Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 2 Transformation of Curves
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ SLS Activity for Learning Experience (Exploring the three Basic Transformations)
- ☐ Quick Concept Check

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 2 Transformation of Curves
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Identify the replacement of variable involved in translation of a graph (i.e. $y=\mathrm{f}\left(x\right)+a$ & $y=\mathrm{f}\left(x+a\right)$) ☐ Draw the graph after applying translation (i.e. $y=\mathrm{f}\left(x\right)+a$ & $y=\mathrm{f}\left(x+a\right)$) and label the corresponding characteristics such as asymptotes, turning points and intersections with the axes after translation. ☐ Identify the replacement of variable involved in stretching of a graph (i.e. $y=a\mathrm{f}\left(x\right)$ & $y=\mathrm{f}\left(\dfrac{x}{a}\right)$) ☐ Draw the graph after applying stretching (i.e. $y=a\mathrm{f}\left(x\right)$ & $y=\mathrm{f}\left(\dfrac{x}{a}\right)$) and label the corresponding characteristics after stretching. ☐ Identify the replacement of variable involved in reflecting a graph either in the x-axis or y-axis (i.e. $y=-\mathrm{f}(x)$ & $y=\mathrm{f}(-x)$). ☐ Draw the graph after applying reflection (i.e. $y=-\mathrm{f}(x)$ & $y=\mathrm{f}(-x)$). and label the corresponding characteristics after reflection. ☐ Draw the graph of $y=\left\|\mathrm{f}(x)\right\|$ and label the characteristics after transformation.	☐ Identify the correct sequence of transformations for graphs with 2 or more transformations. ☐ Draw the graph with 2 or more transformations and label the corresponding characteristics after transformations. ☐ Describe a sequence of transformations given the original and resulting equations. ☐ Determine the resulting equation of graph given a sequence of transformations.	☐ Draw the graph of $y=\mathrm{f}\left(\left\|x\right\|\right)$ and label the characteristics after transformation. ☐ Draw the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ and label the corresponding characteristics such as asymptotes, turning points and intersections with axes after transformation.

Definition of Learning (John Hattie):
The process of developing sufficient surface knowledge to then move to deeper understanding such that one can appropriately transfer this learning to new tasks and situations.

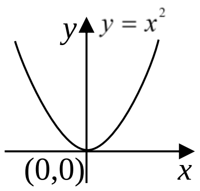
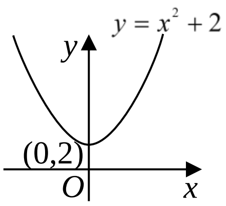
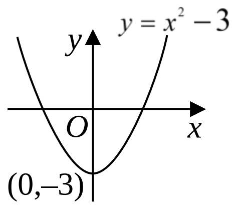
Basic Transformations

§1 Translation

Translating a graph in the direction of an axis is to move the graph in the direction of the axis, **without** changing its shape or size.

1.1 Translation in the *y*-direction

Let $y=\mathrm{f}(x)=x^2$. Using GC, sketch the graphs of $y=\mathrm{f}(x)+a$ for $a=2$ and $a=-3$. How does the graph of $y=\mathrm{f}(x)=x^2$ change with different values of *a*?

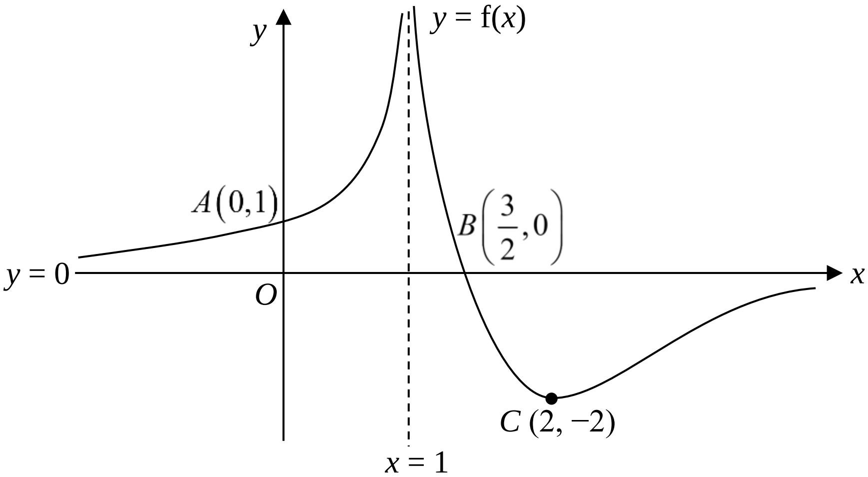
$y=\mathrm{f}(x)$	Transformed equation $y=\mathrm{f}(x)+a$	Replacement	Geometrical description of transformation
$y=x^2$ 	$y=x^2+2$ 	Replace *y* by $y-2$	**<u>Translation</u>** of the graph $y=x^2$ by 2 units in the **<u>positive</u>** *y*-direction. i.e. *y* is replaced by (*y* − 2), thus we add 2 to all the original *y*-coordinates
	$y=x^2-3$ 	Replace *y* by $y-(-3)=y+3$	**<u>Translation</u>** of the graph $y=x^2$ by 3 units in the **<u>negative</u>** *y*-direction. i.e. *y* is replaced by (*y* − (−3)), thus we add −3 to all the original *y*-coordinates.

To get $y=\mathrm{f}(x)+a$ from the graph of $y=\mathrm{f}(x)$, where $a>0$,
- Replace *y* by $y-a$
- $y=\mathrm{f}(x)\longrightarrow y-a=\mathrm{f}(x)\Rightarrow y=\mathrm{f}(x)+a$
- The graph of $y=\mathrm{f}(x)+a$ is a **translation** of the graph of $y=\mathrm{f}(x)$ by *a* units in the **positive** *y*-direction.
- $(x_i, \mathrm{f}(x_i))\longrightarrow(x_i, \mathrm{f}(x_i)+a)$

To get $y=\mathrm{f}(x)-a$ from the graph of $y=\mathrm{f}(x)$, where $a>0$,
- Replace *y* by $y-(-a)=y+a$
- $y=\mathrm{f}(x)\longrightarrow y+a=\mathrm{f}(x)\Rightarrow y=\mathrm{f}(x)-a$
- The graph of $y=\mathrm{f}(x)-a$ is a **translation** of the graph of $y=\mathrm{f}(x)$ by *a* units in the **negative** *y*-direction.
- $(x_i, \mathrm{f}(x_i))\longrightarrow(x_i, \mathrm{f}(x_i)-a)$

Example 1

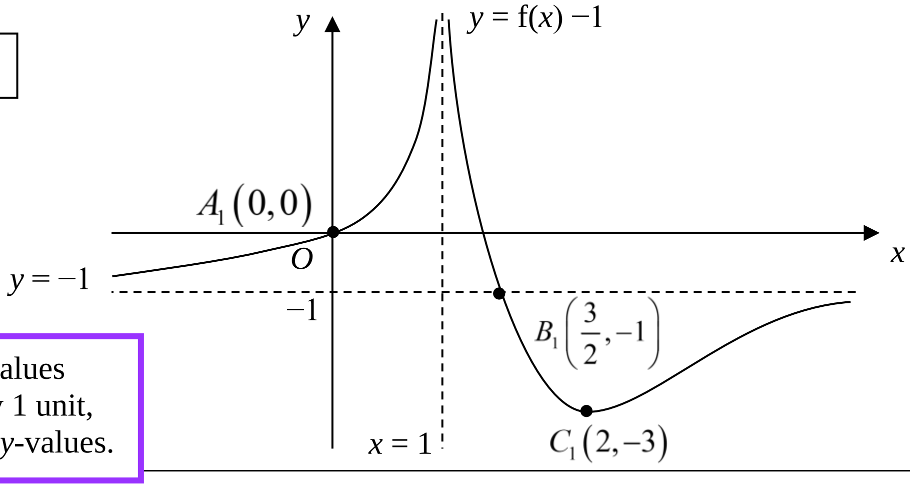
The graph of $y = \mathrm{f}(x)$ is shown. Sketch the graph of $y = \mathrm{f}(x) - 1$.



Solution:

$y = \mathrm{f}(x) - 1$

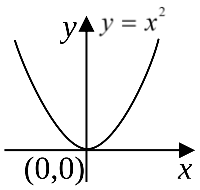
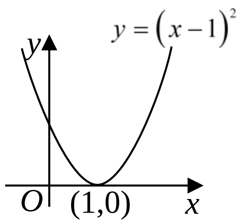
Replace y by $y+1$

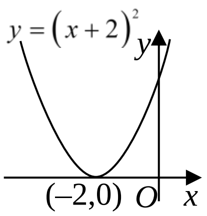


Notice that all y-values have decreased by 1 unit, i.e. minus 1 to all y-values.

1.2 <u>Translation in the *x*-direction</u>

Let $y=\mathrm{f}(x)=x^2$. Using GC, sketch the graphs of $y=\mathrm{f}\left(x+a\right)$ for $a=-1$ and $a=2$. How does the graph of $y=\mathrm{f}(x)=x^2$ change with different values of *a*?

