Preliminary Examination Paper 1
Three Musketeers Exam (final)

Time: 2 hours 15 minutes

Name: ....

Marks: **90**

(Paper 1 Question Paper)

Topics: the triple threat

(*trigonometry*, *differentiation*, *integration*)

Pages: **30**

READ THE INSTRUCTIONS FIRST

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question. The use of an approved scientific calculator is expected, where appropriate. You are reminded of the need for clear presentation in your answers.

90

Formula List:

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

To all viewers (and holy grail moderators):
setter of this paper is a sec 4 student who has taken a deep and fond interest in my beloved, amath
1) thank you holy grail mods for removing the previous copy!
2) this paper, if you are using it as practice, is much more difficult from the regular question types. in my opinion, amath questions has gotten more vanilla and less interesting once you start to try more papers, hence this paper showcases how some amath questions can truly be very difficult
this amath paper solely consists of the three big chapters, and it encourages critical thinking
3) note that the heading should be 4049/1 instead of 4047/1
4) similarly, marks might not be given as fairly in this paper as in normal papers
5) if you really want to do it timed, I suggest giving more time (about 15 minutes) to balance the fairness of this paper
inspiration for some qns:
nchs, tkss, bbss, blss, nhhs

Answer **all** questions :)

Do not use a calculator for the whole of this question.

1 (a) Find the value of $\cot\left[\,2\cos^{-1}\left(-\frac{4}{5}\right)\right]$. [2]

(b) It is given that $\cot(-B) < 0$ and $\sec(-B) < 0$, where $0 \le B \le 2\pi$.

(i) Explain why $\sin\frac{B}{2} > 0$. [1]

(ii) Given that $\cot(-B) = -\frac{12}{5}$, find the exact value of $\cos\left(\frac{B-\pi}{2}\right)$. [2]

(c) **(i)** Find the exact value of $\operatorname{cosec}^2\left(\frac{7\pi}{12}\right)$. [2]

(ii) Hence, find $\sec^2\left(\frac{7\pi}{12}\right)$. [1]

2 **(a)** Show, using the aid of a diagram, that $\int_{-k}^{k} 3\sqrt{k^2 - x^2}\ \mathrm{d}x = \frac{3\pi k^2}{2}$, for $k > 0$. [2]

(b) Hence, find in terms of k and/or π, the value of:

(i) $\int_{k}^{0} \sqrt{4k^2 - 4x^2} - 2\pi \cos\left(\frac{\pi x}{2k}\right) \mathrm{d}x$. [2]

(ii) $\int_{0}^{-k} \sqrt{k^2 - x^2} + \pi \tan^2\left(\frac{\pi x}{3k}\right) \mathrm{d}x$. [2]

3* **(a)** **(i)** Prove that $\dfrac{1-\cos\theta+\sin\theta}{1+\cos\theta+\sin\theta}=\tan\dfrac{\theta}{2}$. [1]

(ii) Prove that $\dfrac{4\sin\theta+4\sin^2\theta}{\sec\theta+\tan\theta}=2\sin 2\theta$. [1]

(b) Hence, solve $\dfrac{\sin 3\theta+\sin^2 3\theta}{\sec 3\theta+\tan 3\theta}=\dfrac{1+\cos 12\theta+\sin 12\theta}{1-\cos 12\theta+\sin 12\theta}$, for $-1\le\theta\le 1$. [3]

4 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x} x^6 \ln x^4 = ax^5 \ln x^3 + bx^5$, where $x > 0$, and a and b are constants to be found. [1]

(ii) **Only by expressing** $\int ax^5 \ln x^3 + bx^5 \, \mathrm{d}x = x^6 \ln x^4 + c_1$, where c_1 is an arbitrary constant, evaluate $\int x^5 \ln x^3 \, \mathrm{d}x$. [2]

(iii) Look at Diagram **4** below. The shaded area can be expressed in two different ways.
One way in expressing the shaded area is first making x the subject, then expressing it as a bound integral below:

$\int_n^m x \, dy$, where m and n are constants.

