2023 RVHS JC1 H2 FM Promo Exam (Solutions)

1	Solution [7]	
(a)	Let P_n be the statement "5^n-2^n is divisible by 3" for $n\in\mathbb{Z}^+$. When n = 1: $5^n-2^n=5-2=3$, which is divisible by 3. Hence, P_1 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e. 5^k-2^k is divisible by 3, or equivalently $5^k-2^k=3q$, where $q\in\mathbb{Z}$. Want to show P_{k+1} is true , i.e. $5^{k+1}-2^{k+1}$ is divisible by 3. $5^{k+1}-2^{k+1}=5\cdot5^k-2\cdot2^k$ $=5\left(5^k-2^k\right)+3\cdot2^k$ $=5\left(3q\right)+3\cdot2^k$ (by inductive hypothesis) $=3\left(5q+2^k\right)$, which is divisible by 3 since $5q+2^k\in\mathbb{Z}$. $\therefore$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.	
(b)	$2\left(5^{n+1}\right)-5\left(2^{n+1}\right)=2\left(5\cdot5^n\right)-5\left(2\cdot2^n\right)$ $=10\left(5^n-2^n\right)$ $=10\left(3p\right)$, where $p\in\mathbb{Z}$ $=30p$ Since any number in the set is of the form 30p, the greatest common divisor is **<u>30</u>**.	

2	Solution [5 marks]	
(i)	Let $\mathrm{f}(x)=\dfrac{2}{\sqrt{8x-x^2-7}}$. By Simpson's rule, $I\approx\left(\dfrac{0.25}{3}\right)\left[\mathrm{f}(4)+4\mathrm{f}(4.25)+2\mathrm{f}(4.5)+4\mathrm{f}(4.75)+\mathrm{f}(5)\right]$ ≈ 0.6796763482 $=0.6797\ (4\text{ d.p.})$	
(ii)	$I=\int_4^5\dfrac{2}{\sqrt{8x-x^2-7}}\,\mathrm{d}x$ $=2\int_4^5\dfrac{1}{\sqrt{9-(x-4)^2}}\,\mathrm{d}x$ $=2\left[\sin^{-1}\left(\dfrac{x-4}{3}\right)\right]_4^5$ $=2\sin^{-1}\left(\dfrac{1}{3}\right)$ $2\sin^{-1}\left(\dfrac{1}{3}\right)\approx 0.679676$ $\sin^{-1}\left(\dfrac{1}{3}\right)\approx 0.3398$	