<table>
<tr><td>$y=\mathrm{f}(x)$</td><td>Transformed equation
$y=\mathrm{f}(x+a)$</td><td>Replacement</td><td>Geometrical description of transformation</td></tr>
<tr><td rowspan="2">$y=x^2$
</td><td>$y=\left(x-1\right)^2$
</td><td>Replace x by $x-1$</td><td><u>Translation</u> of the graph $y=x^2$ by 1 unit in the <u>positive</u> x-direction.

i.e. x is replaced by ($x-1$), thus we add 1 to all the original x-coordinates.</td></tr>
<tr><td>$y=\left(x+2\right)^2$
</td><td>Replace x by
$x-\left(-2\right)=x+2$</td><td><u>Translation</u> of the graph $y=x^2$ by 2 units in the <u>negative</u> x-direction.

i.e. x is replaced by ($x+2$), thus we add –2 to all the original x-coordinates.</td></tr>
</table>

To get $y=\mathrm{f}\left(x-a\right)$ from the graph of $y=\mathrm{f}\left(x\right)$, where $a>0$,

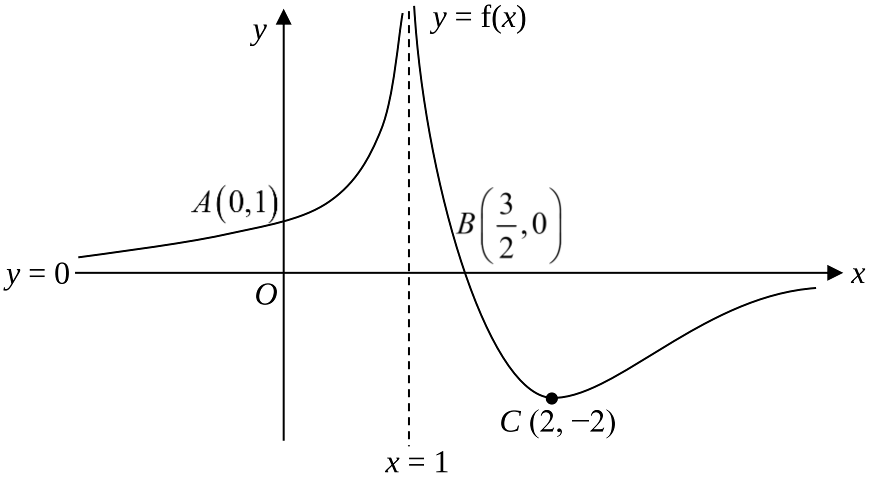
- Replace *x* by $x-a$
- $y=\mathrm{f}\left(x\right)\longrightarrow y=\mathrm{f}\left(x-a\right)$
- The graph of $y=\mathrm{f}(x-a)$ is a **translation** of the graph $y=\mathrm{f}(x)$ by *a* units in the **positive** *x*-direction.
- $\left(x_i,\mathrm{f}\left(x_i\right)\right)\rightarrow\left(x_i+a,\mathrm{f}\left(x_i\right)\right)$

To get $y=\mathrm{f}\left(x+a\right)$ from the graph of $y=\mathrm{f}\left(x\right)$, where $a>0$,

- Replace *x* by $x-\left(-a\right)=x+a$
- $y=\mathrm{f}\left(x\right)\longrightarrow y=\mathrm{f}\left(x+a\right)$
- The graph of $y=\mathrm{f}(x+a)$ is a **translation** of the graph $y=\mathrm{f}(x)$ by *a* units in the **negative** *x*-direction.
- $\left(x_i,\mathrm{f}\left(x_i\right)\right)\longrightarrow\left(x_i-a,\mathrm{f}\left(x_i\right)\right)$

Example 2

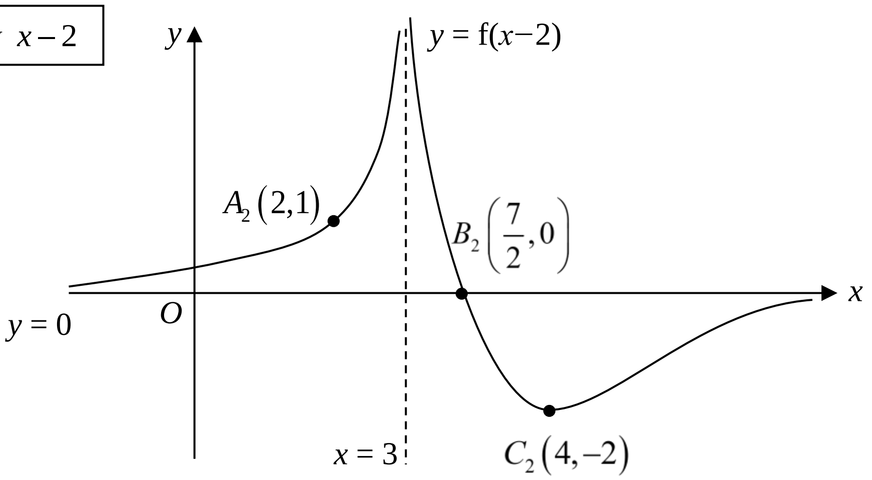
The graph of $y=\mathrm{f}\left(x\right)$ is shown. Sketch the graph of $y=\mathrm{f}(x-2)$.



Solution:

$y=\mathrm{f}(x-2)$

Replace x by $x-2$

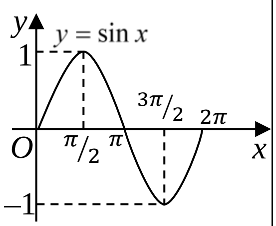
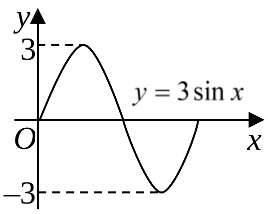
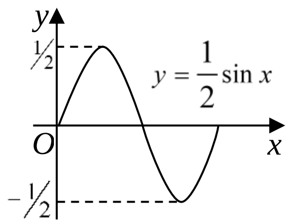


Notice that all x-values have increased by 2 unit, i.e. add 2 to all x-values.

§2 Stretch/Scale

2.1 Stretch parallel to the y-axis

Let $y = \mathrm{f}(x) = \sin x$. Using GC, sketch the graphs of $y = a\mathrm{f}(x)$ for $a = 3$ and $a = \frac{1}{2}$. How does the graph of $y = \mathrm{f}(x) = \sin x$ change with different values of a ?

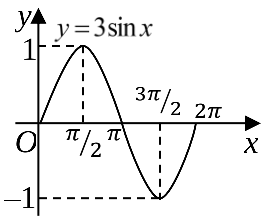
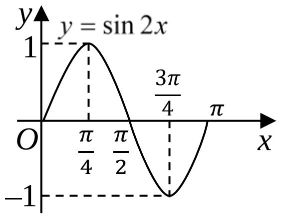
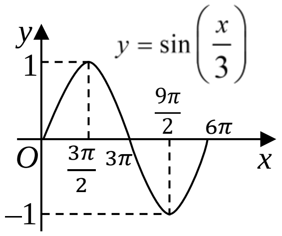
$y = \mathrm{f}(x)$	Transformed equation $y = a\mathrm{f}(x),\ a > 0$	Replacement	Geometrical description of transformation
$y = \sin x$ 	$y = 3\sin x$ 	Replace y by $\frac{y}{3}$	**<u>Stretch</u>** the graph of $y = \sin x$ by **<u>factor</u>** 3 parallel to the **<u>*y*-axis</u>**. i.e. replace y by $\frac{y}{3}$, thus we multiply 3 to all the original y-coordinates.
	$y = \frac{1}{2}\sin x$ 	Replace y by $\frac{y}{\left(\frac{1}{2}\right)}$	**<u>Stretch</u>** the graph $y = \sin x$ by **<u>factor</u>** $\frac{1}{2}$ parallel to the **<u>*y*-axis</u>**. i.e. replace y by $\frac{y}{\left(\frac{1}{2}\right)}$, thus we multiply $\frac{1}{2}$ to all the original y-coordinates.

To get $y = a\mathrm{f}(x)$ from the graph of $y = \mathrm{f}(x)$, where $a > 0$,

- Replace y by $\frac{y}{a}$
- $y = \mathrm{f}(x) \longrightarrow \frac{y}{a} = \mathrm{f}(x) \Rightarrow y = a\mathrm{f}(x)$
- The graph of $y = a\mathrm{f}(x)$ is a **stretch** of the graph $y = \mathrm{f}(x)$ by factor a parallel to the ***y*-axis**.
- $(x_i, \mathrm{f}(x_i)) \longrightarrow (x_i, a\mathrm{f}(x_i))$

2.2 <u>Stretch parallel to the *x*-axis</u>

Let $y = \mathrm{f}(x) = \sin x$. Using GC, sketch the graphs of $y = \mathrm{f}\left(\frac{x}{a}\right)$ for $a = \frac{1}{2}$ and $a = 3$. How does the graph of $y = \mathrm{f}(x) = \sin x$ change with different values of *a*?

$y = \mathrm{f}(x)$	Transformed equation $y = \mathrm{f}\left(\frac{x}{a}\right)$, $a > 0$	Replacement	Geometrical description of transformation
$y = \sin x$ 	$y = \sin 2x$ 	Replace *x* by $\frac{x}{\left(\frac{1}{2}\right)}$	**<u>Stretch</u>** the graph $y = \sin x$ by **<u>factor</u>** $\frac{1}{2}$ parallel to the **<u>*x*-axis</u>**. i.e replace *x* by $\frac{x}{\left(\frac{1}{2}\right)}$, thus we multiply $\frac{1}{2}$ to all the original *x*-coordinates.
	$y = \sin\left(\frac{x}{3}\right)$ 	Replace *x* by $\frac{x}{3}$	**<u>Stretch</u>** the graph $y = \sin x$ by **<u>factor</u>** 3 parallel to the **<u>*x*-axis</u>**. i.e. replace *x* by $\frac{x}{3}$, thus we multiply 3 to all the original *x*-coordinates.

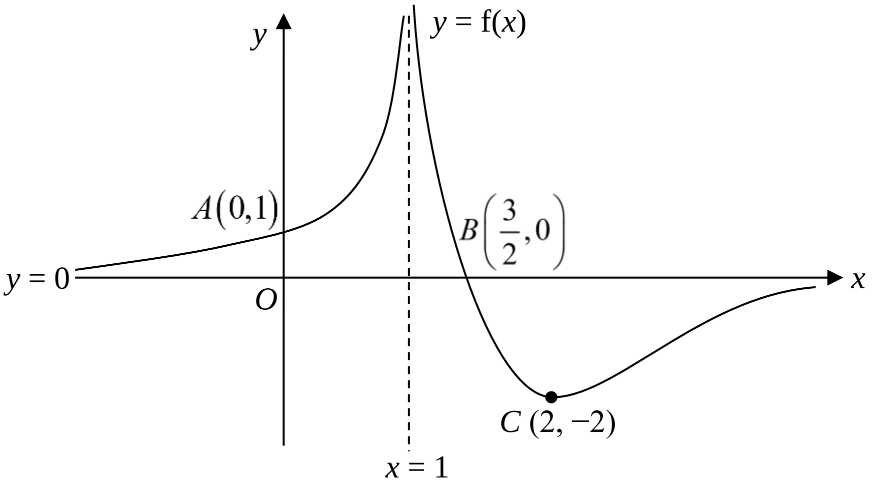
To get $y = \mathrm{f}\left(\frac{x}{a}\right)$ from the graph of $y = \mathrm{f}(x)$, where $a > 0$,

- Replace *x* by $\frac{x}{a}$.
- $y = \mathrm{f}(x) \longrightarrow y = \mathrm{f}\left(\frac{x}{a}\right)$
- The graph of $y = \mathrm{f}\left(\frac{x}{a}\right)$ is a **stretch** of the graph $y = \mathrm{f}(x)$ by **factor** a parallel to the ***x*-axis**.
- $(x_i, \mathrm{f}(x_i)) \longrightarrow (ax_i, \mathrm{f}(x_i))$

