2020 A Level H3 Mathematics (9820/01)		
Qn	Suggested Solution	Remarks
1i	Apply the AM-GM inequality to yield the following. $$\frac{(n-1)x+y}{n} \geq \sqrt[n]{x^{n-1}y}$$ $$\frac{((n-1)x+y)^n}{n^n} \geq x^{n-1}y$$ $$((n-1)x+y)^n \geq n^n x^{n-1}y$$	
1ii	Let $n=2$, $x=1$ and $y=a$ in **(i)** to get: $(1+a)^2 \geq 2^2 a$. Let $n=3$, $x=\frac{1}{2}$ and $y=b$ in **(i)** to get: $(1+b)^3 \geq 3^3\left(\frac{1}{2}\right)^2 b$. Let $n=4$, $x=\frac{1}{3}$ and $y=c$ in **(i)** to get: $(1+c)^4 \geq 4^4\left(\frac{1}{3}\right)^3 c$. In each case, equality holds only when $x=y$. Hence, $(1+a)^2(1+b)^3(1+c)^4 \geq (2^2 a)\left(3^3\left(\frac{1}{2}\right)^2 b\right)\left(4^4\left(\frac{1}{3}\right)^3 c\right)$ $=4^4 abc$ $=256 \qquad (\because abc=1)$ with equality only if $a=1$, $b=\frac{1}{2}$ and $c=\frac{1}{3}$ which is impossible since $abc=1$. Therefore, $(1+a)^2(1+b)^3(1+c)^4 > 256$.	
2i	Let $y=\mathrm{f}(x)=\frac{1}{ax+b}$, $x \neq -\frac{b}{a}$. $ax+b=\frac{1}{y}$ $x=\frac{1}{a}\left(\frac{1}{y}-b\right)$ $\mathrm{f}^{-1}(x)=\frac{1}{a}\left(\frac{1}{x}-b\right), \qquad x \neq 0$	
2ii	$\mathrm{f}^3(p)=p$ $\frac{1}{a\left(\frac{ap+b}{a+b(ap+b)}\right)+b}=p \qquad$ ----------(*) $a^2p^2+abp=a+abp+b^2-abp-ab^2p^2-b^3p$ $(a+b^2)ap^2+b(a+b^2)p-(a+b^2)=0$ $(a+b^2)(ap^2+bp-1)=0 \qquad$ ----------(**) $a+b^2=0 \qquad$ or $\qquad ap^2+bp-1=0$	

	Case 1: $a+b^2=0$. Then equation (**), and hence equation (*), holds true for all values of p for which $\mathrm{f}^3(p)$ exists. Hence, $\mathrm{f}^3(x)=x$ for all $x\in\mathbb{R}$ such that $\mathrm{f}^3(x)$ exists. Case 2: $ap^2+bp-1=0$. Then $p=\dfrac{1}{ap+b}=\mathrm{f}(p)$, i.e. p is a fixed point of f.	
2iii	$Ax_nx_{n+1}+Bx_{n+1}=1 \quad \Rightarrow \quad x_{n+1}=\dfrac{1}{Ax_n+B}$ From **(ii)**, $\mathrm{f}^2(x)=x$ when $b=0$ for all x such that $\mathrm{f}^2(x)$ exists; and $\mathrm{f}^3(x)=x$ when $a+b^2=0$ for all x such that $\mathrm{f}^3(x)$ exists. Period 2: Set $A=1$ and $B=0$ to get the recurrence relation $x_nx_{n+1}=1$, with the condition that $x_1\neq -1,0,1$. Period 3: Set $A=-1$ and $B=1$ to get the recurrence relation $-x_nx_{n+1}+x_{n+1}=1$, with the condition that $x_1\neq 0,1$.	Compare A and B in **(iii)** to a and b in **(ii)** respectively.
