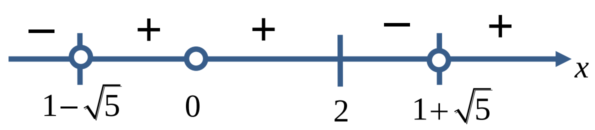
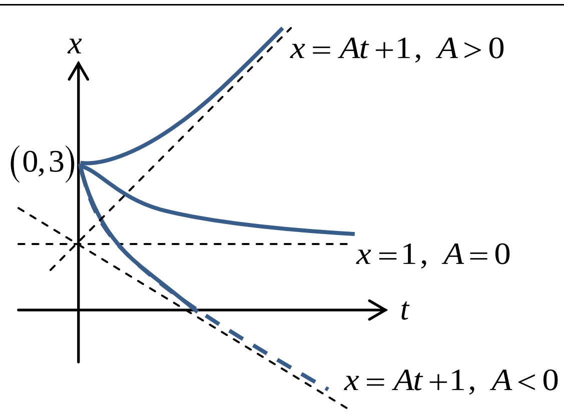
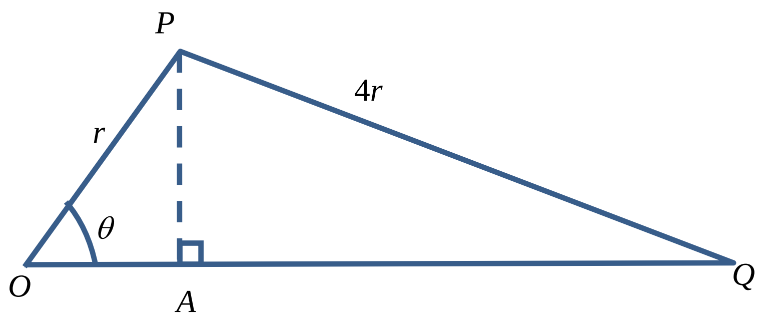
<u>**2015 NYJC JC2 Prelim Exam 9740/1 Solutions**</u>

Qn	Solutions
1	$\dfrac{4x}{x^2-2x-4} > -x$ $\Rightarrow \dfrac{4x}{x^2-2x-4} + x > 0$ $\Rightarrow \dfrac{x(x^2-2x-4)+4x}{x^2-2x-4} > 0$ $\Rightarrow \dfrac{x(x^2-2x)}{(x-1)^2-5} > 0$ $\Rightarrow \dfrac{x^2(x-2)}{(x-1-\sqrt{5})(x-1+\sqrt{5})} > 0$  $\Rightarrow 1-\sqrt{5} < x < 2$ or $x > 1+\sqrt{5}$ and $x \neq 0$
2	**(i)** $\dfrac{d^2x}{dt^2} = \dfrac{4}{(t+1)^3}$ $\dfrac{dx}{dt} = -\dfrac{2}{(t+1)^2} + A$ $x = \dfrac{2}{t+1} + At + B$ When $t = 0$, $x = 3$ $3 = 2 + B \quad \Rightarrow \quad B = 1$ Hence $x = \dfrac{2}{t+1} + At + 1$
	(ii) 

Qn	Solutions

3(i)



$OA = r\cos\theta$

$AP = r\sin\theta$

$OQ = x - r\cos\theta$

By Pythagoras Theorem,

$AP^2 + AQ^2 = PQ^2$

$\Rightarrow r^2\sin^2\theta + (x - r\cos\theta)^2 = 16r^2$

$\Rightarrow x - r\cos\theta = \sqrt{16r^2 - r^2\sin^2\theta}$ (since $x - r\cos\theta > 0$)