Example 3

The graph of $y = \mathrm{f}(x)$ is shown. On separate diagrams, sketch the graphs of

(i) $y = \frac{1}{3}\mathrm{f}(x)$ **(ii)** $y = \mathrm{f}(3x)$

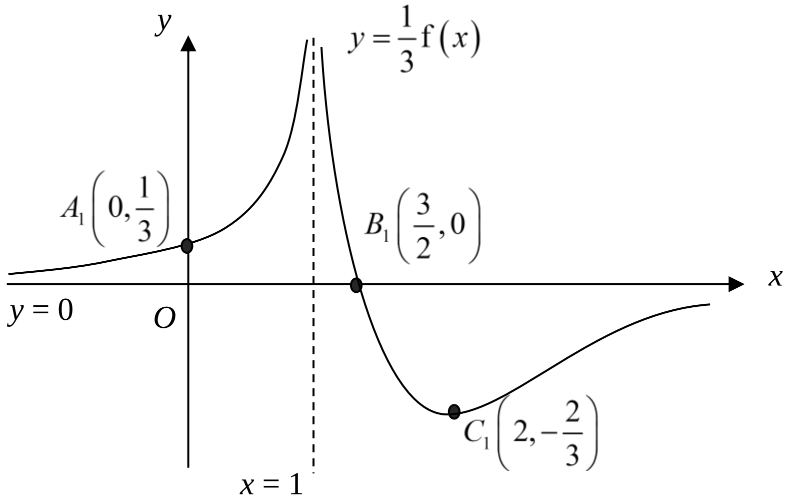


Solution:

(i) $y = \frac{1}{3}\mathrm{f}(x)$

Replace y by $\frac{y}{\left(\frac{1}{3}\right)}$

Notice that all y-values are multiplied by $\frac{1}{3}$.

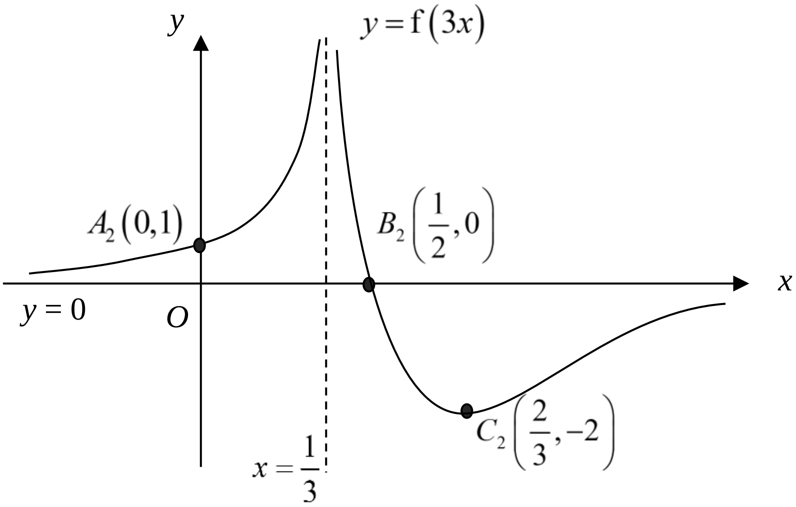


Note: For stretching parallel to the y-axis, vertical asymptotes remain unchanged and horizontal asymptotes are stretched accordingly.

(ii) $y = \mathrm{f}(3x)$

Replace x by $\frac{x}{\left(\frac{1}{3}\right)}$

Notice that all x-values are multiplied by $\frac{1}{3}$.

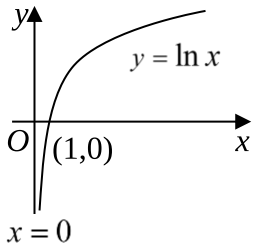
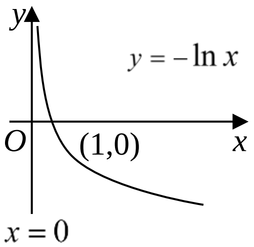


Note: For stretching parallel to the x-axis, horizontal asymptotes remain unchanged and vertical asymptotes are stretched accordingly.

§3 Reflection

3.1 Reflection in the *x*-axis

Let $y=\mathrm{f}(x)=\ln x$. Using GC, sketch the graphs of $y=-\mathrm{f}\left(x\right)=-\ln x$. What do you think is the relationship between the two graphs?

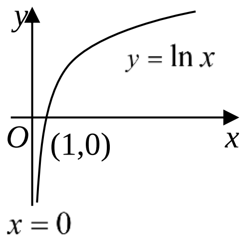
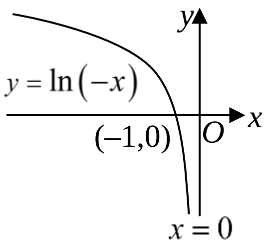
$y=\mathrm{f}(x)$	Transformed equation $y=-\mathrm{f}\left(x\right)$	Replacement	Geometrical description of transformation
$y=\ln x$ 	$y=-\ln x$ 	Replace y by $-y$	**<u>Reflection</u>** of the graph $y=\ln x$ in the **<u>*x*-axis</u>**. i.e. replace y by $-y$, thus we multiply -1 to all the original y-coordinates.

To get $y=-\mathrm{f}\left(x\right)$ from the graph of $y=\mathrm{f}\left(x\right)$,

- Replace y by $-y$
- $y=\mathrm{f}\left(x\right)\longrightarrow y=-\mathrm{f}\left(x\right)$
- The graph of $y=-\mathrm{f}(x)$ is a **reflection** of the graph $y=\mathrm{f}(x)$ in the ***x*-axis**.
- $\left(x_i,\mathrm{f}\left(x_i\right)\right)\longrightarrow\left(x_i,-\mathrm{f}\left(x_i\right)\right)$

3.2 Reflection in the *y*-axis

Let $y=\mathrm{f}(x)=\ln x$. Using GC, sketch the graphs of $y=\mathrm{f}\left(-x\right)=\ln\left(-x\right)$. What do you think the relationship between the two graphs is?

$y=\mathrm{f}(x)$	Transformed equation $y=\mathrm{f}\left(-x\right)$	Replacement	Geometrical description of transformation
$y=\ln x$ 	$y=\ln(-x)$ 	Replace x by $-x$	**<u>Reflection</u>** of the graph $y=\ln x$ in the **<u>*y*-axis</u>**. i.e. replace x by $-x$, thus we multiply -1 to all the original x-coordinates.

To get $y=\mathrm{f}\left(-x\right)$ from the graph of $y=\mathrm{f}\left(x\right)$,

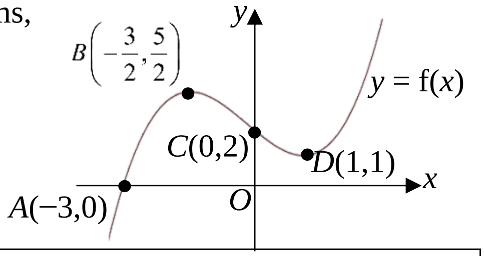
- Replace x by $-x$
- $y=\mathrm{f}(x)\longrightarrow y=\mathrm{f}(-x)$
- The graph of $y=\mathrm{f}(-x)$ is a **reflection** of the graph $y=\mathrm{f}(x)$ in the ***y*-axis**.
- $\left(x_i,\mathrm{f}\left(x_i\right)\right)\longrightarrow\left(-x_i,\mathrm{f}\left(x_i\right)\right)$

Example 4

Given the graph of $y = \mathrm{f}(x)$, sketch, on separate diagrams, the graphs of

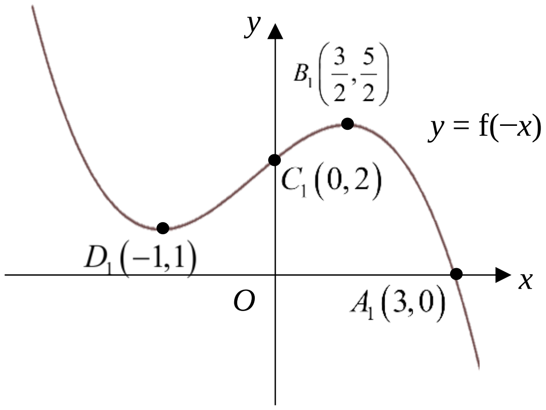
(i) $y = \mathrm{f}(-x)$,

(ii) $y = -\mathrm{f}(x)$.

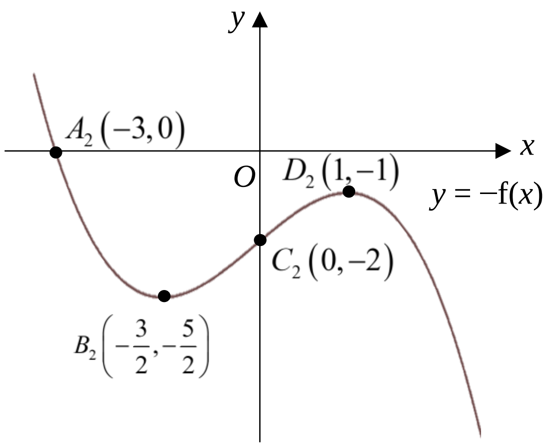


Solution:

(i) $y = \mathrm{f}(-x)$ $\boxed{\text{Replace } x \text{ by } -x}$



(ii) $y = -\mathrm{f}(x)$ $\boxed{\text{Replace } y \text{ by } -y}$



Example 5 (Combining, and Determining Sequence of, Basic Transformations)
2011/MJC Prelim/I/10a

State a sequence of transformations which transform the graph of $x^2+y^2=1$ to the graph of $(x-1)^2+y^2=4$.