Find the values of m and n, and the shaded area below. [2]

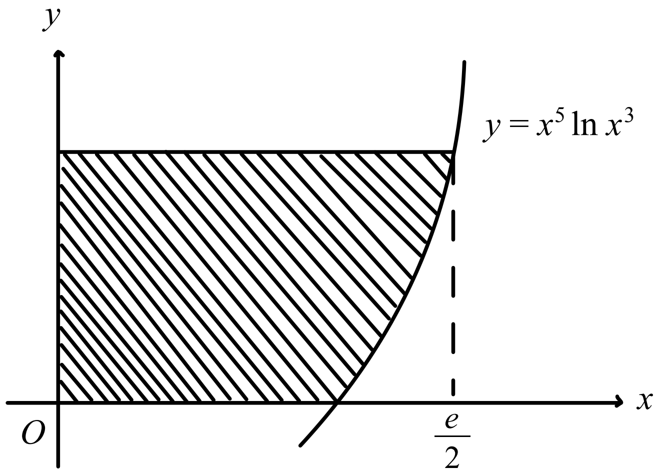


Diagram **4**

(iv) It is given that $\frac{dy}{dx} = \frac{k}{4x^2 - 9}$. At $x = -\frac{1}{2}$, the equation of the normal is parallel to $8y - 9x = 24$, and at $x = 2$, $y = k$. Form an equation for y. [3]

5 It is given that $f''(x) = 10\sin^2 3x - 6\tan^2 6x - 11$.

(i) Given that the normal of the line $y = f(x)$ at $x = 0$ is perpendicular to the x-axis, find and simplify an expression for $f'(x)$. [3]

(ii) Find an expression for $\frac{d}{dx}\ln(2 - 2\sin^2 nx)$, where n is a constant. [1]

(iii) Hence, given that $y = \mathrm{f}(x)$ intersects the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, find an expression for $\mathrm{f}(x)$. [3]

(iv) In the case where $y = \mathrm{f}'(x)$, the gradient of y at $x = 0$ is equal to the minimum gradient of $y = ax^3 + 3x^2 - 5x + 2$, $a \neq 0$. Find a. [1]

6 **(a)** It is given that $f(x) = \frac{2}{e^{kx}} - 3e^{3x} + e^{\ln x}$.

If $-3f'(x) + f''(x) + 3 = 0$, find the exact value(s) of k. [2]

(b) The equation of a curve is given by $y = 3x^2 e^{-\frac{1}{2}x}$.

A point P lies on the curve such that the normal to the curve at P has a negative gradient.

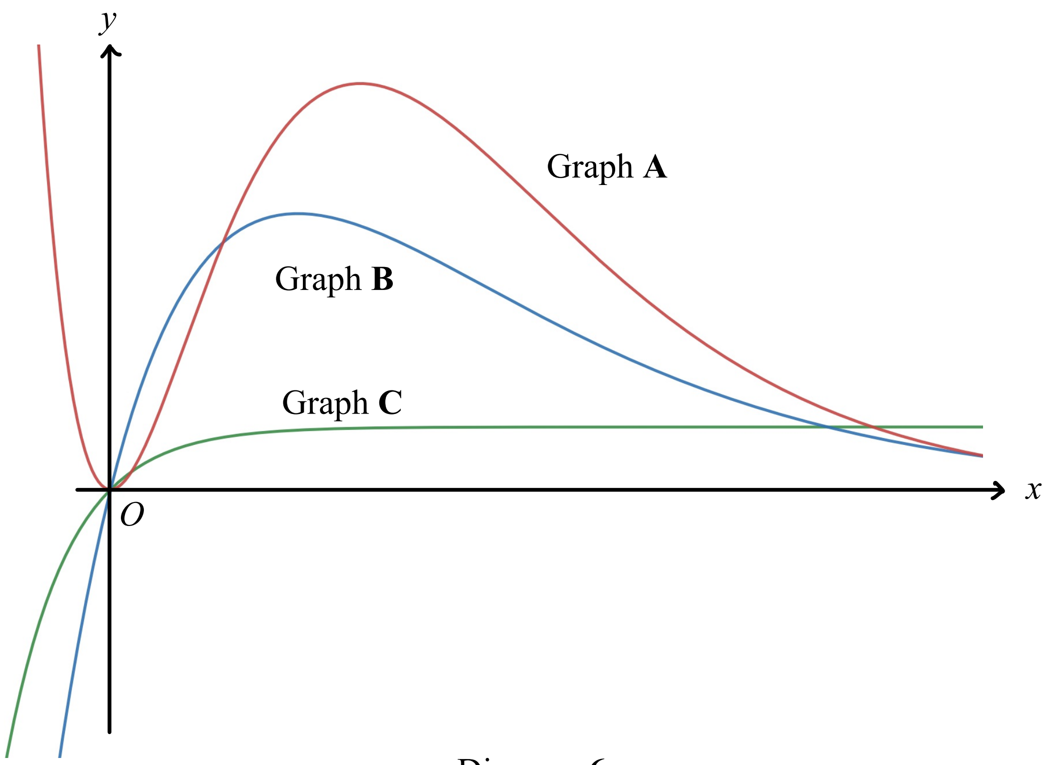
(i) State the range of values of the possible x coordinates of P. [2]