$\Rightarrow x = r\left(\cos\theta + \sqrt{16 - \sin^2\theta}\right)$

<u>Alternative solution</u>

Using Cosine rule,

$(4r)^2 = r^2 + x^2 - 2rx\cos\theta$

$x^2 - 2rx\cos\theta = 15r^2$

$(x - r\cos\theta)^2 - r^2\cos^2\theta = 15r^2$

$(x - r\cos\theta)^2 = 15r^2 + r^2\cos^2\theta$

$(x - r\cos\theta)^2 = 15r^2 + r^2\left(1 - \sin^2\theta\right)$

$(x - r\cos\theta)^2 = r^2\left(16 - \sin^2\theta\right)$

$x - r\cos\theta = r\sqrt{16 - \sin^2\theta}$ (Since $x - r\cos\theta > 0$)

$\therefore x = r\left(\cos\theta + \sqrt{16 - \sin^2\theta}\right)$

3(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$= r\left(-\sin\theta + \frac{1}{2}\left(16 - \sin^2\theta\right)^{-\frac{1}{2}}\left(-2\sin\theta\cos\theta\right)\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

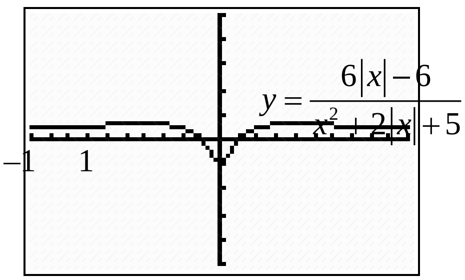
$$= -r\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{16 - \sin^2\theta}}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Qn	Solutions
	Given that $\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.5\text{ rad/s}$ and when $\theta=\frac{2\pi}{3}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=-r\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{\left(-\frac{1}{2}\right)}{\sqrt{16-\left(\frac{3}{4}\right)}}\right)\left(\frac{1}{2}\right)\approx -0.378r\text{ cm/s}$ P is moving towards O at a rate of $0.378r$ cm/s.
4	(a) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}\Rightarrow \mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{a}=\mathbf{0}\Rightarrow 2\mathbf{a}\times\mathbf{b}=\mathbf{0}\ \therefore\ \mathbf{a}\times\mathbf{b}$ is the zero vector. (a) Since the vector perpendicular to both $\mathbf{a}$ ($\overrightarrow{OA}$) and $\mathbf{b}$ ($\overrightarrow{OB}$) is also perpendicular to $\mathbf{c}$ $\overrightarrow{OC}$, $(\mathbf{a}\times\mathbf{b}).\mathbf{c}=\mathbf{0}$ implies that the four points are coplanar. Vector normal to plane ABC = $\begin{pmatrix}5\\-2\\1\end{pmatrix}$ Equation of plane ABC is $\mathbf{r}\cdot\begin{pmatrix}5\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}5\\-2\\1\end{pmatrix}=0$ Cartesian equation is $5x-2y+z=0$ Foot of perpendicular of P to plane ABC is given by $\overrightarrow{FP}=\left(\overrightarrow{OP}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}=\frac{\overrightarrow{OP}\cdot\begin{pmatrix}5\\-2\\1\end{pmatrix}}{\sqrt{30}}\frac{\begin{pmatrix}5\\-2\\1\end{pmatrix}}{\sqrt{30}}=\frac{1}{2}\begin{pmatrix}5\\-2\\1\end{pmatrix}$ $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$ $=\overrightarrow{OP}+2\overrightarrow{PF}$ $=\begin{pmatrix}2\\-1\\3\end{pmatrix}-\begin{pmatrix}5\\-2\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\2\end{pmatrix}$