Solution:

$(x-1)^2+y^2=4$

$\Rightarrow\left(\frac{x-1}{2}\right)^2+\left(\frac{y}{2}\right)^2=1$

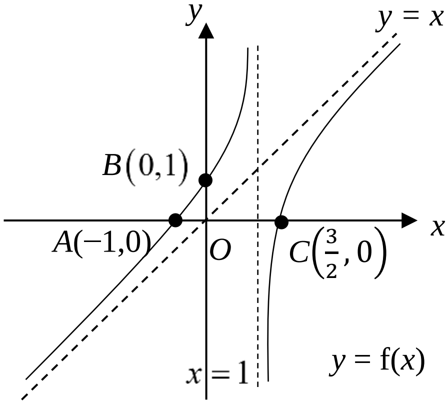
Method 1 (Stretch, Translate, Stretch)	**Method 2 (Translate, Stretch, Stretch)**
$x^2+y^2=1$ $\downarrow$ Replace x by $\frac{x}{2}$ $\left(\frac{x}{2}\right)^2+y^2=1$ $\downarrow$ Replace x by $x-1$ $\left(\frac{x-1}{2}\right)^2+y^2=1$ $\downarrow$ Replace y by $\frac{y}{2}$ $\left(\frac{x-1}{2}\right)^2+\left(\frac{y}{2}\right)^2=1$ $\Rightarrow(x-1)^2+y^2=4$ The transformations are in the following order: (1) Stretch by factor 2 parallel to the x-axis. (2) Translation of 1 unit in the positive x-direction. (3) Stretch by factor 2 parallel to the y-axis.	$x^2+y^2=1$ $\downarrow$ Replace x by $x-\frac{1}{2}$ $\left(x-\frac{1}{2}\right)^2+y^2=1$ $\downarrow$ Replace x by $\frac{x}{2}$ $\left(\frac{x}{2}-\frac{1}{2}\right)^2+y^2=1$ $\downarrow$ Replace y by $\frac{y}{2}$ $\left(\frac{x-1}{2}\right)^2+\left(\frac{y}{2}\right)^2=1$ $\Rightarrow(x-1)^2+y^2=4$ The transformations are in the following order: (1) Translation of $\frac{1}{2}$ units in the positive x-direction. (2) Stretch by factor 2 parallel to the x-axis. (3) Stretch by factor 2 parallel to the y-axis.

There are usually many ways to combine basic transformations to achieve the final expression, Use appropriate replacements (e.g. in Method 1, Replace x by $\frac{x}{2}$, Replace x by $x-1$ then Replace y by $\frac{y}{2}$) to check that the suggested combination of basic transformations indeed works.

Example 6
(Sketching graphs involving combination of Basic Transformations)

The diagram shows the graph of $y = \mathrm{f}(x)$.

Sketch the graph of $y = 2\mathrm{f}(-x)+1$.



Solution:

$y = \mathrm{f}(x)$

$\downarrow$ (1) Replace x by $-x$ — Reflection in the y-axis

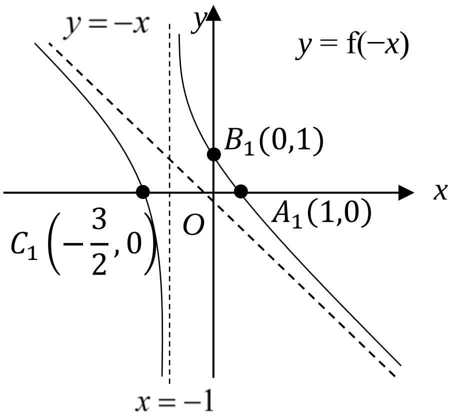
$y = \mathrm{f}(-x)$

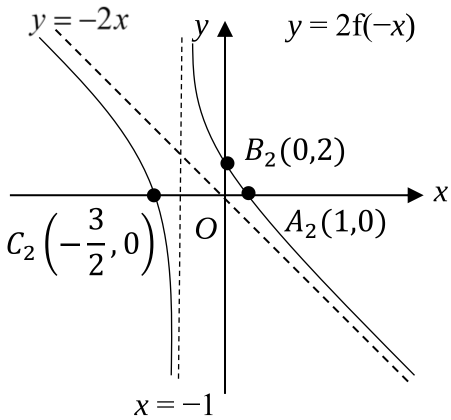
$\downarrow$ (2) Replace y by $\frac{y}{2}$ — Stretch of factor 2 parallel to the y-axis

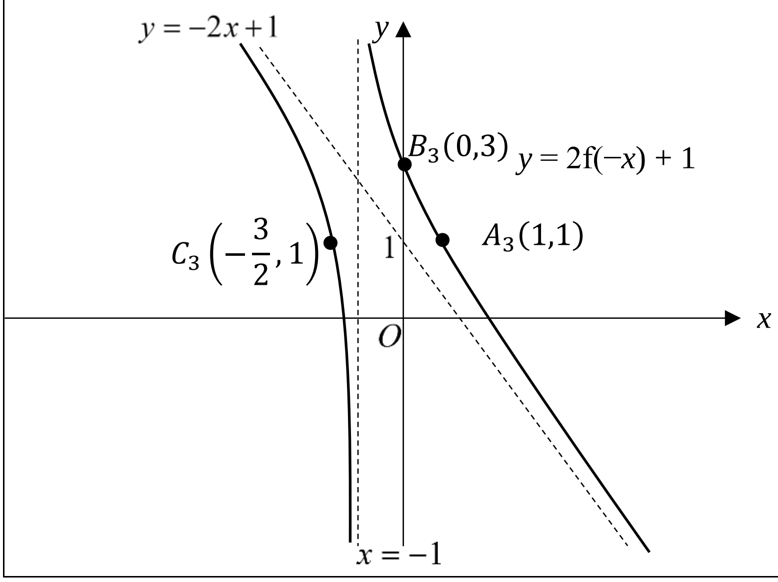
$\frac{y}{2} = \mathrm{f}(-x) \Rightarrow y = 2\mathrm{f}(-x)$

$\downarrow$ (3) Replace y by $y-1$ — Translation of 1 unit in the positive y-direction

$y-1 = 2\mathrm{f}(-x) \Rightarrow y = 2\mathrm{f}(-x)+1$

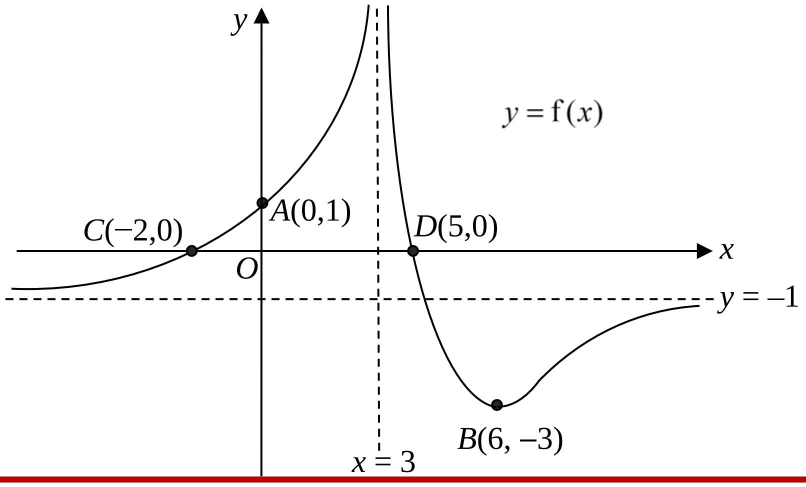






Example 7 (Sketching graphs involving combination of Basic Transformations)

Given the graph of $y = \mathrm{f}(x)$, sketch the graph of $y = \mathrm{f}(2x+1)$. You should show the coordinates of the points corresponding to the points *A*, *B*, *C* and *D*, and the asymptotes.



First determine the correct series of Basic Transformations, using replacement to check. Next, you can try to sketch intermediate graph(s) of the corresponding Basic Transformation before reaching the required final graph. Ensure that all relevant points and asymptotes are labelled in the graph(s).

Solution:

Method 1: Translation followed by Stretch

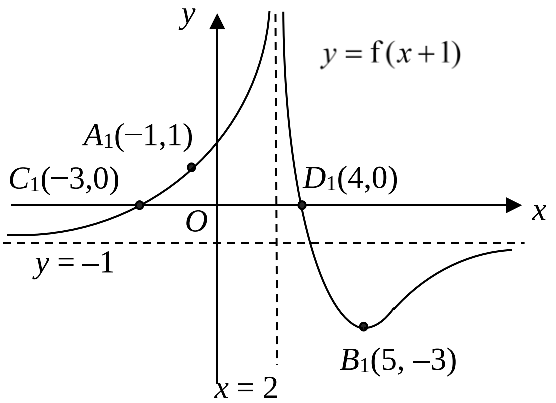
$y = \mathrm{f}(x)$

$\downarrow$ (1) Replace x by $x+1$ — Translation of 1 unit in the negative x-direction

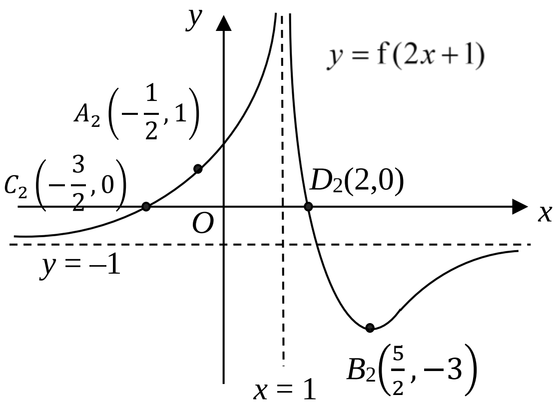
$y = \mathrm{f}(x+1)$

$\downarrow$ (2) Replace x by $2x = \dfrac{x}{\left(\frac{1}{2}\right)}$ — Stretch of factor $\dfrac{1}{2}$ parallel to x-axis

$y = \mathrm{f}(2x+1)$



(1) Replace x by $x + 1$



(2) Replace x by $2x = \dfrac{x}{\left(\frac{1}{2}\right)}$

Method 2: Stretch followed by Translation

$y = \mathrm{f}(x)$

$\downarrow$ (1) Replace x by $2x = \dfrac{x}{\left(\frac{1}{2}\right)}$ ⟸ Stretch of factor $\frac{1}{2}$ parallel to x-axis

$y = \mathrm{f}(2x)$

$\downarrow$ (2) Replace x by $x + \frac{1}{2}$ ⟸ Translation of $\frac{1}{2}$ unit in the negative x-direction

$y = \mathrm{f}(2x+1)$

Example 8 (Determine resulting expression after series of Basic Transformations)

A curve $y = \ln x$ undergoes the following transformations *A*, *B*, *C*, *D*, *E* in succession.

A: A translation of magnitude 2 units in the negative x-direction.
B: A stretch of factor 2 parallel to the x-axis.
C: A reflection in the y-axis.
D: A reflection in the x-axis.
E: A stretch of factor $\frac{1}{2}$ parallel to the y-axis.

Determine the equation of the resulting curve.