3i	Let $Q(n)$ be the statement: $\int_0^t x^n\mathrm{e}^{-x}\,\mathrm{d}x=n!\left(1-\mathrm{e}^{-t}\mathrm{P}_n(t)\right)$, $n\in\mathbb{Z}_0^+$. LHS of $Q(0)=\int_0^t x^0\mathrm{e}^{-x}\,\mathrm{d}x=\left[-\mathrm{e}^{-x}\right]_0^t=1-\mathrm{e}^{-t}$ RHS of $Q(0)=0!\left(1-\mathrm{e}^{-t}\mathrm{P}_0(t)\right)=1-\mathrm{e}^{-t}\left(\dfrac{t^0}{0!}\right)=1-\mathrm{e}^{-t}$ Therefore, $Q(0)$ is true. Assume $Q(k)$ is true for some $k\in\mathbb{Z}_0^+$, i.e. $\int_0^t x^k\mathrm{e}^{-x}\,\mathrm{d}x=k!\left(1-\mathrm{e}^{-t}\mathrm{P}_k(t)\right)$. LHS of $Q(k+1)=\int_0^t x^{k+1}\mathrm{e}^{-x}\,\mathrm{d}x$ $=\left[-x^{k+1}\mathrm{e}^{-x}\right]_0^t-\int_0^t-(k+1)x^k\mathrm{e}^{-x}\,\mathrm{d}x$ $=-t^{k+1}\mathrm{e}^{-t}+(k+1)\int_0^t x^k\mathrm{e}^{-x}\,\mathrm{d}x$ $=-t^{k+1}\mathrm{e}^{-t}+(k+1)k!\left(1-\mathrm{e}^{-t}\mathrm{P}_k(t)\right)$ (by inductive hypothesis) $=-(k+1)!\dfrac{t^{k+1}\mathrm{e}^{-t}}{(k+1)!}+(k+1)!\left(1-\mathrm{e}^{-t}\mathrm{P}_k(t)\right)$ $=(k+1)!\left[1-\mathrm{e}^{-t}\left(\mathrm{P}_k(t)+\dfrac{t^{k+1}}{(k+1)!}\right)\right]$ $=(k+1)!\left[1-\mathrm{e}^{-t}\left(\sum_{i=0}^{k}\dfrac{t^i}{i!}+\dfrac{t^{k+1}}{(k+1)!}\right)\right]$ $=(k+1)!\left[1-\mathrm{e}^{-t}\sum_{i=0}^{k+1}\dfrac{t^i}{i!}\right]=(k+1)!\left(1-\mathrm{e}^{-t}\mathrm{P}_{k+1}(t)\right)=$ RHS of $Q(k+1)$	

	Since $Q(k)$ is true implies that $Q(k+1)$ is true, and $Q(0)$ is true, then by PMI, $Q(n)$ is true for all $n \in \mathbb{Z}_0^+$.	
3ii	For any fixed $n \in \mathbb{Z}_0^+$, $\mathrm{e}^{-t}\mathrm{P}_n(t) = \dfrac{1+t+\frac{t^2}{2!}+\ldots+\frac{t^n}{n!}}{\mathrm{e}^t}$ $= \dfrac{1+t+\frac{t^2}{2!}+\ldots+\frac{t^n}{n!}}{1+t+\frac{t^2}{2!}+\ldots+\frac{t^n}{n!}+\ldots}$ $= \dfrac{\frac{1}{t^n}+\frac{1}{t^{n-1}}+\frac{1}{2!t^{n-2}}+\ldots+\frac{1}{n!}}{\frac{1}{t^n}+\frac{1}{t^{n-1}}+\frac{1}{2!t^{n-2}}+\ldots+\frac{1}{n!}+\frac{t}{(n+1)!}+\frac{t^2}{(n+2)!}+\ldots} \xrightarrow[t\to\infty]{} 0.$ $\therefore \int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\left(\int_0^t x^n\mathrm{e}^{-x}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left[n!\left(1-\mathrm{e}^{-t}\mathrm{P}_n(t)\right)\right] = n!$	
3iii	For all $n \in \mathbb{Z}$ with $n > t > 0$, we have the following results. $\left(1+\frac{t}{n}\right)^n = 1+\binom{n}{1}\left(\frac{t}{n}\right)+\binom{n}{2}\left(\frac{t}{n}\right)^2+\ldots+\binom{n}{n}\left(\frac{t}{n}\right)^n$ $= 1+t+\frac{n(n-1)}{n^2}\frac{t^2}{2!}+\ldots+\frac{n!}{n^n}\frac{t^n}{n!}$ $\le 1+t+\frac{t^2}{2!}+\ldots+\frac{t^n}{n!} = \mathrm{P}_n(t)$ $\left(1-\frac{t}{n}\right)^{-n} = 1+(-n)\left(-\frac{t}{n}\right)+\frac{(-n)(-n-1)}{2!}\left(-\frac{t}{n}\right)^2+\ldots \quad \left(\because n>t>0 \Rightarrow \frac{t}{n}<1\right)$ $= 1+t+\frac{n(n+1)}{n^2}\frac{t^2}{2!}+\ldots+\frac{n(n+1)\ldots(2n-1)}{n^n}\frac{t^n}{n!}+\ldots$ $> 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots+\frac{t^n}{n!} = P_n(t)$ $\therefore \left(1+\frac{t}{n}\right)^n \le \mathrm{P}_n(t) < \left(1-\frac{t}{n}\right)^{-n}$	Convergence only holds for $\lvert r\rvert < 1$.