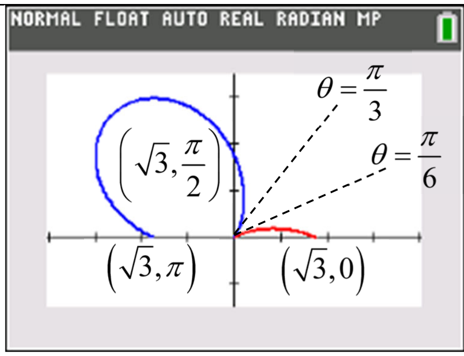
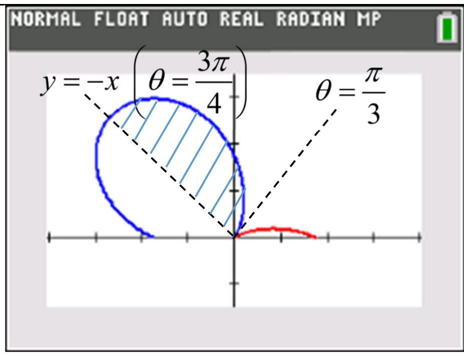
3	Solution [7 marks]	
	Characteristic equation: $m^2-4=0$ $m=\pm 2$ C.F. is $z_c = Ae^{2x}+Be^{-2x}$ Let P.I. be $z_p = kxe^{2x}+\alpha\cos 2x+\beta\sin 2x$ $z'_p = 2kxe^{2x}+ke^{2x}-2\alpha\sin 2x+2\beta\cos 2x$ $z''_p = 4kxe^{2x}+4ke^{2x}-4\alpha\cos 2x-4\beta\sin 2x$ Sub into DE: $4kxe^{2x}+4ke^{2x}-4\alpha\cos 2x-4\beta\sin 2x$ $-4\left(kxe^{2x}+\alpha\cos 2x+\beta\sin 2x\right)$ $=8\mathrm{e}^{2x}-\cos 2x+4\sin 2x$ Comparing coefficients: $4k=8\Rightarrow k=2$ $-8\alpha=-1\Rightarrow \alpha=\frac{1}{8}$ $-8\beta=4\Rightarrow \beta=-\frac{1}{2}$ $\therefore z = Ae^{2x}+Be^{-2x}+2xe^{2x}+\frac{1}{8}\cos 2x-\frac{1}{2}\sin 2x$ when $x=0$, $z=0$: $A+B=-\frac{1}{8}$ $\frac{\mathrm{d}z}{\mathrm{d}x}=2Ae^{2x}-2Be^{-2x}+4xe^{2x}+2e^{2x}-\frac{1}{4}\sin 2x-\cos 2x$ when $x=0$, $\frac{\mathrm{d}z}{\mathrm{d}x}=1$: $2A-2B+2+2\left(-\frac{1}{2}\right)=1$ $\Rightarrow A-B=0$ $\therefore A=B=-\frac{1}{16}$ $\therefore z=-\frac{1}{16}e^{2x}-\frac{1}{16}e^{-2x}+2xe^{2x}+\frac{1}{8}\cos 2x-\frac{1}{2}\sin 2x$	

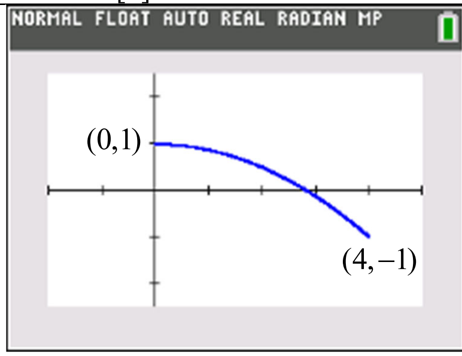
4	Solution [7 marks]	
(i)	$P(B') = \dfrac{15}{25} = \dfrac{3}{5}$ $P(G) = \dfrac{10}{25} = \dfrac{2}{5}$ $P(G \cap B') = \dfrac{6}{25} = P(B') \times P(G)$ Thus *G* and *B'* are independent.	
(ii)	P(exactly 1 A and at least 1 R) = P(1AR, 1A'R') + P(1AR', 1A'R) + P(1AR, 1A'R) $= \dfrac{\binom{2}{1}\binom{14}{1} + \binom{2}{1}\binom{7}{1} + \binom{2}{1}\binom{7}{1}}{\binom{25}{2}} = \dfrac{56}{300}$ P(exactly 1 A | at least 1 Red) $= \dfrac{\text{P(exactly 1 A and at least 1 R)}}{\text{P(at least 1 R)}}$ $= \dfrac{\frac{56}{300}}{\left(1 - \frac{\binom{16}{2}}{\binom{25}{2}}\right)}$ $= \dfrac{\frac{56}{300}}{\left(1 - \frac{120}{300}\right)}$ $= 0.311$ or $\dfrac{14}{45}$	

5	Solution [7]	
	$x_n - 4x_{n-1} + 4x_{n-2} = \dfrac{9}{2^n} - - - (*)$ Substitute $x_n = y_n + 2^{-n}$ into (*): $y_n + 2^{-n} - 4\left(y_{n-1} + 2^{-n+1}\right) + 4\left(y_{n-2} + 2^{-n+2}\right) = \dfrac{9}{2^n}$ $y_n + 2^{-n} - 4y_{n-1} - 8\left(2^{-n}\right) + 4y_{n-2} + 16\left(2^{-n}\right) = 9\left(2^{-n}\right)$ $y_n - 4y_{n-1} + 4y_{n-2} = 0$ (shown) The characteristic equation of the recurrence relation is	