Solution:

$y = \ln(x)$

$\downarrow A$: Replace x by $x + 2$

$y = \ln(x+2)$

$\downarrow B$: Replace x by $\dfrac{x}{2}$

$y = \ln\left(\dfrac{x}{2}+2\right)$

$\downarrow C$: Replace x by $-x$

$y = \ln\left(\dfrac{-x}{2}+2\right)$

$\downarrow D$: Replace y by $-y$

$-y = \ln\left(\dfrac{-x}{2}+2\right) \Rightarrow y = -\ln\left(\dfrac{-x}{2}+2\right)$

$\downarrow E$: Replace y by $\dfrac{y}{\frac{1}{2}} = 2y$

$2y = -\ln\left(\dfrac{-x}{2}+2\right) \Rightarrow y = -\dfrac{1}{2}\ln\left(\dfrac{-x}{2}+2\right)$

Equation of the resulting curve: $y = -\dfrac{1}{2}\ln\left(\dfrac{-x}{2}+2\right)$.

For each Basic Transformation, just use the corresponding Replacement of variables, step by step.

Example 9 (Reversing Transformations to determine original expression) [2014/MJC Promo/4]

A curve undergoes, in succession, the following transformations:
A: A reflection in the *y*-axis
B: A translation of 3 units in the negative *y*-direction
C: A translation of 4 units in the positive *x*-direction
D: A scaling parallel to the *x*-axis by a factor of 2

The equation of the resulting curve is $y=\sqrt{1-\frac{x^2}{4}}$.

Determine the equation of the original curve before the transformations were carried out.

Solution:

$y=\sqrt{1-\frac{x^2}{4}}$

D': Replace x by $\frac{x}{\left(\frac{1}{2}\right)}=2x$

$y=\sqrt{1-\frac{(2x)^2}{4}}=\sqrt{1-x^2}$

C': Replace x by $(x+4)$

$y=\sqrt{1-(x+4)^2}$

B': Replace y by $(y-3)$

$y=\sqrt{1-(x+4)^2}+3$

A': Replace x by $(-x)$

$y=\sqrt{1-(-x+4)^2}+3$

Strategy: We use the "last in, first out" idea. So since *D* is that last transformation, we will find the reverse transformation D' first, followed by C', B' then A'.

D':
Scaling parallel to the *x*-axis by a factor of $\frac{1}{2}$

C':
Translation of 4 units in the negative *x*-direction

B':
Translation of 3 units in the positive *y*-direction

A':
Reflection in the *y*-axis
(**Idea:** To reverse *A*: Reflection in the *y*-axis, we should just Reflection in the *y*-axis again.)

Equation of the original curve: $y=\sqrt{1-(4-x)^2}+3$.

§4 Graphs of $y=|\mathrm{f}(x)|$ and $y=\mathrm{f}(|x|)$

4.1 The sketch of $y=|\mathrm{f}(x)|$

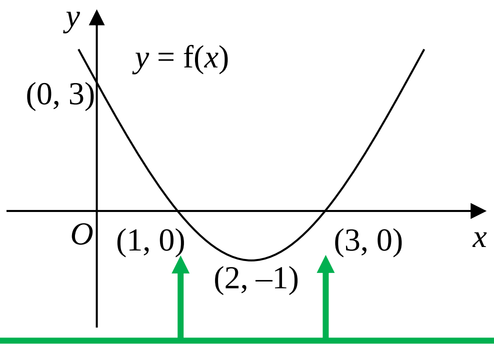
Recall the definition $|x|=\begin{cases} x & \text{, if } x\geq 0 \\ -x & \text{, if } x<0 \end{cases}$

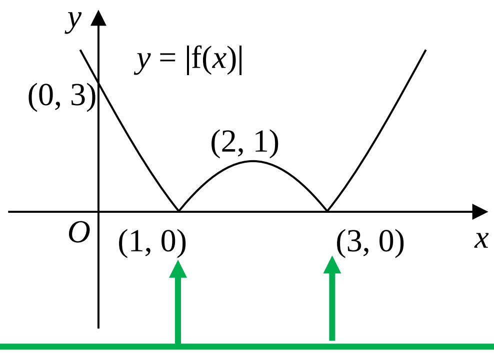
Thus we have $\boxed{y=|\mathrm{f}(x)|=\begin{cases} \mathrm{f}(x) & \text{, if } \mathrm{f}(x)\geq 0 \\ -\mathrm{f}(x) & \text{, if } \mathrm{f}(x)<0 \end{cases}}$

Example 10

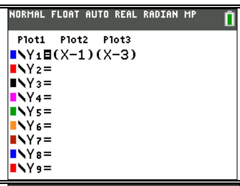
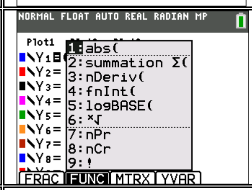
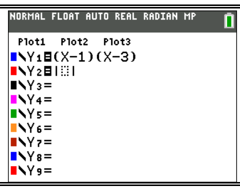
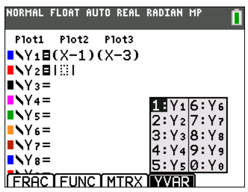
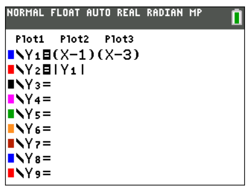
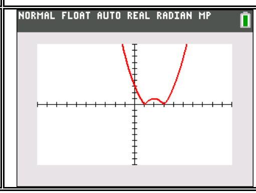
Sketch the graphs of $y=\mathrm{f}(x)=(x-1)(x-3)$ and $y=|\mathrm{f}(x)|=|(x-1)(x-3)|$ on separate diagrams.

Solution:





Note the gradient of $y = \mathrm{f}(x)$ graph at the x-axis, so these parts of $y = |\mathrm{f}(x)|$ graph must be "sharp".

GC Skills	
Step 1: Press [Y=]. Key in f(x) for Y_1. Press [ENTER].	
Step 2: For Y_2, **(i)** Press [ALPHA][WINDOW] **(ii)** Select **1:abs(**	 
Step 3: **(i)** Press [ALPHA][TRACE] **(ii)** Select $\mathbf{Y_1}$, and press [ENTER].	 
Step 4: Press [GRAPH]. (Use the zoom commands, if necessary)	 $y=\lvert\mathrm{f}(x)\rvert=\lvert(x-1)(x-3)\rvert$

A procedure to sketch $y = |\mathrm{f}(x)|$ from the graph of $y = \mathrm{f}(x)$:

Step 1: Retain the portion above the x-axis (where $y \geq 0$) of the original graph of $y = \mathrm{f}(x)$.

Step 2: Reflect the portion below the x-axis (where $y < 0$) in the x-axis.

Note: For the graph of $y = |\mathrm{f}(x)|$, $y \geq 0$ for all x.

4.2 The sketch of $y = \mathrm{f}(|x|)$

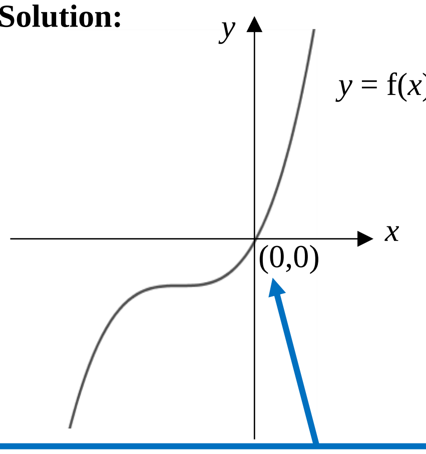
Recall the definition
$$|x| = \begin{cases} x & \text{, if } x \geq 0 \\ -x & \text{, if } x < 0 \end{cases}$$

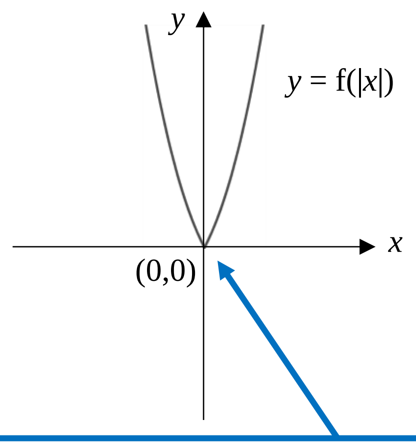
Thus we have
$$\boxed{y = \mathrm{f}(|x|) = \begin{cases} \mathrm{f}(x) & \text{, if } x \geq 0 \\ \mathrm{f}(-x) & \text{, if } x < 0 \end{cases}}$$

Example 11

Sketch the graphs of $y = \mathrm{f}(x) = (x+1)^3 - 1$ and $y = \mathrm{f}(|x|) = (|x|+1)^3 - 1$ on separate diagrams.

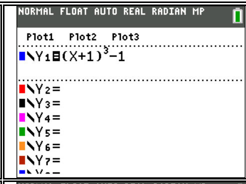
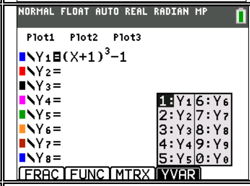
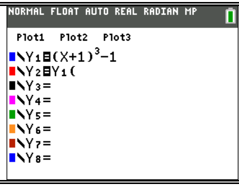
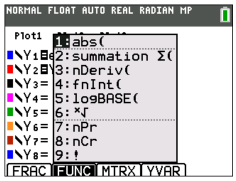
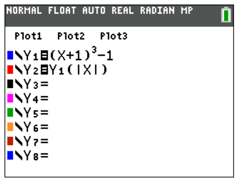
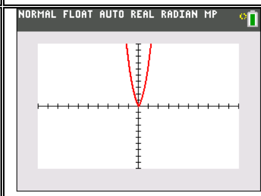
Solution:





Note the gradient of $y = \mathrm{f}(x)$ graph at the y-axis, so this part of $y = \mathrm{f}(|x|)$ graph must be "sharp".

Observe: The graph of $y = \mathrm{f}(|x|)$ is symmetrical about the y-axis.

GC Skills	
Step 1: Press [Y=]. Key in f(x) for Y_1. Press [ENTER].	
Step 2: For Y_2, **(i)** Press [ALPHA] [TRACE] **(ii)** Select **Y_1** and [( ]	 
Step 3: **(i)** Press [ALPHA] [WINDOW] **(ii)** Select **1:abs(** , [X,T,Θ,*n*], [▶], [ )] and [ENTER].	 
Step 4: Press [GRAPH]. (Use the zoom commands, if necessary)	 $y=\mathrm{f}\left(\lvert x\rvert\right)=\left(\lvert x\rvert+1\right)^3-1$

A procedure to sketch $y=\mathrm{f}\left(\lvert x\rvert\right)$ from the graph of $y=\mathrm{f}(x)$:

<u>Step 1</u>: Retain the portion of $y=\mathrm{f}(x)$ for $x\geq 0$.

<u>Step 2</u>: Duplicate and reflect the retained portion in the *y*-axis.

Note: The graph of $y=\mathrm{f}\left(\lvert x\rvert\right)$ is always symmetrical about the *y*-axis.