(ii) Find the equation of the normal at P, given that the x coordinate of P is 1. [2]

(iii) Look at the diagram below. It shows three different graphs, Graph **A**, **B**, and **C**.

Determine whether which graph is most suitable for the shape of the line $y = 3x^2 e^{-\frac{1}{2}x}$.

Explain your answer with reasonings.



[2]

Diagram **6**

7 It is given that $y = \dfrac{2 \sin x}{\cos x - 1}$, for $0 < x < 2\pi$.

(i) Explain, with reasoning, whether y is increasing or decreasing. [2]

(ii) Find the range of values of x for which $\dfrac{dy}{dx}$ is increasing. [2]

8 In the diagram, a surveillance camera is mounted at a point S that is 7.5 m towards the left of point L_0. A lizard crawls from point L_0 along an upward course at a speed of 1.2 m/s. The surveillance camera tracks the motion of the lizard by panning upwards from the fixed point S.

(a) Find the rate of change of the angle that the surveillance camera makes with the lizard and the wall of the building when the lizard is 10 m from L_0. Give your answer in radians per second.

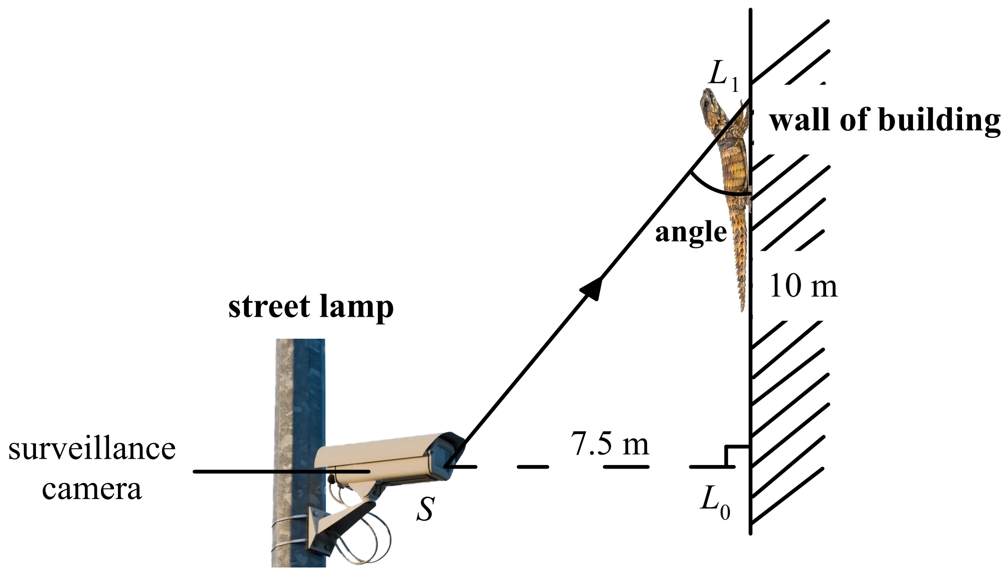


Diagram **8** [3]

(b) In another case, the speed of the lizard v m/s varies and is related to the the angle the surveillance camera makes with the lizard, x radians in the equation, with respect to t seconds:

$$v = k\frac{dx}{dt}$$

Show that $k < -7$. [1]

9 To study the effects of water waves, a wave generator and a rubber duck were placed in a water tank as shown in the diagram above. The height, h metres, from the bottom of water tank was modelled by $h = a \cos kt + b$, where t is the time in hours after midnight and a, b and k are constants. The motion of the rubber duck was observed for 60 hours.
The minimum height of 1.4 m from bottom of water tank was first recorded at midnight on Day 1.
The duck reaches minimum height again at 16 00 on Day 2.
After 60 hours, the duck reaches a height of 2.4 m.