	$m^2-4m+4=0$ $\Rightarrow (m-2)^2=0$ $\Rightarrow m=2$ (repeated roots) $\therefore y_n=(A+Bn)\left(2^n\right)$ $\Rightarrow x_n=y_n+2^{-n}=(A+Bn)\left(2^n\right)+2^{-n}$ $x_0=1:\ A+1=1\Rightarrow A=0$ $x_1=3:\ 2B+0.5=3\Rightarrow B=1.25$ Hence, $x_n=1.25n\left(2^n\right)+2^{-n}$	

6	Solution [8]	
(i)	Let the equation of H_1 be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Difference in distances of ship from the two stations $=0.3\times 140$ $=42$ km Hence, $a=\frac{42}{2}=21$ $c=\frac{50}{2}=25$ $\Rightarrow 25^2=21^2+b^2$ $\therefore b^2=184$ Equation of H_1 is $\frac{x^2}{441}-\frac{y^2}{184}=1$	
(ii)	The equation of H_2 with reference to the coordinate axes of A and B is, $\frac{(x-45)^2}{324}-\frac{y^2}{76}=1\Rightarrow y=\pm\sqrt{76\left(\frac{(x-45)^2}{324}-1\right)}$ $H_1:\frac{x^2}{441}-\frac{y^2}{184}=1\Rightarrow y=\pm\sqrt{184\left(\frac{x^2}{441}-1\right)}$ Solving $y=\sqrt{76\left(\frac{(x-45)^2}{324}-1\right)}$ and $y=\sqrt{184\left(\frac{x^2}{441}-1\right)}$, $x=23.027586$ and $y=6.1029768$ Hence, the coordinates of the location of the ship with respect to coordinate axes of A and B is $\left(23.0, 6.10\right)$ (to 3sf).	

7	Solution [8]	
	$\sqrt{3}-2\sin 2\theta \geq 0$ $\sin 2\theta \leq \frac{\sqrt{3}}{2}$ $0 \leq 2\theta \leq \frac{\pi}{3}$ or $\frac{2\pi}{3} \leq 2\theta \leq 2\pi$ $0 \leq \theta \leq \frac{\pi}{6}$ or $\frac{\pi}{3} \leq \theta \leq \pi$	
(i)		
(ii)	 Required area $=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \left(\sqrt{3}-2\sin 2\theta\right)^2 \, \mathrm{d}\theta$ $=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \left(3-4\sqrt{3}\sin 2\theta + 4\sin^2 2\theta\right) \mathrm{d}\theta$ $=\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \left(3-4\sqrt{3}\sin 2\theta + 2 - 2\cos 4\theta\right) \mathrm{d}\theta$ $=\frac{1}{2}\left[5\theta + 2\sqrt{3}\cos 2\theta - \frac{1}{2}\sin 4\theta\right]_{\frac{\pi}{3}}^{\frac{3\pi}{4}}$ $=\frac{1}{2}\left(\frac{15\pi}{4} - \frac{5\pi}{3} + \sqrt{3} - \frac{\sqrt{3}}{4}\right)$ $=\frac{25\pi + 9\sqrt{3}}{24}$ units^2	

<table>
<tr><td>8</td><td>Solution [8]</td><td></td></tr>
<tr><td></td><td>


</td><td></td></tr>
<tr><td>(i)</td><td>

$$A = 2\pi \int_0^{\frac{\pi}{2}} x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \, \mathrm{d}t$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 4\sin t \sqrt{16\cos^2 t + 4\sin^2 2t} \, \mathrm{d}t$$

$$= 8\pi \int_0^{\frac{\pi}{2}} \sin t \sqrt{16\cos^2 t + 16\sin^2 t \cos^2 t} \, \mathrm{d}t$$

$$= 32\pi \int_0^{\frac{\pi}{2}} \sin t \cos t \sqrt{1+\sin^2 t} \, \mathrm{d}t \text{ (shown)}$$