Extension: Given the graph of $y=\mathrm{f}(x)$, how do you sketch the graph of $y=\mathrm{f}\left(-\lvert x\rvert\right)$?

It can be obtained by a sequence of 2 transformations:

1) Reflection of the graph of $y=\mathrm{f}(x)$ in the *y*-axis $\boxed{\text{Replace } x \text{ by } -x}$ followed by

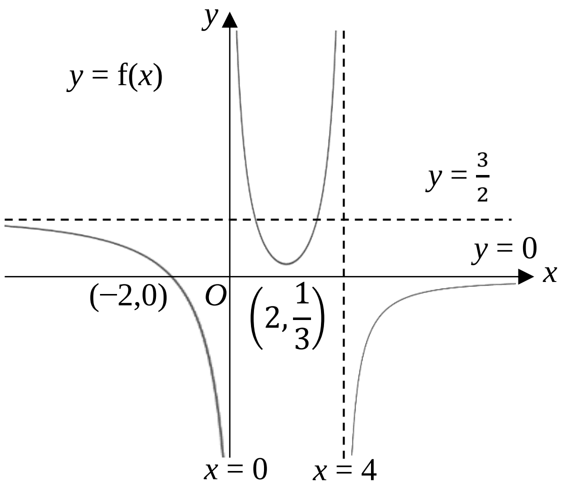
2) $\lvert x\rvert$ transformation on the resulting graph $\boxed{\text{Replace } x \text{ by } \lvert x\rvert}$

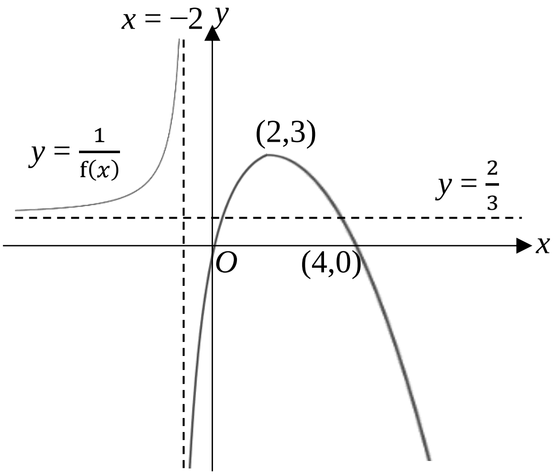
Note: The $\lvert x\rvert$ transformation cannot be done first as the desired graph of $y=\mathrm{f}\left(-\lvert x\rvert\right)$ will not be obtained $y=\mathrm{f}(x)\xrightarrow[\text{by } \lvert x\rvert]{\text{Replace } x} y=\mathrm{f}\left(\lvert x\rvert\right)\xrightarrow[\text{by } -x]{\text{Replace } x} y=\mathrm{f}\left(\lvert -x\rvert\right)=\mathrm{f}\left(\lvert x\rvert\right)$

§5 Graph of $y = \dfrac{1}{f(x)}$ (i.e reciprocal graph of $y = f(x)$)

Example 12

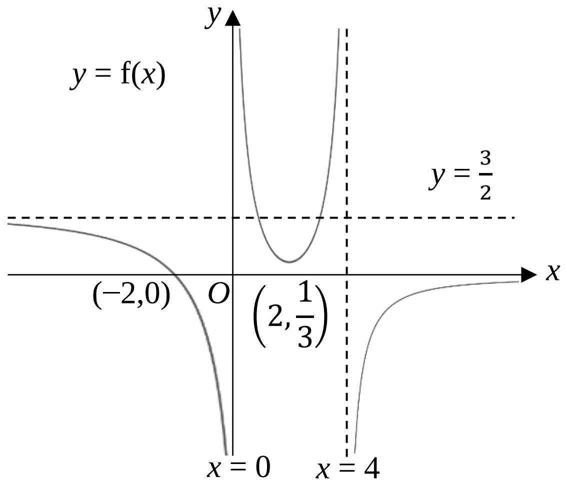
The diagrams show the graph of $y = f(x)$ and the graph of $y = \dfrac{1}{f(x)}$.

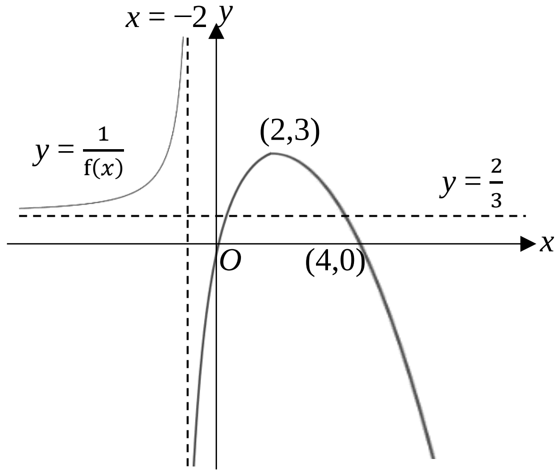




The main idea of reciprocal graph $y = \dfrac{1}{f(x)}$ is really to take reciprocal of the ***y*-values** from the graph of $y = f(x)$. So for example, the point $\left(2, \dfrac{1}{3}\right)$ in the graph of $y = f(x)$ becomes the point $(2,3)$ in the graph of $y = \dfrac{1}{f(x)}$.

Taking reciprocal retains the sign. For example, reciprocal of 1000 is $\dfrac{1}{1000}$ and the reciprocal of $-\dfrac{1}{500}$ is -500. From these two examples, you can also notice that reciprocal of a big number gives a small number, and vice versa.





Observe both graphs and notice the changes to the following features after transforming the graph $y=\mathrm{f}(x)$ to the graph of $y=\dfrac{1}{\mathrm{f}(x)}$.

	Features on $y=\mathrm{f}(x)$	**Features on** $y=\dfrac{1}{\mathrm{f}(x)}$
1	Minimum point $\left(2,\dfrac{1}{3}\right)$	Becomes the maximum turning point $(2,3)$
2	Horizontal asymptote $y=\dfrac{3}{2}$	Becomes the horizontal asymptote $y=\dfrac{2}{3}$
3	As the y-values in $y=\mathrm{f}(x)$ increases (the curve rises)	y-values in $y=\dfrac{1}{\mathrm{f}(x)}$ decrease (the curve drops)
4	As the y-values in $y=\mathrm{f}(x)$ decreases (the curve drops)	y-values in $y=\dfrac{1}{\mathrm{f}(x)}$ increases (the curve rises)
5	As the y-values in $y=\mathrm{f}(x)$ approaches $+\infty$	y-values in $y=\dfrac{1}{\mathrm{f}(x)}$ approaches 0^+
6	As the y-values in $y=\mathrm{f}(x)$ approaches $-\infty$	y-values in $y=\dfrac{1}{\mathrm{f}(x)}$ approaches 0^-
7	Horizontal asymptote $y=0$ (Observe the part of the curve tending towards $y=0$, on the right.)	This part of the curve now tends towards $-\infty$.
8	Vertical asymptotes $x=0$ and $x=4$	Corresponds to x-intercepts $(0,0)$ and $(4,0)$ respectively.
9	x-intercept $(-2,0)$	Corresponds to the vertical asymptote $x=-2$

Table of changes for Reciprocal Graph

	$y=\mathrm{f}(x)$	$y=\dfrac{1}{\mathrm{f}(x)}$
Asymptotes	$x=a$ is a vertical asymptote i.e. $\mathrm{f}(a)$ is undefined	$\dfrac{1}{\mathrm{f}(a)}=0$ $\Rightarrow$ $(a,0)$ is an intercept
	$y=a,\ a\neq 0$ is a horizontal asymptote	$y=\dfrac{1}{a}$ is a horizontal asymptote
	y = 0 is a horizontal asymptote	$\dfrac{1}{\mathrm{f}(x)}\to\infty$ (if $\mathrm{f}(x)>0$) or $\dfrac{1}{\mathrm{f}(x)}\to-\infty$ (if $\mathrm{f}(x)<0$)
	$y=ax+b$ is an oblique asymptote	$y=0$ is a horizontal asymptote
Intercepts	x-intercept at $(a,0)$	$x=a$ is a vertical asymptote.
	y-intercept at $(0,b)$, $b\neq 0$	y-intercept at $\left(0,\dfrac{1}{b}\right)$
Max/Min points (if they do not lie on the *x*-axis)	Max. point $(a,b), b\neq 0$	Min. point $\left(a,\dfrac{1}{b}\right)$
	Min. point $(a,b), b\neq 0$	Max. point $\left(a,\dfrac{1}{b}\right)$
Shape – Behaviour of graph	$\mathrm{f}(x)\to\infty$	$\dfrac{1}{\mathrm{f}(x)}\to 0^{+}$ y = 0 is a horizontal asymptote
	$\mathrm{f}(x)\to-\infty$	$\dfrac{1}{\mathrm{f}(x)}\to 0^{-}$ y = 0 is a horizontal asymptote
Shape – *y*-values	$\mathrm{f}(x)>0$	$\dfrac{1}{\mathrm{f}(x)}>0$
	$\mathrm{f}(x)<0$	$\dfrac{1}{\mathrm{f}(x)}<0$
	$\mathrm{f}(x)$ increases	$\dfrac{1}{\mathrm{f}(x)}$ decreases
	$\mathrm{f}(x)$ decreases	$\dfrac{1}{\mathrm{f}(x)}$ increases

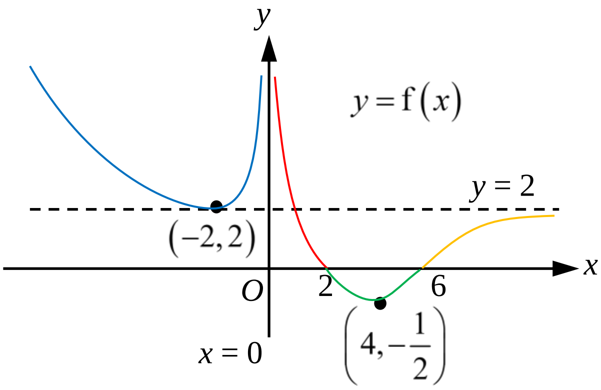
Note: In general, $\boxed{\dfrac{1}{\mathrm{f}(x)}\neq \mathrm{f}^{-1}(x)}$

Reciprocal   Inverse

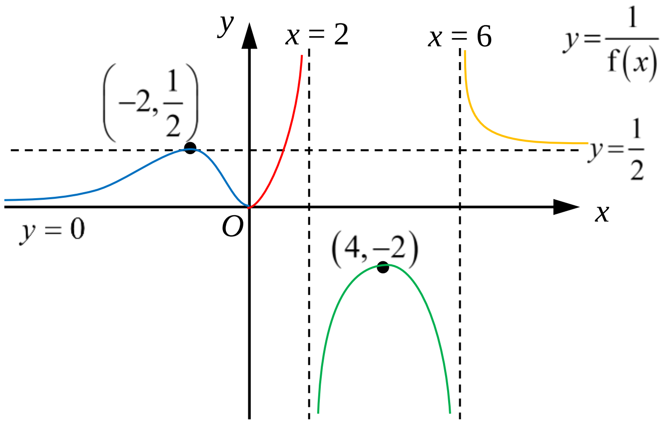
Example 13 [2015/ DHS Promo/4 (part)]

The diagram below shows the graph of $y = \mathrm{f}(x)$ with asymptotes $y = 2$ and $x = 0$.

