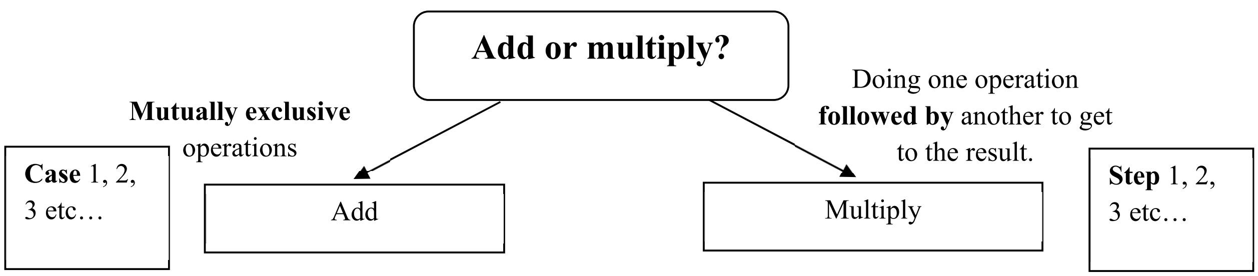
RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 13 (Summary and Tutorial)

Topic: Permutations and Combinations, Probability

Summary for Permutations and Combinations

Definitions - A **permutation** is an ***ordered arrangement*** of objects
- A **combination** is a ***selection of objects*** in which the order of selection ***does not matter***.



Permutations

Given	Objects taken	No of Permutations
n distinct objects	n	$n!$
n distinct objects	r ***no repetitions***	${}^nP_r$ or ${}^nC_r\,r!$
n distinct objects	r ***with repetitions***	n^r
n objects, not all distinct (*n_1 of type 1, n_2 of type 2, …, n_k of type k, where* $n = n_1 + n_2 + \ldots + n_k$)	n	$\dfrac{n!}{n_1!n_2!\ldots n_k!}$
n objects, not all distinct	r	Involves combinations & permutations See section on ***Combinations***.

Useful Techniques when dealing with restrictions:

1. Grouping or Slotting.
   (a) Grouping ("must be together", "cannot be separated" etc.).
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are together $= (5!)\times(3!)$
   [abc, d, e, f, g: 5 items → 5! ways, within abc → 3! ways]
   (b) Slotting ("cannot be together", "must be separated" etc.)
   Eg. No of ways to arrange the letters a, b, c, d, e, f, g such that a, b & c are not adjacent to each other $= (4!)\times\left({}^5P_3\right)$
   [arrange d, e, f, g first → 4! ways, slot and permute a, b, c → ${}^5P_3$ ways]

2. Taking Complement ("two items cannot be together" or "at least 1").
   Use this technique with care. Note that for Eg in 1(b) above, a, b, c not adjacent to each other is NOT the complement of abc together.

Combinations

Notation: ${}^nC_r$ can also be written as $\binom{n}{r}$

Note:

- $\binom{n}{r} = \binom{n}{n-r} = \dfrac{n!}{r!(n-r)!}$. E.g. $\binom{9}{2} = \binom{9}{7}$
- ${}^nP_r = {}^nC_r r!$

No of ways to select ***r objects***	Given : ***n distinct*** objects $\binom{n}{r}$	Given: ***n*** objects, not all distinct ***No direct way to calculate.*** Need to consider different cases: (i) Start with case where the r selected objects are ***all distinct.*** (ii) Next, consider case(s) with pair(s) of identical objects, and so on. (iii) **No of ways = Sum of the different cases**
Eg: To choose 3 letters (***arrangements*** not required)	Given: Letters a,b,c,d,e,f No of ways = $\binom{6}{3} = 20$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1 way. Case 2: Contains an identical pair (aab, aac, bba, bbc) – 2 x 2 = 4 ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 1 + 4 +1 = 6
Eg. Find the no of 3-letter codes (***arrangements*** to be considered)	Given: Letters a,b,c,d,e,f No of ways = $\binom{6}{3}(3!) = 120$ OR: ${}^6P_3 = 6\times5\times4 = 120$	Given: a, a, a, b, b, c Case 1: All distinct (abc) – 1x3! = 6 ways. Case 2: Contains an identical pair (aab, aac, bba, bbc) – $2\times2\times\frac{3!}{2!} = 12$ ways. Case 3: All identical (aaa) – 1 way. Total no of ways = 6 + 12 +1 = 19

Remark: Before doing any calculations, it is important to establish if

- the problem involves ***permutations only***, or ***combinations only***, or ***both***.
- the objects or items given are ***all distinct***. (see Eg in above table).
- ***repetitions*** are allowed.