4i	$39x+23y+65z = 1872 = 2^4 \times 3^2 \times 13$ Hence, $13 \mid (39x+23y+65z)$. Since $13 \mid 39x$ and $13 \mid 65z$, we must have $13 \mid 23y$. Since $\gcd(13,23)=1$, $13 \mid y$. Finally, since y is prime, $y = 13$.	
4ii (a)	Given $y=13$, then we have $39x+65z=1573$ which yields $3x+5z=121$. ----------(*) Taking modulo 5 on both sides of (*) gives $3x \equiv 1 \pmod 5$, and so $x \equiv x+5x = 6x \equiv 2 \pmod 5$. Similarly, taking modulo 3 on both sides of (*) gives $5z \equiv 1 \pmod 3$, and so $z \equiv 3(3z)+z = 10z \equiv 2 \pmod 3$.	

4ii (b)	From **(a)**, $3x+5z=121$, $x=5m+2$ and $z=3n+2$ for some $m,n\in\mathbb{Z}$. $\therefore 3(5m+2)+5(3n+2)=121 \Rightarrow 15(m+n)=105 \Rightarrow m+n=7$ Hence, $\lvert z-x\rvert=\lvert 3n+2-(5m+2)\rvert=\lvert 3n-5m\rvert=\lvert 3(7-m)-5m\rvert=\lvert 21-8m\rvert$. Minimal value of $\lvert z-x\rvert$ is 3 when $m=3$, which in turn gives $n=4$. Thus, the required solution is $x=17$, ($y=13$) and $z=14$.	
4iii	From **(i)**, if y is prime, then $y=13$ which leads to $3x+5z=121$. From **(ii)**, $(x,z)=(17,14)$ is a solution to $3x+5z=121$. Since $\gcd(3,5)=1$, integer solutions to $3x+5z=121$ are $$\begin{cases} x=17+5r \\ z=14-3r \end{cases} \quad \text{for } r\in\mathbb{Z}.$$ When r is even, z is even; and when r is odd, x is even. It is thus impossible for x and z to be both prime unless one of them is 2. When $x=2$, $r=-3$ and $z=23$ which is prime. When $z=2$, $r=4$ and $x=37$ which is prime. Hence, solutions with x, y and z prime are $(x,y,z)=(2,13,23)$ or $(x,y,z)=(37,13,2)$.	
5a (i)	For $x\neq 0$, $f(x)+2f\left(\frac{1}{x}\right)=3x$. ----------- (1) Since $x\neq 0$, $\frac{1}{x}$ is defined and non-zero as well. Replacing x with $\frac{1}{x}$ in (1) gives $f\left(\frac{1}{x}\right)+2f(x)=\frac{3}{x}$. ----------- (2)	
5a (ii)	$2\times(2)-(1)$: $3f(x)=\frac{6}{x}-3x$ Hence, $f(x)=\frac{2}{x}-x$, $x\neq 0$.	