The curve has turning points at (−2, 2) and $\left(4, -\dfrac{1}{2}\right)$. Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$.



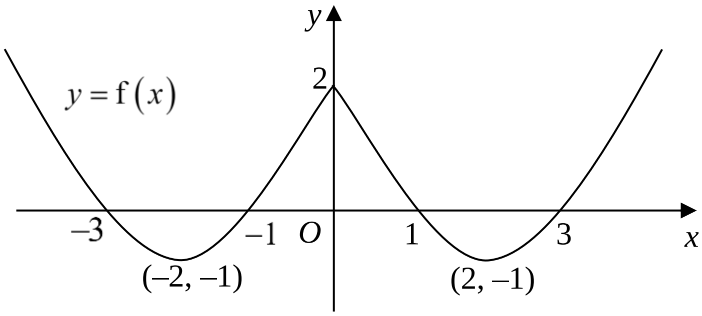
Solution:



H2 Mathematics (9758)
Chapter 2 Transformations of Curves
Discussion Questions

Level 1

1 Describe a single transformation that will change $y=\mathrm{f}(x)$ into each of the following functions. The diagram shows the graph of $y=\mathrm{f}(x)$.



On separate diagrams, sketch the graphs of the following, showing clearly the coordinates of intersection with the axes.

(a) $y=\mathrm{f}(x)-1$ **(b)** $y=\mathrm{f}(x-2)$ **(c)** $y=2\mathrm{f}(x)$ **(d)** $y=\mathrm{f}\left(\frac{x}{3}\right)$

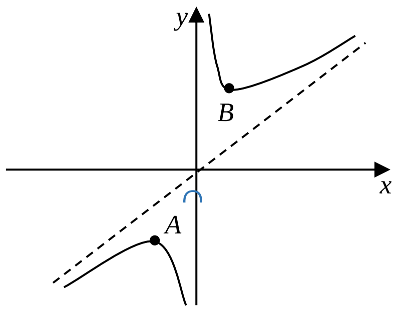
(e) $y=-\mathrm{f}(x)$ **(f)** $y=\mathrm{f}(-x)$ **(g)** $y=|\mathrm{f}(x)|$

Level 2

2 The diagram shows the graph of $y=\mathrm{f}(x)$ with asymptotes $x=0$ and $y=x$. The points A, B, O have coordinates $(-1,-2)$, $(1,2)$, $(0,0)$ respectively. Sketch on separate diagrams the graph of

(i) $y=3\mathrm{f}\left(\frac{1}{2}x\right)$, **(ii)** $y=2\mathrm{f}(x+4)$,

(iii) $y=\mathrm{f}(2x+1)$,



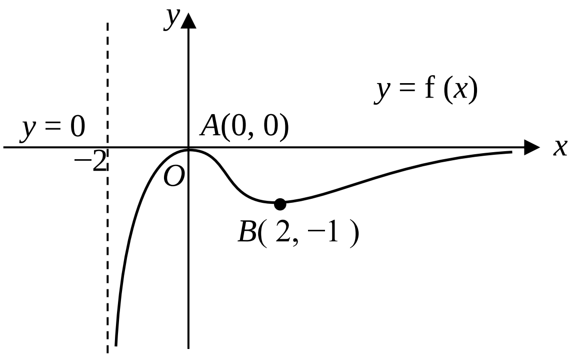
showing in each case the coordinates of the points corresponding to A, B and O and the equations of the asymptotes where applicable.

3 The diagram shows the graph of $y=\mathrm{f}(x)$. On separate diagrams, sketch the graphs of

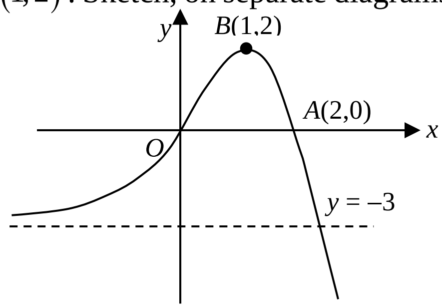
(i) $y=\mathrm{f}(2x+1)+1$,

(ii) $\dfrac{y}{3}=-\mathrm{f}\left(|x|\right)$,

labelling each graph clearly, showing the asymptotes (if any) and the coordinates of the points corresponding to *A* and *B*.



4 The diagram shows the graph of $y=\mathrm{f}\left(x\right)$. The curve crosses the *x*-axis at the origin *O* and the point $A\left(2,0\right)$, and has a maximum point at $B\left(1,2\right)$. Sketch, on separate diagrams, the graphs of



(i) $y=\left|\mathrm{f}\left(x\right)\right|$, **(ii)** $y=\mathrm{f}\left(-|x|\right)$,

(iii) $y=\dfrac{1}{\mathrm{f}\left(x\right)}$,

indicating in each case the coordinates of the axial intercepts and turning points, and the equation of the asymptotes where applicable.

5 **2016/Specimen Paper/I/2**

The curve *C* with equation $y=x^3$ is transformed onto the curve with equation $y=\mathrm{f}\left(x\right)$ by a translation of 2 units in the negative *x*-direction, followed by a stretch of factor $\dfrac{1}{2}$ parallel to the *y*-axis, followed by a translation of 1 unit in the positive *y*-direction.

(i) Write down the equation of the new curve. [1]

(ii) Sketch *C* and the curve with equation $y=\mathrm{f}\left(x\right)$ on the same diagram, stating the exact values of the coordinates of the points where $y=\mathrm{f}\left(x\right)$ crosses the *x*- and *y*-axes. Find the *x*-coordinate(s) of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

6 A curve $y = \mathrm{f}\left(x\right)$ undergoes, in succession, the following transformations.

A: A translation of magnitude 2 units in the positive *x*-direction.

B: A stretching parallel to the *x*-axis by a factor $\dfrac{1}{2}$.

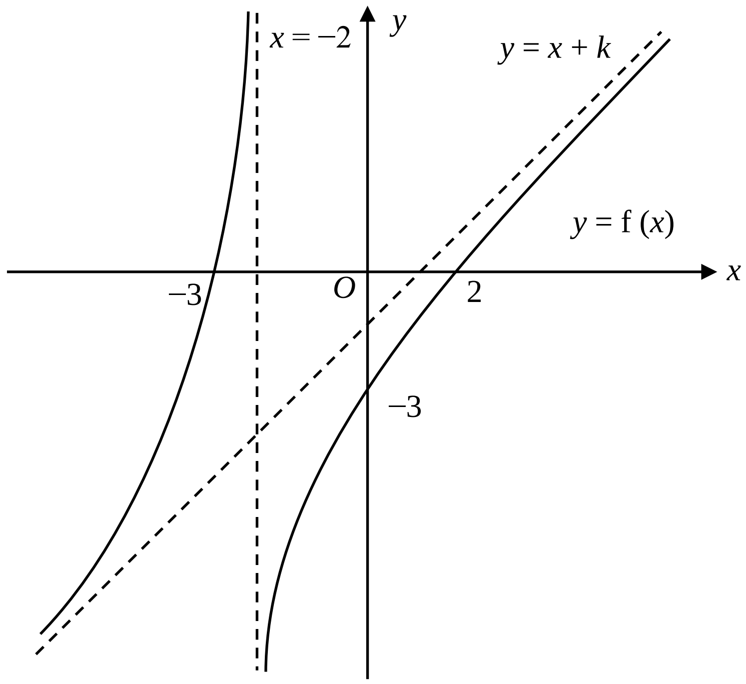
C: A reflection in the *y*-axis.

D: A translation of magnitude 1 unit in the positive *y*-direction.

The resulting curve has equation $y = 1 + \mathrm{e}^{2(x+1)}$. Determine the equation of the curve before the four transformations were effected.

7 **2013/SRJC Prelim/I/3 (Modified)**

The graph of the function $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = \dfrac{(x-a)(x+b)}{cx+d}$, $x \neq -\dfrac{d}{c}$, $a, b, c, d \in \mathbb{R}^+$, is shown below. The asymptotes are $y = x + k$ and $x = -2$ where k is a constant. The curve cuts the *x*-axis at –3 and 2 and the *y*-axis at -3.



(i) Find the values of *a*, *b*, *c*, *d* and *k*. [5]

(ii) Sketch on a separate diagram, the graph of $y = \dfrac{1}{\mathrm{f}(x)}$ [3]

Your sketch should clearly show any axial intercepts and equations of asymptotes.

8 **2015(9740)/I/5**

(i) State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = \frac{1}{4}(x-3)^2$. [3]

A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ \frac{1}{4}(x-3)^2 & \text{for } 1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Sketch the curve for $-1 \le x \le 4$. [3]

(iii) On a separate diagram, sketch the curve with equation $y = 1 + \mathrm{f}\left(\frac{1}{2}x\right)$, for $-1 \le x \le 4$. [2]

9 A curve C has equation $x^2 + 6x + 5 = \frac{(y-5)^2}{-4}$. Describe fully a sequence of three transformations which would transform C on to the curve $x^2 + y^2 = 4^2$.

<u>Level 3</u>

10 **2016(9740)/I/3**

The curve $y = x^4$ is transformed onto the curve with equation $y = \mathrm{f}(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = \mathrm{f}(x)$. The curve $y = \mathrm{f}(x)$ also passes through the point with coordinates $(0, c)$. Given that $\mathrm{f}(x)$ has the form $k(x-l)^4 + m$ and that a, b and c are positive constants with $c > b$, express k, l and m in terms of a, b and c. [2]

By sketching the curve $y = \mathrm{f}(x)$, or otherwise, sketch the curve $y = \frac{1}{\mathrm{f}(x)}$. State, in terms of a, b and c, the coordinates of any points where $y = \frac{1}{\mathrm{f}(x)}$ crosses the axes and of any turning points. [4]

H2 Mathematics (9758)
Chapter 2 Transformation of Curves
Extra Practice Questions

1 **2020/VJC Prelim/8b**

The transformations A, B and C are given as follows:

A: A translation of 3 units in the negative x-direction.