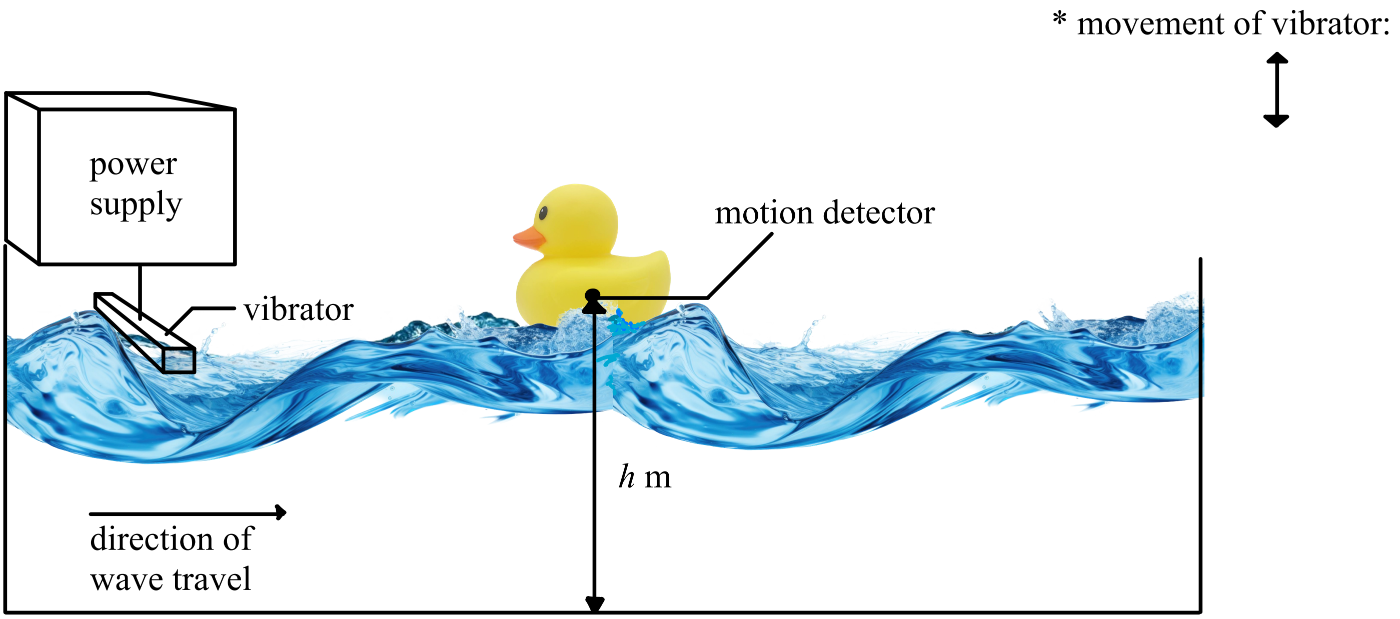


Diagram **9**

(i) Explaining each of your values, find a, b, and k. [3]

(ii) Hence, sketch the graph of $h = a \cos kt + b$, for $0 \le t \le 60$. [2]

(iii) How long does the rubber duck remain above 2.0 m during this period of 60 hours? Round your answer to the nearest minute. [2]

(iv) It is further given that after 60 hours, the operator adjusted the vibrator such that it oscillates up-and-down at a greater displacement from equilibrium and at a faster rate.
The graph starts from $t = 0$ again at the lowest point.
State and explain how the values of a, b, and k change, if at all. [1]

10 Look at the figure below. In the diagram, two circles with centres A and B and radii r cm and 5 cm, are right next to each other as shown. Lines CD and EP are tangents to both circles, and AEB is a right angle. $AP = r$ cm, $BC = 5$ cm, and $CD = 8$ cm.
Angle $EAP = \theta$, and θ is always an acute angle.