Qn	Solutions
5(i)	$\mathrm{f}(r-2)=\dfrac{1}{2r-4+1}=\dfrac{1}{2r-3}$ $\mathrm{f}(r-2)-\mathrm{f}(r)=\dfrac{1}{2r-3}-\dfrac{1}{2r+1}=\dfrac{1+3}{(2r-3)(2r+1)}=\dfrac{4}{(2r-3)(2r+1)}$
5(ii)	$\dfrac{1}{1\times5}+\dfrac{1}{3\times7}+\dfrac{1}{5\times9}+\cdots=\dfrac{1}{4}\sum_{r=2}^{n+1}\left(\mathrm{f}(r-2)-\mathrm{f}(r)\right)$ $=\dfrac{1}{4}\big[\mathrm{f}(0)-\mathrm{f}(2)$ $+\mathrm{f}(1)-\mathrm{f}(3)$ $+\mathrm{f}(2)-\mathrm{f}(4)$ $\vdots$ $+\mathrm{f}(n-3)-\mathrm{f}(n-1)$ $+\mathrm{f}(n-2)-\mathrm{f}(n)$ $+\mathrm{f}(n-1)-\mathrm{f}(n+1)\big]$ $=\dfrac{1}{4}\left[\mathrm{f}(0)+\mathrm{f}(1)-\mathrm{f}(n)-\mathrm{f}(n+1)\right]$ $=\dfrac{1}{4}\left[1+\dfrac{1}{3}-\dfrac{1}{2n+1}-\dfrac{1}{2n+3}\right]$ $=\dfrac{1}{3}-\dfrac{1}{4}\left(\dfrac{2n+3+2n+1}{(2n+1)(2n+3)}\right)$ $=\dfrac{1}{3}-\dfrac{n+1}{(2n+1)(2n+3)}$
5(iii)	As $n\to\infty$, $\dfrac{n+1}{(2n+1)(2n+3)}\to 0$, therefore the series converges. The sum to infinity = $\dfrac{1}{3}$
5(iv)	$\dfrac{3}{5^2}+\dfrac{3}{7^2}+\dfrac{3}{9^2}+\cdots<\dfrac{3}{1\times5}+\dfrac{3}{3\times7}+\dfrac{3}{5\times9}+\cdots=3\left(\dfrac{1}{3}\right)=1$
6(i)	$R^2=(h-R)^2+r^2$ $R^2=h^2-2hR+R^2+r^2$ $\therefore r^2=2hR-h^2$ (shown)

Qn	Solutions
6(ii)	Let V be the volume of the cone. $V=\frac{1}{3}\pi r^2 h$ $=\frac{1}{3}\pi\left(2Rh-h^2\right)h=\frac{2}{3}\pi Rh^2-\frac{1}{3}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{4}{3}\pi Rh-\pi h^2$ For max. volume, set $\frac{\mathrm{d}V}{\mathrm{d}h}=0.$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{4}{3}\pi Rh-\pi h^2=0$ $h\left(\frac{4}{3}R-h\right)=0$ $h=0$ (rejected) or $h=\frac{4}{3}R$ When $h=\frac{4}{3}R$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2}=\frac{4}{3}\pi R-2\pi h$ $=\frac{4}{3}\pi R-2\pi\left(\frac{4}{3}R\right)=-\frac{4}{3}\pi R<0$ The volume of the cone is a maximum when $h=\frac{4}{3}R$.
6(iii)	$r^2=2hR-h^2$ $=2\left(\frac{4}{3}R\right)R-\left(\frac{4}{3}R\right)^2=\frac{8}{9}R^2$ $\therefore r=\frac{2\sqrt{2}}{3}R$ $(\because r>0)$ Ratio $\frac{r}{h}=\frac{\left(\frac{2\sqrt{2}}{3}R\right)}{\left(\frac{4}{3}R\right)}=\frac{\sqrt{2}}{2}$
7(i)	$\mathrm{f}'(0)=2,\quad a=\frac{\mathrm{f}''(0)}{2!}$ and $b=\frac{\mathrm{f}'''(0)}{3!}$ $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ---(1) When $x=0$, from (1), we have

