answer.

(iii)	$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=0$ $\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}=0$ $\lvert\mathbf{u}\rvert^2-\lvert\mathbf{v}\rvert^2=0$ $\lvert\mathbf{v}\rvert=\lvert\mathbf{u}\rvert$ since $\lvert\mathbf{v}\rvert\geq 0$ $\lvert\mathbf{v}\rvert=\left\lvert\begin{pmatrix}2\\-1\\2\end{pmatrix}\right\rvert$ $=\sqrt{2^2+(-1)^2+2^2}$ $=3$ Properties of scalar product: $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2$

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Vectors **a**, **b** and **c** are such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$.

(i) Show that $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is a constant. [2]

(ii) It is now given that **a** and **c** are unit vectors, that the modulus of **b** is 4 and that the angle between **b** and **c** is 60°. Using a suitable scalar product, find exactly the two possible values of λ. [5]

14	**Solution**
(i)	$\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$ **Useful technique**: Shift all terms to one side so that a zero vector is "created" on the other side. $\mathbf{a} \times 3\mathbf{b} - 2\mathbf{a} \times \mathbf{c} = \mathbf{0}$ $\mathbf{a} \times (3\mathbf{b} - 2\mathbf{c}) = \mathbf{0}$ $\mathbf{0}$ is the zero **vector**. Learning point: Vector (Cross) product gives a vector. Learning point: $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ $\Rightarrow$ **a** is parallel to **b** **OR** $\mathbf{a} = \mathbf{0}$ **OR** $\mathbf{b} = \mathbf{0}$ $3\mathbf{b} - 2\mathbf{c} = \mathbf{0}$ or $3\mathbf{b} - 2\mathbf{c}$ is parallel to **a** $\therefore 3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a},\ \lambda \in \mathbb{R}$.
(ii)	We are given that **a** and **c** are unit vectors, that the modulus of **b** is 4 and that the angle between **b** and **c** is 60°. Meaning: $\lvert\mathbf{a}\rvert = 1,\ \lvert\mathbf{c}\rvert = 1,\ \lvert\mathbf{b}\rvert = 4$ and $\mathbf{b}\bullet\mathbf{c} = \lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\cos 60^0$ $\because\ 3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$ Recall: $\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ where θ is the angle between the two non-zero vectors **a** and **b** **Useful technique** of applying the concept $\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2$ $(3\mathbf{b} - 2\mathbf{c})\bullet(3\mathbf{b} - 2\mathbf{c}) = \lambda\mathbf{a}\bullet\lambda\mathbf{a}$ $9\mathbf{b}\bullet\mathbf{b} - 6\mathbf{b}\bullet\mathbf{c} - 6\mathbf{c}\bullet\mathbf{b} + 4\mathbf{c}\bullet\mathbf{c} = \lambda^2\mathbf{a}\bullet\mathbf{a}$ $9\lvert\mathbf{b}\rvert^2 - 12\mathbf{b}\bullet\mathbf{c} + 4\lvert\mathbf{c}\rvert^2 = \lambda^2\lvert\mathbf{a}\rvert^2$ $9\lvert\mathbf{b}\rvert^2 - 12\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\cos 60^0 + 4\lvert\mathbf{c}\rvert^2 = \lambda^2\lvert\mathbf{a}\rvert^2$ $9(4)^2 - 12(4)(1)\cos 60^0 + 4(1)^2 = \lambda^2(1)^2$ $144 - 48\cos 60^\circ + 4 = \lambda^2$ $\lambda^2 = 124$ $\lambda = \pm 2\sqrt{31}$ $\therefore \lambda = -2\sqrt{31}$ or $\lambda = 2\sqrt{31}$. Properties of scalar product: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ $\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2$