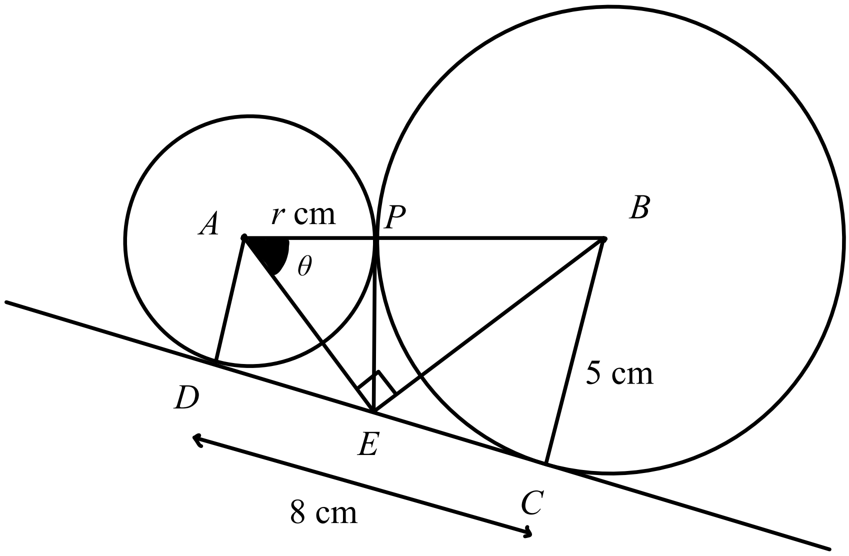


Diagram **10**

(i) By finding DE and CE in terms of r and/or θ,

show that the area of trapezium $ABCD$, $A = 4(\frac{8\tan\theta - 5}{\tan^2\theta} + 5)$ cm ^{2}. [2]

(ii) Hence, show that $\dfrac{dA}{d\theta} = \dfrac{4(-8\sin\theta + 10\cos\theta)}{\sin^3\theta}$.

Explain the significance of this expression. [3]

(iii) It is given that as θ varies, the value of $\frac{dA}{d\theta} = \frac{4}{\sin^3\theta}$.

By expressing the **numerator of the expression of** $\frac{dA}{d\theta}$ **from** (**ii**) in the form

$R\cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, solve for the area of the trapezium. [4]

11 The diagram below shows the values of the gradient function and its second derivative function of a real function $y = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a function not to be determined.

Value	**0^-**	**0**	**0^+**	**1^-**	**1**	**1^+**	**2^-**	**2**	**2^+**	**3^-**	**3**	**3^+**
Value of $\frac{dy}{dx}$	+	+	+	+	0	–	–	–	–	–	0	+
Value of $\frac{d^2y}{dx^2}$	–	–						–	–	+		

Legend:
'+' denotes **positive**
'–' denotes **negative**

Diagram **11**

(i) In Diagram 11, fill in the empty boxes with the correct symbol '+', '–', or 0. (not marked)
Using Diagram 11, identify the stationary point(s) of $y = \mathrm{f}(x)$ from $0 \le x \le 3.1$ and determine the nature of the stationary point(s). [1]

(ii) Determine the number of stationary point(s) of the graph $y = \mathrm{f}'(x)$, if any, from $0 \le x \le 3.1$ and determine the nature of the stationary point(s), if any. [1]

(Updated)

12* In the diagram, A and B are two fixed points on a horizontal ground and a projector L is positioned on the ground at L which is x m away from B. The projector casts a beam of light on a screen CD, of fixed height 5 m. C is the bottom of the screen, where $BC = 4$ m.
A partial virtual image is shown on the screen.
Angle CLD is $\alpha°$ and Angle BLC is $\beta°$. Assume that the thickness of the screen is negligible.

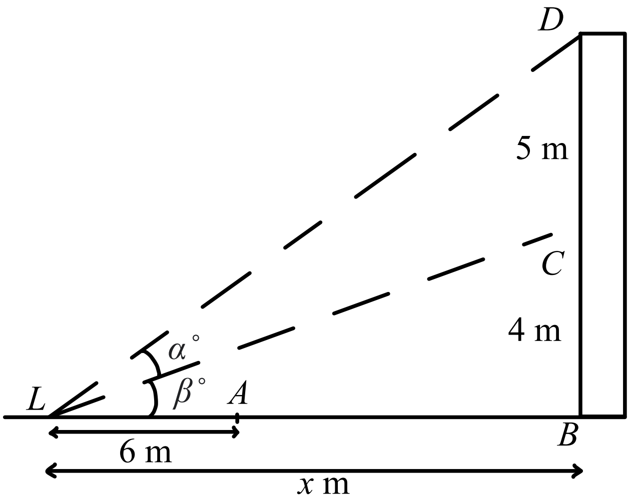


Diagram **12**

(i) Express $\tan \alpha°$ in terms of x. [2]