Using GC to solve, $A = 61.3 \text{ units}^2$
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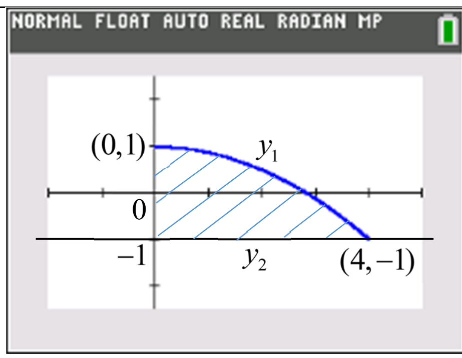
<tr><td>(ii)</td><td>


Required volume

$$= 2\pi \int_0^4 x\left(y_1 - y_2\right) \mathrm{d}x$$

$$= 2\pi \int_0^{\frac{\pi}{2}} x\left(y_1 - y_2\right) \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$$

$$= 2\pi \int_0^{\frac{\pi}{2}} 4\sin t\left(\cos 2t - (-1)\right) 4\cos t \, \mathrm{d}t$$

$$= 64\pi \int_0^{\frac{\pi}{2}} \sin t \cos^3 t \, \mathrm{d}t$$

$$= 64\pi \left[-\frac{\cos^4 t}{4}\right]_0^{\frac{\pi}{2}}$$

$$= 16\pi \text{ units}^3$$
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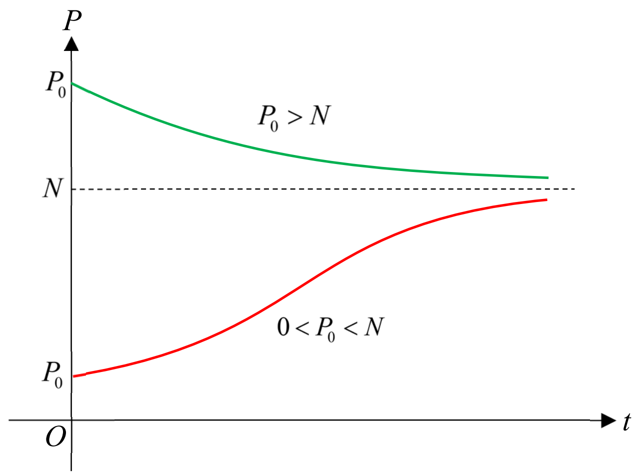
9.	Solution [9 marks]	
(i)	N is the carrying capacity, which is the largest population of white-tailed deer the ecosystem of the state of Kentucky can support.	
(ii)		
(iii)	$\frac{dP}{dt} = \frac{1}{5}P(N-P) - 100H$ $\frac{dP}{dt} = \frac{1}{5}P(900-P) - 100H = 0$ $900P - P^2 - 500H = 0$ $P^2 - 900P + 500H = 0$ $P = \frac{900 \pm \sqrt{900^2 - 4(500H)}}{2} = 450 \pm \frac{\sqrt{810000 - 2000H}}{2}$ For population to not become extinct, there should be at least 1 equilibrium point, i.e. $810000 - 2000H \geq 0$ $\Rightarrow H \leq 405$ and $P_0 = 150 \geq 450 - \frac{\sqrt{810000-2000H}}{2}$ $\sqrt{810000-2000H} \geq 600$ $810000 - 2000H \geq 360000$ $H \leq 225$ Thus, max H is 225.	

10	Solution [9 marks]	
(i)	$\frac{dy}{dx} = \frac{4e^{-x} - (x+5)y}{x+3} = g(x, y)$ $h = 0.25$	

x_n	$y_n = y_{n-1} + h\, g(x_{n-1}, y_{n-1})$
0	$\frac{2}{3}$
0.25	0.722222
0.5	0.670187