Qn	Solutions
	$\mathrm{f}''(0)+2\mathrm{f}'(0)=0$ $\Rightarrow \mathrm{f}''(0)=-2\mathrm{f}'(0)=-2(2)=-4$ $\therefore a=-\dfrac{4}{2!}=-2$ (shown) Differentiate (1) w.r.t. x: $(1+2x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $(1+2x)\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+4\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ---(2) When $x=0$, from (1), we have $\mathrm{f}'''(0)+4\mathrm{f}''(0)=0$ $\Rightarrow \mathrm{f}'''(0)=-4\mathrm{f}''(0)=-4(-4)=16$ $\therefore b=\dfrac{16}{3!}=\dfrac{8}{3}$
7(ii)	$\dfrac{2x-2x^2+\frac{8}{3}x^3}{\sqrt[3]{8+x}}$ $=\left(2x-2x^2+\dfrac{8}{3}x^3\right)(8+x)^{-\frac{1}{3}}$ $=\left(2x-2x^2+\dfrac{8}{3}x^3\right)(8)^{-\frac{1}{3}}\left(1+\dfrac{x}{8}\right)^{-\frac{1}{3}}$ $=\dfrac{1}{2}\left(2x-2x^2+\dfrac{8}{3}x^3\right)\left(1+\left(-\dfrac{1}{3}\right)\left(\dfrac{x}{8}\right)+\dfrac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\dfrac{x}{8}\right)^2+\ldots\right)$ $=\left(x-x^2+\dfrac{4}{3}x^3\right)\left(1-\dfrac{1}{24}x+\dfrac{1}{288}x^2+\ldots\right)$ $\approx x-\dfrac{1}{24}x^2+\dfrac{1}{288}x^3-x^2+\dfrac{1}{24}x^3+\dfrac{4}{3}x^3$ $=x-\dfrac{25}{24}x^2+\dfrac{397}{288}x^3$
8	**(i)** $6x-6=3(2x+2)-12$ $A=3,\ B=-12$

Qn	Solutions
	$\int_0^1 \frac{6x-6}{x^2+2x+5}\,\mathrm{d}x = \int_0^1 \frac{3(2x+2)-12}{x^2+2x+5}\,\mathrm{d}x$ $= 3\int_0^1 \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x - 12\int_0^1 \frac{1}{(x+1)^2+2^2}\,\mathrm{d}x$ $= \left[3\ln\left\|x^2+2x+5\right\| - 6\tan^{-1}\left(\frac{x+1}{2}\right)\right]_0^1$ $= \left[3\ln 8 - 6\tan^{-1}1\right] - \left[3\ln 5 - 6\tan^{-1}\frac{1}{2}\right]$ $= 3\left(\ln\frac{8}{5} - \frac{\pi}{2} + 2\tan^{-1}\frac{1}{2}\right)$
	(ii)  Required area $= -\int_{-1}^1 \frac{6\|x\|-6}{x^2+2\|x\|+5}\,\mathrm{d}x$ $= -2\int_0^1 \frac{6x-6}{x^2+2x+5}\,\mathrm{d}x$ $= -6\left(\ln\frac{8}{5} - \frac{\pi}{2} + 2\tan^{-1}\frac{1}{2}\right)$
9(a)	(i) $x_5(x_{21}) = 4096 \Rightarrow ar^4\left(ar^{20}\right) = 4096 \Rightarrow a^2r^{24} = 4096\cdots(1)$ $ar^{12} = \sqrt{4096} = 64$ $\sum_{k=1}^{25} \log_4 x_k = \log_4 x_1 + \log_4 x_2 + \log_4 x_3 + ... + \log_4 x_{25}$ $= \log_4\left(x_1 x_2 x_3 ... x_{25}\right)$ $= \log_4\left(a(ar)\left(ar^2\right)...\left(ar^{24}\right)\right)$ $= \log_4\left(a^{25} r^{1+2+...+24}\right)$ $= \log_4\left(a^{25} r^{\frac{25(24)}{2}}\right)$ $= \log_4\left(a^{25} r^{25\times 12}\right)\cdots(2)$

Qn	Solutions
	$= \log_4 \left(ar^{12}\right)^{25} = 25\log_4\left(ar^{12}\right)$ $= 25\log_4(64) = 25\log_4\left(4^3\right)$ $= 25 \times 3 = 75$
	(ii) $y_n - y_{n-1} = \log_4(x_n) - \log_4(x_{n-1}) = \log_4 \dfrac{x_n}{x_{n-1}}$ $= \log_4 \dfrac{ar^{n-1}}{ar^{n-2}} \quad = \log_4 r$ is a constant free from n. Hence, $\{ y_n \}$ is an arithmetic sequence.
	(b) Let the amount of oil mined in the first year be a. The maximum total amount of oil mined $= a + 0.94a + 0.94^2 a + 0.94^3 a + ...$ $= \dfrac{a}{1-0.94}$ □ $16.66a < 17a$ Let n be the number of year at which the mine will be in operation.