(ii) Due to numerous faults in projector L, projector L is replaced with projector M that has a much wider range of projection.
In order for projector M to cast its full image on a screen, the screen itself has to extend long enough in order to capture the full virtual image, and the image now reaches point B, the lowest point on the screen. The projector is able to cast its image, yet some of the image is still cut off and unavailable to view.
The angle the projector needs to cast such that a full virtual image is shown on the screen is given by acute $(\alpha + 2\beta)^\circ$.
Hence, the operator moves the projector in front by 6 m to position A, and now a full image is shown on the screen.
Find the value of x. [2]

(Updated)

13* **(a)** It is given that $x = \mathrm{f}(y)$ is drawn on a graph.

(i) Given that $\int_{a}^{2a} x \, \mathrm{d}y = 0$, what will this imply about the geometrical interpretation of the graph of $\mathrm{f}(y)$? [1]

(ii) Given further that $3x = -8 \sin^2 y \cos^2 y + 1$, what is the largest possible value of a given $a < 0$? [1]

(b) Look at the diagram below. It shows the graph of $y = 2 \cos x - 3 \cos (3x)$ plotted for $x > 0$.

Two vertical lines, $x = \dfrac{\pi}{3}$ and $x = \dfrac{5\pi}{6}$ are shown below.

Calculate the **exact** shaded area bounded by the two lines and the curve below. [4]

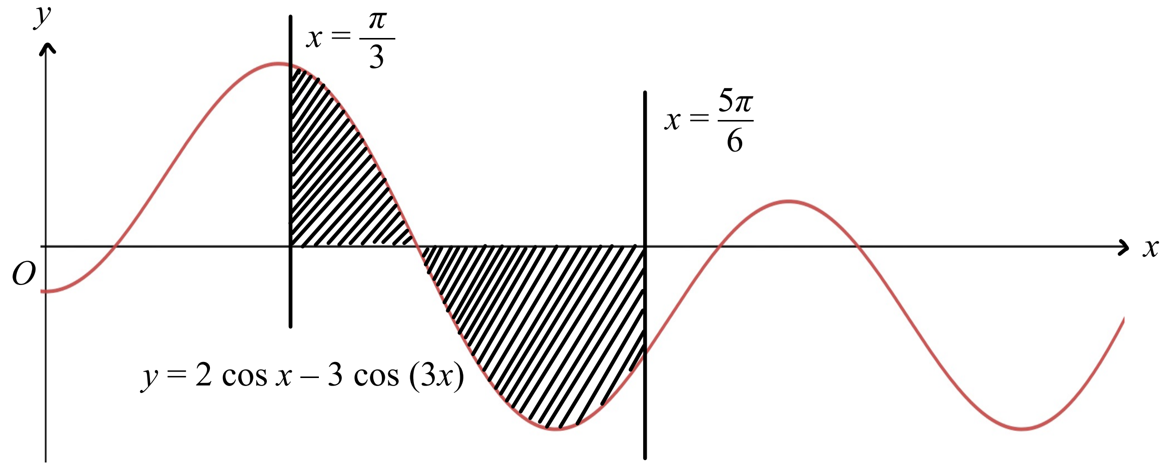


Diagram **13.1**

(Continuation of working space for question **13 (b)**)

(c) The diagram below shows part of a curve of $y = \dfrac{1}{x^2}$.