Circular Permutations – Arranging *n* distinct objects in a Circle

- In row arrangement, such as in a queue, there is a *first* & a *last* position. No of ways = $n!$
- In a circle, such as a round table, there is no starting or ending point.

  No of ways = $\frac{n!}{n} = (n-1)!$
  - No. of ways to seat 2 couples and a boy at a round table = $(5-1)! = 24$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together = $(3-1)!2!2! = 8$

    [couple 1, couple 2, boy round the round → $(3-1)!$ ways, each couple → $2!$ ways]
- If the seats of the round table are numbered, we will perform the necessary calculations assuming the seats are not numbered, and multiply the value with the number of seats
  - No. of ways to seat 2 couples and a boy at a round table with numbered seats = $(5-1)! \times 5 = 120$
  - No. of ways to seat 2 couples and a boy at a round table if each couple is seated together, and the seats are numbered = $(3-1)!2!2! \times 5 = 40$

Reminder: Questions on P&C may involve probability. Read the questions carefully.

Summary for Probability

If the sample space S consists of a finite number of **equally likely** outcomes, then the probability of event A written as $P(A)$ is defined as

$$\mathrm{P}(A)=\frac{\text{no. of elements in } A}{\text{no. of elements in } S}=\frac{\mathrm{n}(A)}{\mathrm{n}(S)}.$$

In layman definition, $\text{Required probability}=\dfrac{\text{outcomes we are interested in}}{\text{total possible outcomes}}$.

Important Results

1. $0 \le \mathrm{P}(A) \le 1$, answers should always be between 0 and 1 inclusive AND exact where possible.
2. $\mathrm{P}(A')=1-\mathrm{P}(A)$
3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$
   With the help of venn diagram, others (besides Results 2 and 3) can also be easily deduced. Some examples as follow:
   $\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(A' \cap B)$
   $\mathrm{P}(A)=\mathrm{P}(A \cap B)+\mathrm{P}(A \cap B')$
   or any others involving $A' \cup B, A \cup B', A' \cup B', A \cap B, A' \cap B, A \cap B', A' \cap B'$ and even 3 sets A, B, C.
4. If A and B are **mutually exclusive**, then $\mathrm{P}(A \cap B) = \mathrm{P}(\phi) = 0$, thus $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Conditional Probability

If A and B are two events and $\mathrm{P}(B) \neq 0$, then the probability of A, given that B has already occurred, is written as $\mathrm{P}(A \mid B)$ and is calculated using the formula:

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Note:

1. $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$, where $\mathrm{P}(A) \neq 0$.
2. In general, $\mathrm{P}(A|B) \neq \mathrm{P}(B|A)$.
3. $\mathrm{P}(A|B)$ can be considered as the probability of occurrence of event $A \cap B$ with respect to the sample space B i.e. reduced sample space.
   Example (part of **AJC Prelim 9740/2008/02/Q7**):

   3 machines A, B and C produce 25%, 35% and 40% respectively of the golf balls manufactured by a factory. These balls are either yellow or white. Of the balls produced by A and B, 20% and 30% respectively are yellow. It is known that the probability of picking a yellow ball is 0.355. If 3 balls

are picked randomly, find the probability that at least 1 is yellow given that all the balls picked are from machine A.

Using reduced sample space, $\text{P(at least 1 is yellow|all from A)} = 1 - 0.8^3 = 0.488$

Independent Events

Two events A and B are said to be independent if the occurrence of A does not affect that of B and vice versa.