5b	For $x\neq 0$, $g(x)+g(-x)+g\left(\frac{1}{x}\right)=x$. ----------- (3) Replacing x with $-x$ in (3) gives $g(-x)+g(x)+g\left(-\frac{1}{x}\right)=-x$. --- (4) $(3)-(4)$: $g\left(\frac{1}{x}\right)-g\left(-\frac{1}{x}\right)=2x$ Replacing x with $\frac{1}{x}$ gives $g(x)-g(-x)=\frac{2}{x}$. ----------- (5) $(3)+(5)$: $2g(x)+g\left(\frac{1}{x}\right)=x+\frac{2}{x}$ ----------- (6) Replacing x with $\frac{1}{x}$ gives $2g\left(\frac{1}{x}\right)+g(x)=\frac{1}{x}+2x$. ----------- (7) $2\times(6)-(7)$: $3g(x)=\frac{3}{x}$ Hence, $g(x)=\frac{1}{x}$, $x\neq 0$.	

6i	For each $n \geq 1$: $x_{n+1}{}^2 - x_n x_{n+2} = x_{n+1}{}^2 - x_n\left(dx_{n+1} - x_n\right)$ $\qquad = x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1} \qquad ----------(*)$ We also have the following for each $n \geq 2$: $x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1} = x_n{}^2 + \left(dx_n - x_{n-1}\right)^2 - dx_n\left(dx_n - x_{n-1}\right)$ $\qquad = x_n{}^2 + d^2 x_n{}^2 - 2dx_n x_{n-1} + x_{n-1}{}^2 - d^2 x_n{}^2 + dx_{n-1}x_n$ $\qquad = x_{n-1}{}^2 + x_n{}^2 - dx_{n-1}x_n$ Recursively, we have: $x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1}$ $= x_{n-1}{}^2 + x_n{}^2 - dx_{n-1}x_n$ $= x_{n-2}{}^2 + x_{n-1}{}^2 - dx_{n-2}x_{n-1}$ $= \ldots$ $= x_1{}^2 + x_2{}^2 - dx_1 x_2$ $= x_2{}^2 - x_1 x_3 \qquad (\text{by } (*))$ $= D$ **<u>Alternative Solution</u>** (to show $x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1} = D$) Let $P(n)$ be the statement $x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1} = D$ for $n \in \mathbb{Z}^+$. Since $x_1{}^2 + x_2{}^2 - dx_1 x_2 = x_2{}^2 - x_1 x_3 = D$, $P(1)$ is true. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$, i.e. $x_k{}^2 + x_{k+1}{}^2 - dx_k x_{k+1} = D$. LHS of $P(k+1)$ $= x_{k+1}{}^2 + x_{k+2}{}^2 - dx_{k+1}x_{k+2}$ $= x_{k+1}{}^2 + x_{k+2}{}^2 - \left(x_k + x_{k+2}\right)x_{k+2} \qquad \left(\because dx_{n+1} - x_n = x_{n+2} \text{ for all } n \in \mathbb{Z}^+\right)$ $= x_{k+1}{}^2 - x_k x_{k+2}$ $= x_k{}^2 + x_{k+1}{}^2 - dx_k x_{k+1} \qquad (\text{by } (*))$ $= D \qquad$ (by inductive hypothesis) $= \text{RHS of } P(k+1)$ Since $P(k)$ is true implies that $P(k+1)$ is true, and $P(1)$ is true, then by PMI, $P(n)$ is true for all $n \in \mathbb{Z}^+$. By (*) and $P(n)$, we have $x_{n+1}{}^2 - x_n x_{n+2} = x_n{}^2 + x_{n+1}{}^2 - dx_n x_{n+1} = D$.	
6ii	Suppose $x_m = 0$ for some $m \in \mathbb{Z}^+$, then by **(i)**, $$D = x_{m+1}{}^2 - x_m x_{m+2} = x_{m+1}{}^2,$$ which shows that D is a perfect square.	
6iii	<u>Case 1</u>: $x_m = 0$ for some $m \in \mathbb{Z}^+$. Then by **(ii)**, D is a perfect square.	