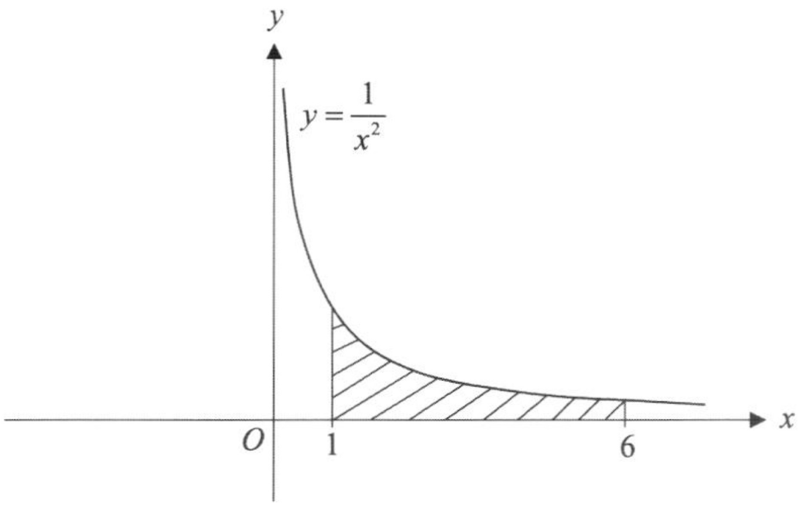


Diagram **13.2**

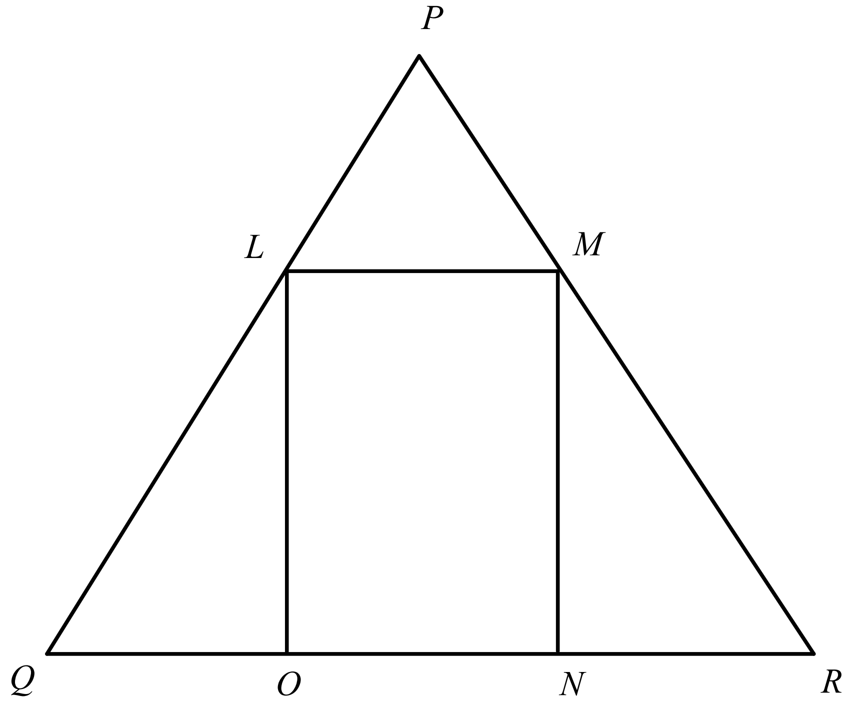
Show that the area of the shaded region is $\dfrac{5}{6}$ units^2.

Hence, without any other calculations, explain why

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}<\frac{5}{6}<\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}$$

You may annotate or sketch out anything on Diagram **13.2** to explain your answer. [2]

14* A farmer owns a triangular shaped area in the form of an equilateral triangle PQR as shown below. He sections off rectangular area $LMNO$ to build his warehouse, as shown below. The remaining triangular areas (PLM, QLO, MNR) is to be fenced.
L, M, N, and O always touch the equilateral triangle where L lies on PQ, M lies on PR and points O and N lie on QR, such that $PL = PM$ and $LQ = MR$. $PQ = 50$ m.



By considering the dimensions of the warehouse, calculate the maximum possible exact area of the warehouse using differentiation.
Show that the area is maximum without any stationary tests. [4]

(Continuation of working of question **14**)

15 In predator-prey relationships, the number of animals in each category tends to vary periodically.

In a certain habitat, the populations of foxes (F) in thousands and rabbits (R) in thousands are modelled by the equations: $F = 300 - 125\sin\frac{\pi t}{4}$ and $R = 1200 - 650\cos\frac{\pi t}{4}$.

where t is the number of months.