$\dfrac{a(1-0.94^n)}{1-0.94} > 16a$	$\dfrac{(1-0.94^n)}{1-0.94} > 16$
$\Rightarrow \quad 0.94^n < 0.04$	
$\Rightarrow \quad n > \dfrac{\lg 0.04}{\lg 0.94} \approx 52.02$	
Smallest n is 53.	

n	$\dfrac{(1-0.94^n)}{1-0.94}$
52	15.999096
53	16.03915

The mine will be closed in the 53th year. Therefore, the mine will be closed in 2049.

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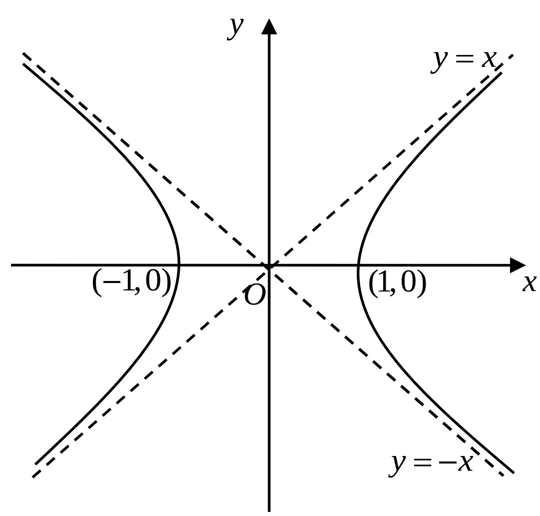

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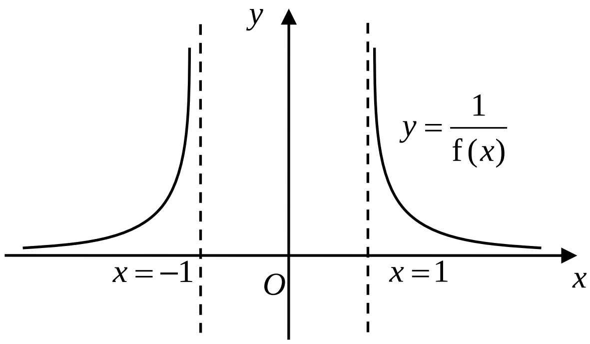
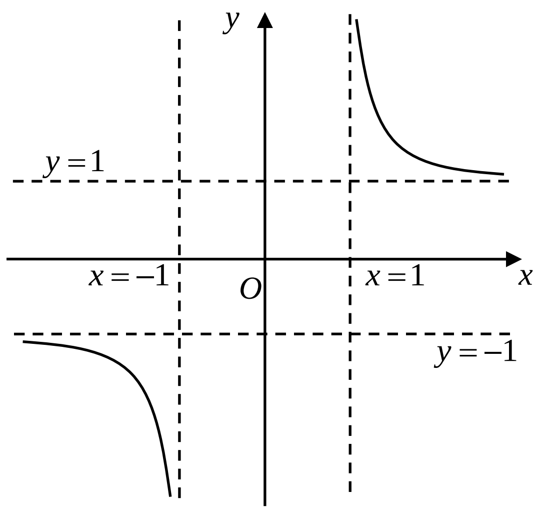
<tr><td>(i)(a)

(i)(b)

(ii)</td><td>




$$x^2-y^2=1 \xrightarrow{T_x \text{ by } 2} (x-2)^2-y^2=1$$
$$\xrightarrow{S_y \text{ by } \frac{1}{2}} (x-2)^2-\frac{y^2}{\left(\frac{1}{2}\right)^2}=1$$
$$\xrightarrow{T_y \text{ by } -6} (x-2)^2-\frac{(y+6)^2}{\left(\frac{1}{2}\right)^2}=1$$
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<tr><td>11(i)</td><td>