B: A reflection about the x-axis.

C: A stretch parallel to the y-axis with a stretch factor of 4.

A curve undergoes in succession, the transformations A, B and C and the equation of the resulting curve is $y=-\dfrac{4(x+3)}{x+2}$.

Determine the equation of the curve before the transformations were effected. [3]

2 **2020/CJC Prelim/1**

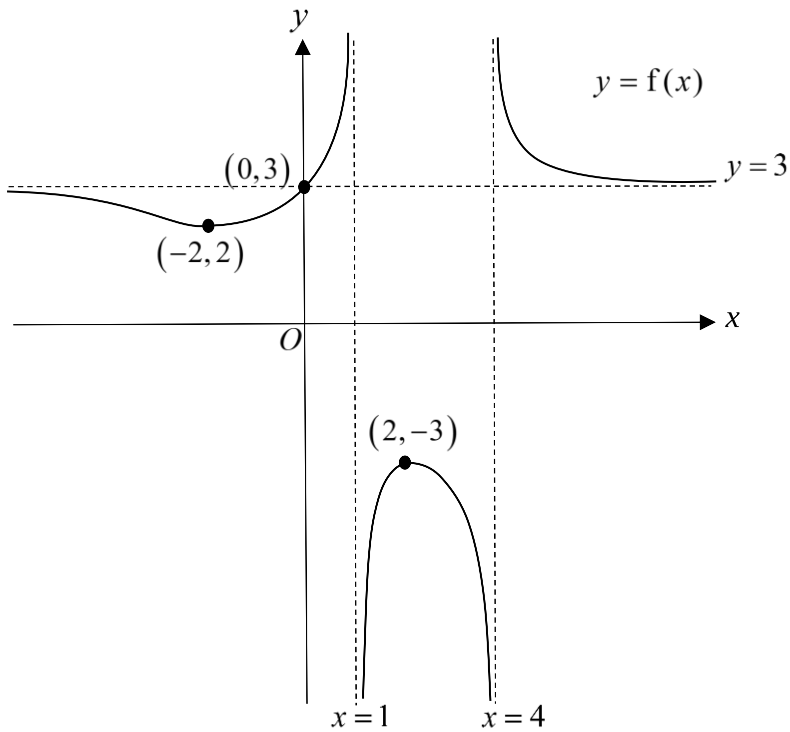
A function f is defined by $\mathrm{f}\left(x\right)=\dfrac{2x+3}{3x+5}$.

(i) Show that $\mathrm{f}\left(x\right)$ can be written in the form $a+\dfrac{b}{3x+5}$, where a and b are constants to be determined. [1]

(ii) Hence describe a sequence of transformations which transforms the graph of $y=\dfrac{1}{x}$ on to the graph of $y=\mathrm{f}\left(x\right)$. [4]

3 2020/RI Prelim/6

The diagram below shows the curve of $y = \mathrm{f}(x)$. The curve has a minimum point at $(-2, 2)$, a maximum point at $(2, -3)$ and cuts the *y*-axis at $(0, 3)$. The lines $x = 1$, $x = 4$ and $y = 3$ are the asymptotes to the curve.



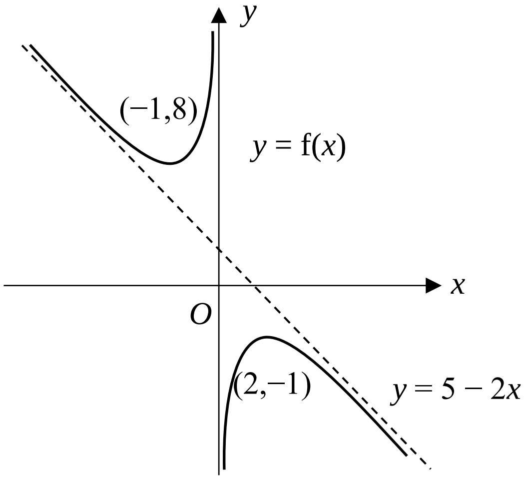
On separate diagrams, draw sketches of the following graphs, stating the exact coordinates of any turning points and/or points of intersection with the axes, and the equations of any asymptotes, where possible.

(a) $y = \mathrm{f}(1 - x)$. [3]

(b) $y = \dfrac{1}{\mathrm{f}(x)}$. [3]

4 2017/MI Promo/10 (modified)

The diagram shows the graph of $y = \mathrm{f}(x)$ with stationary points $(-1, 8)$ and $(2, -1)$. The line $y = 5 - 2x$ and the y-axis are asymptotes of the curve.



Sketch, on separate diagrams, the graphs of

(i) $y = |\mathrm{f}(x)|$, [3]

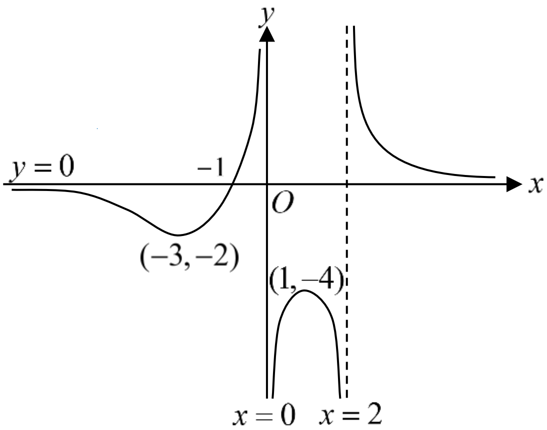
(ii) $y = \mathrm{f}(|x|)$, [3]

(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating clearly the equations of any asymptotes, as well as the coordinates of any axial intercepts and turning points.

5 2012/VJC Prelim/I/9(b) (modified)

The diagram below shows the graph of $y = \mathrm{h}(3 - x)$. On separate diagrams, sketch the graphs of



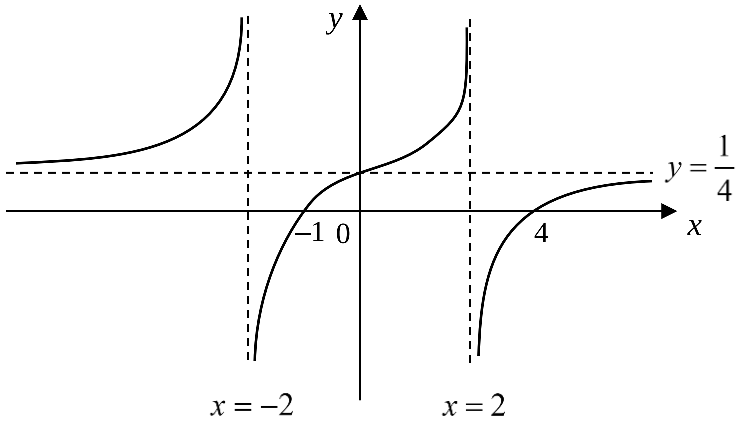
(i) $y = \mathrm{h}(x + 3)$, [2]

(ii) $y = \dfrac{1}{2}\mathrm{h}(x)$, [3]

(iii) $y = \mathrm{h}(-|x| + 3)$. [2]

6 2017/RVHS Promo/6a

The figure below shows a sketch of the curve with equation $y=\dfrac{1}{\mathrm{f}(x)}$. The equations of the vertical asymptotes are $x=-2$ and $x=2$, and horizontal asymptote $y=\dfrac{1}{4}$. It has x-intercepts at -1 and 4, and y-intercept at $\dfrac{1}{4}$.



On separate diagrams, show a sketch of the curve

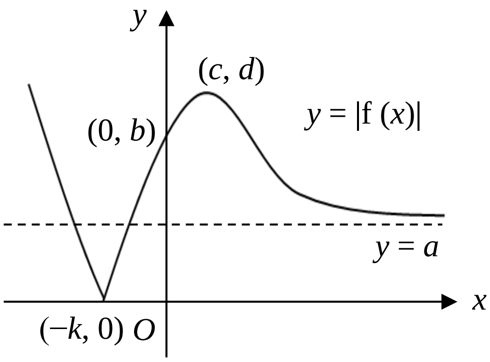
(i) $y=\mathrm{f}\left(x\right)$, [3]

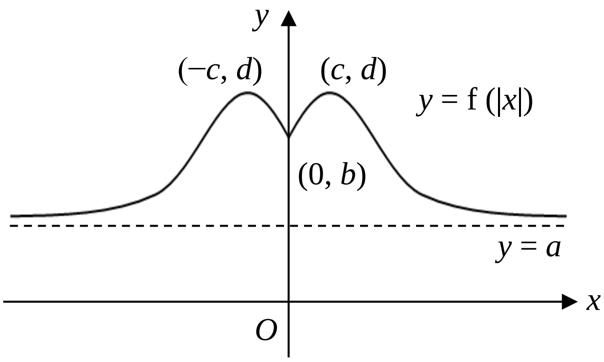
(ii) $2y=\dfrac{1}{\mathrm{f}\left(\left|x\right|\right)}$. [2]

Label clearly on each sketch, the equation of asymptote(s), axial intercept(s) and coordinates of turning point(s) (if any).

7 2018/MI Promo/2

The sketches below show the graphs of $y=\left|\mathrm{f}\left(x\right)\right|$ and $y=\mathrm{f}\left(\left|x\right|\right)$ for a certain function $y=\mathrm{f}\left(x\right)$, where $y=\mathrm{f}\left(x\right)$ is an increasing function for $x<c$ and a, b, c, d and k are positive constants.





Sketch the graph of $y=\mathrm{f}\left(x\right)$, showing clearly the coordinates of any turning points, axial intercepts and the equations of any asymptotes. [3]

8 2021/YIJC Prelim/2

A curve has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \begin{cases} \frac{1}{2}(x+5) & \text{for } x < -3, \\ 1 & \text{for } -3 \le x \le -1, \\ x^2 & \text{for } x > -1. \end{cases}$

(i) Sketch the curve for $-4 \le x \le 2$. [3]

(ii) On a separate diagram, sketch the curve with equation $y = \mathrm{f}(2x-1)$, for $-2 \le x \le \frac{1}{2}$. [2]

Answer Key

No	**Year**	**JC/CI**	**Answers**
1	**2020**	**VJC**	$y = \dfrac{x}{x-1}$
2	**2020**	**CJC**	(i) $\dfrac{2}{3} + \dfrac{-\frac{1}{3}}{3x+5}$