	Case 2: $x_n \neq 0$ for all $n \in \mathbb{Z}^+$. (Equivalently, $x_n^{\ 2} \geq 1$ for all $n \in \mathbb{Z}^+$.) Since there exist positive and negative terms, then $\exists k \in \mathbb{Z}^+$ such that x_k and x_{k+1} are of opposite signs, so that $x_k x_{k+1} < 0$. $$\begin{aligned} D &= x_k^{\ 2} + x_{k+1}^{\ 2} - dx_k x_{k+1} \\ &\geq 1 + 1 + d\left(-x_k x_{k+1}\right) \\ &\geq d + 2 \qquad \left(\because -x_k x_{k+1} \geq 1 \text{ and } d > 0\right) \end{aligned}$$ By cases 1 and 2, D is a perfect square or $D \geq d+2$.	
6iv	A $(3,5)$-sequence satisfies $$\begin{cases} 3x_{n+1} - x_n = x_{n+2},\ n \geq 1 \\ x_2^{\ 2} - x_1 x_3 = 5 \end{cases}.$$ A possible sequence of five consecutive terms having both positive and negative terms is $-1, 1, 4, 11, 29$.	From (iii), the equality $D = d+2$ holds when $x_k x_{k+1} = -1$. Use this hint to see that we can set $x_1 = -1$ and $x_2 = 1$.
7i	$X = \{1, 2, ..., m\}$, $Y = \{1, 2, ..., n\}$, $\mathrm{f}: X \to Y$ Number of different functions mapping X to Y is n^m.	
7ii	For $n \geq m$, number of one-to-one functions is $$n(n-1)(n-2)...(n-m+1) = \frac{n!}{(n-m)!}.$$	
7iii	For $m \geq n$, and for $r = 1, 2, ..., n$, let A_r be the set of functions $\mathrm{f}: X \to Y$ such that $r \notin \mathrm{Im}(\mathrm{f})$, i.e. $\forall x \in X$, $\mathrm{f}(x) \neq r$. Hence, the number of onto functions is given by $\left|\left(A_1 \cup A_2 \cup ... \cup A_n\right)'\right|$. $$\begin{aligned} \left|\left(\bigcup_{i=1}^{n} A_i\right)'\right| &= n^m - \sum_i \left|A_i\right| + \sum_{i \neq j} \left|A_i \cap A_j\right| + ... + (-1)^{n-1} \sum_k \left|\bigcap_{i \neq k} A_i\right| \\ &= \binom{n}{0} n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m + ... + (-1)^{n-1}\binom{n}{n-1} 1^m \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m \end{aligned}$$	Note: sum from $i = 0$ to $i = n$ or $i = n-1$ is the same.
7iv	For $m = n = 5$ and for $r = 1, 2, 3, 4, 5$, let B_r be the set of one-to-one functions $\mathrm{f}: X \to Y$ such that $\mathrm{f}(r) = r$. Number of one-to-one functions that map no element to itself $$\begin{aligned} &= \left|\left(\bigcup_{r=1}^{5} B_r\right)'\right| \\ &= 5! - \sum_i \left|B_i\right| + \sum_{i \neq j} \left|B_i \cap B_j\right| - \sum_{\substack{i \neq j, i \neq k, \\ j \neq k}} \left|B_i \cap B_j \cap B_k\right| + \sum_{i=1}^{5} \left|\bigcap_{k \neq i} B_k\right| - \left|\bigcap_{i=1}^{5} B_i\right| \\ &= 5! - \binom{5}{1} 4! + \binom{5}{2} 3! - \binom{5}{3} 2! + \binom{5}{4} 1! - 1 \\ &= 44 \end{aligned}$$	

8a	An ellipse is symmetrical about its centre. Since F is centred on the origin and contains the point with position vector $\mathbf{x}$, then it must also contain the point with position vector $-\mathbf{x}$. An ellipse is convex – if A and B are points in an ellipse, then the line segment AB lies in the ellipse. Since F contains points with position vectors $-\mathbf{x}$ and $\mathbf{y}=\mathbf{x}+\begin{pmatrix}2a\\2b\end{pmatrix}$, then it must contain the midpoint given by position vector $\begin{pmatrix}a\\b\end{pmatrix}$.	