$y \approx 0.670$

(ii)	$u_1 = \frac{2}{3} + \frac{1}{2}\left(\frac{4-5\left(\frac{2}{3}\right)}{3}\right) = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{9}\right) = \frac{7}{9}$ $y \approx \frac{2}{3} + \left(\frac{1}{2}\right) \cdot \frac{\frac{2}{9} + \frac{\left(4e^{-0.5} - 5.5\left(\frac{7}{9}\right)\right)}{3.5}}{2} = 0.590$	
(iii)	The Euler method is computationally simpler but it lacks accuracy due to less accurate gradients used. The improved Euler's method, while computationally more tedious, is more accurate as it takes the mean of the initial and next gradient, giving a better approximation to the gradient.	
(iv)	$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x+5}{x+3}y = \frac{4\mathrm{e}^{-x}}{x+3}$ I.F. $= \mathrm{e}^{\int \frac{x+5}{x+3}\mathrm{d}x}$ $= \mathrm{e}^{\int 1+\frac{2}{x+3}\mathrm{d}x}$ $= \mathrm{e}^{x+2\ln(x+3)}$ $= \mathrm{e}^{x}(x+3)^2$ $y\mathrm{e}^{x}(x+3)^2 = \int \frac{4\mathrm{e}^{-x}}{x+3}\mathrm{e}^{x}(x+3)^2 \ \mathrm{d}x$ $= 4\int (x+3)\ \mathrm{d}x$ $= 4\left(\frac{x^2}{2} + 3x\right) + C$ Since $y(0) = \frac{2}{3}$, $\frac{2}{3}(9) = C \Rightarrow C = 6$ $y\mathrm{e}^{x}(x+3)^2 = 2x^2 + 12x + 6$ $y = \frac{2x^2+12x+6}{\mathrm{e}^{x}(x+3)^2}$ Thus, $p(x) = 2x^2 + 12x + 6$	

11	Solution [13 marks]	
(a)	By ratio theorem, $\overrightarrow{OR} = \lambda\mathbf{q} + (1-\lambda)\mathbf{p}$ $\overrightarrow{OR} \cdot \overrightarrow{PQ} = 0$ $(\lambda\mathbf{q} + (1-\lambda)\mathbf{p}) \cdot (\mathbf{q} - \mathbf{p})$ $= \lambda\|\mathbf{q}\|^2 - \lambda\mathbf{q}\cdot\mathbf{p} + (1-\lambda)\mathbf{p}\cdot\mathbf{q} - (1-\lambda)\|\mathbf{p}\|^2$ $= \lambda\|\mathbf{q}\|^2 - (1-\lambda)\|\mathbf{p}\|^2 = 0$ $\lambda\|\mathbf{q}\|^2 = (1-\lambda)\|\mathbf{p}\|^2 = \|\mathbf{p}\|^2 - \lambda\|\mathbf{p}\|^2$ $\lambda = \frac{\|\mathbf{p}\|^2}{\|\mathbf{p}\|^2 + \|\mathbf{q}\|^2}$	
(b)(i)	$2x + 2y + z = 10$ $x - 2y + 2z = -7$ By GC, $\mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\\frac{1}{2}\\1\end{pmatrix}$ Thus l_1: $\mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + \lambda'\begin{pmatrix}-2\\1\\2\end{pmatrix}, \lambda' \in \mathbb{R}$	
(ii)	$\begin{pmatrix}-2\\1\\2\end{pmatrix} \cdot \begin{pmatrix}1\\a\\-1\end{pmatrix} = 0$ $\Rightarrow -2 + a - 2 = 0 \Rightarrow a = 4$ $\begin{pmatrix}1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}1\\a\\-1\end{pmatrix} = b$ $\Rightarrow 1 + 4a = b \Rightarrow b = 17$	
(iii)	Let the foot of perpendicular of *B* on π_2 be *N*. Let $l_{BN}: \mathbf{r} = \begin{pmatrix}1\\0\\5\end{pmatrix} + \alpha\begin{pmatrix}1\\-2\\2\end{pmatrix}$ $\left[\begin{pmatrix}1\\0\\5\end{pmatrix} + \alpha\begin{pmatrix}1\\-2\\2\end{pmatrix}\right] \cdot \begin{pmatrix}1\\-2\\2\end{pmatrix} = -7$ $11 + 9\alpha = -7$ $\alpha = -2$	

