Name: ______________________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2024

MATHEMATICS **9820/01**
Higher 3

13 September 2024

Paper 1 **3 hours**

Additional materials: 12-page Answer Booklet
(additional 4-page Answer Booklet(s), if applicable)
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

1 Let a and b be positive integers.

(a) (i) Prove that $\gcd(a,b)=\gcd(a,b-a)$ [3]

(ii) Using the result from part (i), evaluate $\gcd(72,120)$. [2]

(b) Prove that for any integers a, b and c, $\gcd(a,\gcd(b,c))=\gcd(\gcd(a,b),c)$ [5]

2 This question is about the third series of coins in Singapore. This series consists of coins of five different denominations: 5 cents, 10 cents, 20 cents, 50 cents and 1 dollar.

(i) Thirteen coins are to be selected. In how many ways can this be done

(a) if there are no restrictions? [2]

(b) if at least one coin of each denomination and at most three 5 cent coins are selected? [4]

(ii) A sequence of thirteen coins is to be formed. In how many ways can this be done

(a) if no two adjacent coins have the same denominations, [1]

(b) if at least one coin of each denomination must be used? [3]

3 (i) Prove that for any integers x and y and for any prime p,

$$(x+y)^p \equiv x^p + y^p \pmod{p}.$$ [2]

(ii) Using induction, show that for any prime p, $a^p \equiv a \pmod{p}$ for all positive integers a. [4]

(iii) Show that if n is not a multiple of 4, then $\sum_{i=1}^{4} i^n \equiv 0 \pmod{5}$. [4]

4 (a) For $a,b,c,p,q,r \in \mathbb{R}^+$, prove that $\sqrt{ap}+\sqrt{bq}+\sqrt{cr} \le \sqrt{(a+b+c)(p+q+r)}$. [3]

(b) For $x,y,z \in \mathbb{R}^+$, prove that $\dfrac{xyz}{(x+y)(y+z)(x+z)} \le \dfrac{1}{8}$. [5]

(c) For $x,y,z \in \mathbb{R}^+$, and using the results of parts **(a)** and **(b)**, prove that [7]

$$\sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} \le 3.$$

Hint: $\dfrac{2x}{x+y}=\dfrac{2x(y+z)(z+x)}{(x+y)(y+z)(z+x)}$

5 Eugene runs once every day. Each of his runs is either a threshold run, a tempo run or a recovery run. Over a period of n consecutive days, he does not do a threshold run on two consecutive days and he does not do a recovery run for more than two consecutive days.

Let a_n , b_n , c_n be the number of possible ways Eugene can run over a period of n consecutive days where the first day is a threshold run, a tempo run or a recovery run respectively.

(i) For $n \in \mathbb{Z}^+$, explain why

$$a_{n+1} = b_n + c_n,$$
$$b_{n+1} = a_n + b_n + c_n,$$
$$c_{n+2} = a_{n+1} + b_{n+1} + a_n + b_n.$$

[4]

(ii) Hence express a_{n+4} in terms of $a_{n+3}, a_{n+2}, a_{n+1}$ and a_n for $n \in \mathbb{Z}^+$. **[3]**

(iii) Show that $a_5 = 40$. **[2]**

(iv) Find the number of ways Eugene can run over a period of 5 consecutive days. **[1]**

6 **(i)** By using substitution(s), prove that, for $n \geq 1$

$$\int_0^1 \left(1-x^2\right)^n \mathrm{d}x = I_{2n+1},$$
$$\int_0^1 \left(1+x^2\right)^{-n} \mathrm{d}x < I_{2n-2}$$

where $I_k = \int_0^{\frac{\pi}{2}} (\cos\theta)^k \mathrm{d}\theta, k \geq 0$. **[4]**

(ii) Using standard series from the List of Formulae (MF26), show that $1-x^2 \leq \mathrm{e}^{-x^2} \leq \left(1+x^2\right)^{-1}$ for $x \geq 0$. Hence prove that $\sqrt{n} I_{2n+1} \leq \int_0^{\sqrt{n}} \mathrm{e}^{-y^2}\, \mathrm{d}y < \sqrt{n} I_{2n-2}$ for $n \geq 1$. **[5]**

(iii) Given that as $k \to \infty$, $\sqrt{k} I_k \to \sqrt{\frac{\pi}{2}}$, show that $\int_0^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$. **[2]**

(iv) Let $U_n = \int_0^{\infty} x^{2n} \mathrm{e}^{-x^2} \mathrm{d}x, n = 0, 1, 2, \ldots$. Given that $x^{2n-1}\mathrm{e}^{-x^2} \to 0$ as $x \to \infty$, prove that $U_n = \frac{2n-1}{2} U_{n-1}$ for $n \geq 1$. Hence prove that $\int_0^{\infty} x^{2n} \mathrm{e}^{-x^2} \mathrm{d}x = \frac{(2n)!\sqrt{\pi}}{2^{2n+1} n!}$. **[4]**

7 **(i)** Given that $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear points, show that

$$(x_3 - x_2)y_1 + (x_1 - x_3)y_2 + (x_2 - x_1)y_3 = 0.$$ **[2]**

The functions f and g are defined on the real numbers and satisfies the equation

$$g(f(x+y)) = f(x) + (2x+y)g(y).$$

(ii) Show that

$$g(f(-x)) = f(x)$$

and hence prove that $f(-x-y) = f(x) + (2x+y)g(y)$. **[3]**

(iii) For real numbers a, b and c, show that

$$f(-a) = f(-b) + (a-b)g(a+b)$$
$$f(-b) = f(-c) + (b-c)g(b+c)$$
$$f(-c) = f(-a) + (c-a)g(c+a)$$ **[2]**

(iv) By showing that $g(x)$ is linear, prove that

$$f(x) = g(x) = 0 \text{ or } f(x) = x^2 + C \text{ and } g(x) = x$$

where C is a constant. **[8]**

8 **(a)** A sequence $\{x_n\}$ is defined by $x_n = \sum_{m=1}^{n} \frac{1}{m^m}$. Given that $m^m \geq 2^m$ for all $m \geq 2$,

(i) show that the sequence is bounded by $\frac{3}{2}$, **[2]**

(ii) show that the sequence converges. **[2]**

(b) The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$ and, for $n \geq 0, F_{n+2} = F_{n+1} + F_n$.

(i) Prove that $F_r \leq 2^{r-1} F_1$ for all $r \geq 1$. **[2]**

(ii) Let $S_n = \sum_{r=1}^{n} \frac{F_r}{9^r}$.

Show that $\sum_{r=1}^{n} \frac{F_{r+1}}{9^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{9^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{9^{r-1}} = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}$. **[4]**

(iii) Show that $\sum_{r=1}^{\infty} \frac{F_r}{9^r} = \frac{9}{71}$. **[3]**

(iv) Given that $\sum_{r=7}^{\infty} \frac{F_r}{9^r} < 2 \times 10^{-6}$, hence find, with justification, the first six digits after the decimal point in the decimal expansion of $\frac{1}{71}$. **[2]**

End of Paper