(i) Using the above models, determine which month(s) in the first year will the population of rabbits be four times as much as the population of foxes. [2]

(ii) Without using the R Formula, predict the number of months it would take for the total population of foxes and rabbits to be the highest within a period of 8 months, and show your workings clearly. [1]

(iii) The graph below shows the population of foxes F in thousands against time t, from $0 \leq t \leq 16$.

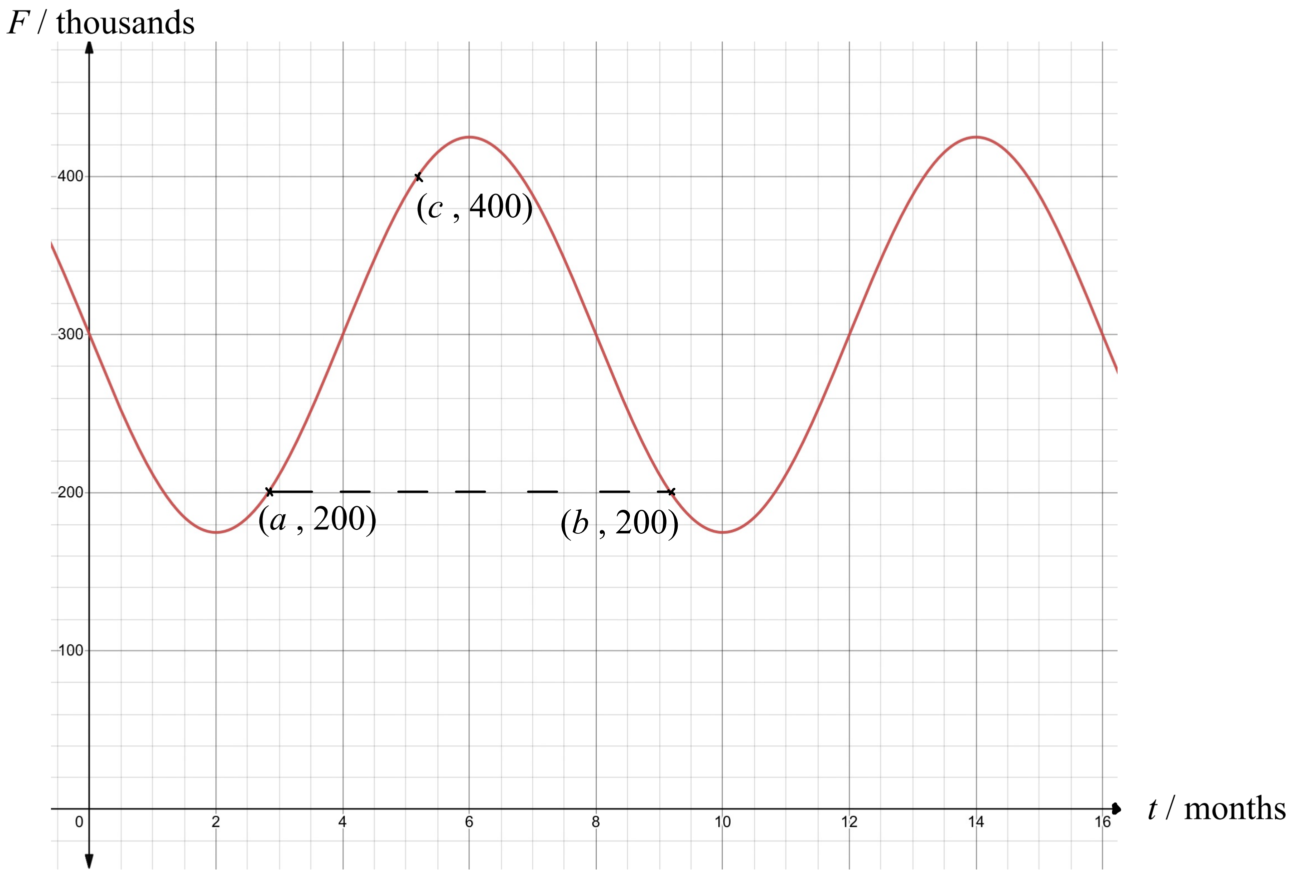


Diagram **15**

The curve passes through the points $(a\ ,200)$, $(b\ ,200)$, and $(c\ ,400)$.

Form an equation connecting b and c. Show all workings. [1]

-End of paper **1**-

You tried your best, and that is all that matters!

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