$w_1=e^{\frac{\pi i}{4}}, w_2=e^{\frac{\pi i}{8}}, w_3=e^{\frac{\pi i}{16}}, w_4=e^{\frac{\pi i}{32}}$
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<tr><td>11(ii)</td><td>

Note that $\theta_1=\frac{\pi}{4}$ and $\arg(w_{n+1})=\frac{1}{2}\arg(w_n)$. Thus $\theta_{n+1}=\frac{1}{2}\theta_n$.

Since $\frac{\theta_{n+1}}{\theta_n}=\frac{1}{2}$ for all $n\geq 1$, thus θ_n is a geometric sequence with common ratio $\frac{1}{2}$.

$\sum_{n=1}^{\infty}\theta_n=\frac{\theta_1}{1-\frac{1}{2}}=\frac{\pi}{2}$
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Qn	**Solutions**
11(iii)	By (i), $\lvert w_3\rvert=\lvert w_4\rvert=1$. Thus the origin satisfies the equation $\lvert z-w_3\rvert=\lvert z-w_4\rvert$. Thus the locus of points satisfying the equation $\lvert z-w_3\rvert=\lvert z-w_4\rvert$ passes through the origin. Let α be the angle between the line and the positive real axis. Since the perpendicular bisector is also the angle bisector, $\alpha=\frac{1}{2}\left(\frac{\pi}{16}+\frac{\pi}{32}\right)=\frac{3\pi}{64}$. Thus the exact Cartesian equation is $y=x\tan\left(\frac{3\pi}{64}\right)$.
12(a)	$\mathbf{r}\cdot(\mathbf{r}-\mathbf{a})=0\Rightarrow\overrightarrow{OP}\cdot\overrightarrow{AP}=0$ ie, $\angle OPA=90^\circ$ Therefore, the locus of P is a sphere with OA as diameter.
12(b)	Observe that $\begin{pmatrix}-3\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-5\\6\end{pmatrix}$ are not parallel since they are not a constant multiplier of one another. By equating l_1 and l_2, $\begin{pmatrix}-3\lambda\\7+2\lambda\\6+2\lambda\end{pmatrix}=\begin{pmatrix}-3+2\mu\\6-5\mu\\-4+6\mu\end{pmatrix}$, use the first two equations to solve for λ and μ and substitute into the third equation to show that there is no unique solutions for λ and μ. Therefore, the two lines are skew. Let $\overrightarrow{OP_1}=\begin{pmatrix}-3\lambda\\7+2\lambda\\6+2\lambda\end{pmatrix}$ and $\overrightarrow{OP_2}=\begin{pmatrix}-3+2\mu\\6-5\mu\\-4+6\mu\end{pmatrix}$ for particular values of λ and μ. Therefore, $\overrightarrow{P_1P_2}=\begin{pmatrix}-3+2\mu+3\lambda\\-1-5\mu-2\lambda\\-10+6\mu-2\lambda\end{pmatrix}$. Vector normal to l_1 and l_2 is $\begin{pmatrix}2\\2\\1\end{pmatrix}$. Since $\overrightarrow{P_1P_2}$ is perpendicular to both l_1 and l_2, we have $k\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}-3+2\mu+3\lambda\\-1-5\mu-2\lambda\\-10+6\mu-2\lambda\end{pmatrix}$

Qn	Solutions
	Solving for the SLE, $\begin{pmatrix} 2k-2\mu-3\lambda \\ 2k+5\mu+2\lambda \\ k-6\mu+2\lambda \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -10 \end{pmatrix}$, we have k = -2, λ = -1 and μ =1. Therefore, $\overrightarrow{OP_1} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\overrightarrow{OP_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. $P_1P_2 = \left\| -2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\| = 2\times\sqrt{2^2+2^2+1^2} = 6$ (AG)