	$\overrightarrow{ON} = \begin{pmatrix}1\\0\\5\end{pmatrix} - 2\begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}-1\\4\\1\end{pmatrix}$ Let B' be the reflected point of B on line of reflection. $\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OB}'\right)$ $\overrightarrow{OB}' = 2\begin{pmatrix}-1\\4\\1\end{pmatrix} - \begin{pmatrix}1\\0\\5\end{pmatrix} = \begin{pmatrix}-3\\8\\-3\end{pmatrix}$ $\overrightarrow{AB}' = \begin{pmatrix}-3\\8\\-3\end{pmatrix} - \begin{pmatrix}1\\4\\0\end{pmatrix} = \begin{pmatrix}-4\\4\\-3\end{pmatrix}$ [Note: A lies on π_2.] Equation of line of reflection: $\mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + \beta\begin{pmatrix}-4\\4\\-3\end{pmatrix}, \beta \in \mathbb{R}$	

12	Solution [12]	
(i)	$u_1 = 20 \times 5 = 100$ $u_2 = 5(100) + 5k = 500 + 5k$ $u_3 = 5(500 + 5k) + 5k = 2500 + 30k$ (shown) Recurrence relation: $u_t = 5u_{t-1} + 5k$, for $t \geq 2$, $u_1 = 100$ $u_t = 5u_{t-1} + 5k$ $= 5\left(5u_{t-2} + 5k\right) + 5k$ $= 5^2 u_{t-2} + 5^5 k + 5k$ $= 5^2\left(5u_{t-3} + 5k\right) + 5^5 k + 5k$ $= 5^3 u_{t-3} + 5^3 k + 5^5 k + 5k$ $= \ldots$ $= 5^{t-1}u_1 + k\left(5^{t-1} + 5^{t-2} + 5^{t-3} + \ldots + 5\right)$ $= (100)5^{t-1} + k\frac{5\left(5^{t-1} - 1\right)}{5-1}$ $= (100)5^{t-1} + 1.25k\left(5^{t-1} - 1\right)$ $= \left(20 + 0.25k\right)5^t - 1.25k$ We have assumed that there is no repetition in person being shared to.	

<table>
<tr><td>(ii)</td><td>Want $u_5 \geq 75000$,
i.e. $\left(20+0.25k\right)5^5 - 1.25k \geq 75000$
Using GC,
<table><tr><td>k</td><td>u_5</td></tr><tr><td>16</td><td>74980 < 75000</td></tr><tr><td>17</td><td>75760 > 75000</td></tr></table>Hence, $k \geq 17$.</td><td></td></tr>
<tr><td>(iii)</td><td>$k = 15:\ u_t = \left(23.75\right)5^t - 18.75$
Want $N = \sum_{t=1}^{n}\left[\left(23.75\right)5^t - 18.75\right] > 100000000$
Using GC,
<table><tr><td>t</td><td>N</td></tr><tr><td>8</td><td>74218600 < 100000000</td></tr><tr><td>9</td><td>417480300 > 100000000</td></tr></table>Hence, the number of days needed for the total number of post shared to exceed 100 million is 9.</td><td></td></tr>
<tr><td>(iv)</td><td>$v_t - 5v_{t-1} - v_{t-2} = 0$
$m^2 - 5m - 1 = 0$
$m = -0.192582$ or $m = 5.192582$
$\therefore v_t = A\left(-0.192582\right)^t + B\left(5.192582\right)^t$
$v_1 = 100 : 100 = A\left(-0.192582\right) + B\left(5.192582\right)$
$v_2 = 550 : 550 = A\left(-0.192582\right)^2 + B\left(5.192582\right)^2$
Solving by GC,
$A = 29.6$ and $B = 20.4$
$\therefore v_t = 29.6\left(-0.193\right)^t + 20.4\left(5.19\right)^t$ (to 3sf)</td><td></td></tr>
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– End of Paper –