8b (i)	Let G be an ellipse with area at least 4. By translating parts of G to fit into the 2 by 2 square given by $0\le x\le 2$ and $0\le y\le 2$ as described in the question, there must be an overlap (otherwise, G cannot have an area at least 4). Let A be a common point in this overlap. Let (p,q) and (r,s) be different points in G that are translated to A, then $$\begin{pmatrix}p\\q\end{pmatrix}+\begin{pmatrix}2m_1\\2n_1\end{pmatrix}=\begin{pmatrix}r\\s\end{pmatrix}+\begin{pmatrix}2m_2\\2n_2\end{pmatrix},$$ for some $m_1,m_2,n_1,n_2\in\mathbb{Z}$ with $(m_1,n_1)\neq(m_2,n_2)$. Let $\mathbf{x}=\begin{pmatrix}p\\q\end{pmatrix}$ and $\mathbf{y}=\begin{pmatrix}r\\s\end{pmatrix}=\mathbf{x}+\begin{pmatrix}2(m_1-m_2)\\2(n_1-n_2)\end{pmatrix}$. By **(a)**, the point (m_1-m_2,n_1-n_2) lies in G and as required, this point has integer coordinates that are not both 0.	
8b (ii)	Suppose on the contrary that there is no such point $(mp-nu,n)$ in C, i.e. for all $m,n\in\mathbb{Z}$, $-2\sqrt{\frac{p}{\pi}}<n<2\sqrt{\frac{p}{\pi}}$, with $(m,n)\neq(0,0)$, $(mp-nu)^2+n^2>\frac{4p}{\pi}$. $p^2m^2+u^2n^2-2upnm+n^2-\frac{4p}{\pi}>0$ $p^2m^2-2upnm+(u^2+1)n^2-\frac{4p}{\pi}>0$ Since this is true for all m, the discriminant $4u^2p^2n^2-4p^2\left[(u^2+1)n^2-\frac{4p}{\pi}\right]<0$. $4p^2\left(\frac{4p}{\pi}-n^2\right)<0$ $\frac{4p}{\pi}-n^2<0$ $n<-2\sqrt{\frac{p}{\pi}}$ or $n>2\sqrt{\frac{p}{\pi}}$ $(\rightarrow\leftarrow)$	

	Alternative Let T be a *linear transformation* with matrix $T=\begin{pmatrix} p & -u \\ 0 & 1 \end{pmatrix}$. Then $\det(T)=p\neq 0$ so T^{-1} exists and has determinant $\det(T^{-1})=\frac{1}{p}>0$. Let E be defined as $E=T^{-1}(C)$. Then E has equation $(px-uy)^2+y^2=\frac{4p}{\pi}$ and is an ellipse also centred on the origin. Area of E $=\det(T^{-1})\times(\text{Area of }C)=\frac{1}{p}\times\pi\left(2\sqrt{\frac{p}{\pi}}\right)^2=4$. By **(b(i))**, E contains a point $(m,n)\neq(0,0)$ with $m,n\in\mathbb{Z}$. Now by transforming E back to C, via T, the point (m,n) in E is mapped to $(mp-nu,n)$, which is not the origin, in C.	
8c	Let $p,u\in\mathbb{Z}^+$ and C be the circle centred on the origin with radius $2\sqrt{\frac{p}{\pi}}$. From **(b(ii))**, $(mp-nu,n)\neq(0,0)$ lies in C. Let the integers $x=mp-nu$ and $y=n$, so we get $$0<x^2+y^2<\left(2\sqrt{\frac{p}{\pi}}\right)^2=\frac{4p}{\pi}. \qquad ---------(*)$$ Since $u^2+1=kp$ for some $k\in\mathbb{Z}^+$, $$\begin{aligned} x^2+y^2&=(mp-nu)^2+n^2\\ &=m^2p^2-2mpnu+n^2(u^2+1)\\ &=p(m^2p-2mnu+n^2k).\end{aligned}$$ By (*), $0<p(m^2p-2mnu+n^2k)<\frac{4p}{\pi}<2p$, which gives $$0<m^2p-2mnu+n^2k<2.$$ $$\therefore m^2p-2mnu+n^2k=1$$ Hence, $x^2+y^2=p(m^2p-2mnu+n^2k)=p$.	