To prove independence of events A and B, we can show either one of the following

1. $\mathrm{P}(A \mid B) = \mathrm{P}(A)$
2. $\mathrm{P}(B \mid A) = \mathrm{P}(B)$
3. $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

It can be easily shown that if A and B are independent, then the following pairs are also independent
A and B', A' and B', A' and B

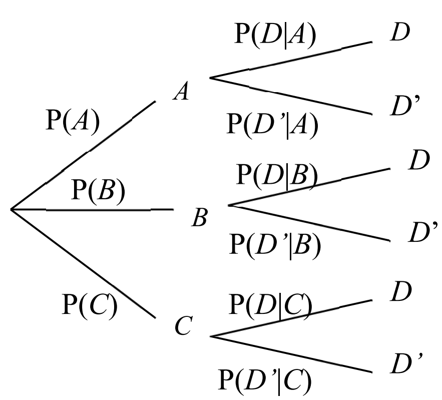
We DO NOT assume independence unless we have proven it or condition is given in the question.

Common mistakes:

Mixed up the tests of "mutually exclusive" and "independence".
Taking $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$ even when A and B are NOT independent.

Techniques in calculating probabilities

1. Systematic listing – example 9740/2014/P2/Q10.
2. Venn diagram – example 9740/2015/P2/Q9. This question involves 3 sets.
3. Tree diagram – though a useful technique, many students are lazy to draw the diagram and hence made careless mistakes.



$\mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) = 1$

$\mathrm{P}(A \cap D) = \mathrm{P}(A) \times \mathrm{P}(D \mid A)$

4. P&C – do not be confused when applying this technique. The counting for denominator has to be consistent with that of numerator. If order matters for denominator, then it has to matter for numerator.

Reminder: Probability could easily be embedded in another Statistics question due to its nature. Read question carefully.

<u>Revision Tutorial Questions</u>

Source of Question: YJC JC2 CT1 9758/2018/Q1

1 3 families went to a cinema to watch a movie together. Each family consists of a father, mother, child and domestic helper. The surnames of the three families are Chua, Lee and Wong. The 12 people are seated in a row of 12 seats. Find the number of arrangements if

(i) the children are seated next to each other, [1]

(ii) the 4 members in each family are seated next to each other, [2]

(iii) each domestic helper is seated next to the child in the family, and each family is seated together, [2]

(iv) all three mothers are next to each other, all four members of the Chua family are next to each other and all four members of the Wong family are next to each other. [3]

After the movie, the 3 families proceed for a dinner at a round table with 12 seats.

(v) In how many ways can the 12 people sit so that no two children are next to each other?

[2]

Solutions:

(i)	No of ways $=10!\times 3!=21\,772\,800$
(ii)	No of ways $=3!\times (4!)^3 \quad =82\,944$
(iii)	No of ways $=(3!\times 2!)^3\times 3! \; =10\,368$
(iv)	No of ways $=2!\times 3!\times 3!\times 4!=1728$
(v)	No of ways $=(9-1)!\times {}^9P_3=20\,321\,280$

Source of Question: DHS Prelim 9758/2017/02/Q10(a)

2 The word DISTRIBUTION has 12 letters.

(i) Find the number of different arrangements of the 12 letters that can be made. [1]

(ii) Find the number of different arrangements which can be made if there are exactly 8 letters between the two Ts. [3]

One of the Is is removed from the word and the remaining letters are arranged randomly.

(iii) Find the probability that no adjacent letters are the same. [4]

Solution:

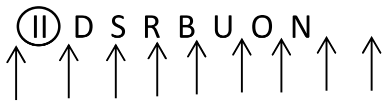
(i)	Number of ways = $\dfrac{12!}{3!2!} = 39916800$
(ii)	Method 1 ↑ I S D R I B U I ↑ O N I ↑ S D R I B U I O ↑ N I S ↑ D R I B U I O N ↑ There are 3 ways to slot in the 2 T's Total number of ways = total number of ways to arrange the remaining ten letters $\times 3$ $= \dfrac{10!}{3!} \times 3 = 1814400$
	<u>Method 2</u> [T _ _ _ _ _ _ _ _ T] [_] [_] <u>Case 1</u>: One I included between the two T's Number of ways = ${}^{7}C_{7} \times 8! \times \dfrac{3!}{2!} = 120960$ <u>Case 2</u>: Two I's included between the two T's Number of ways = ${}^{7}C_{6} \times \dfrac{8!}{2!} \times 3! = 846720$ <u>Case 3</u>: Three I's included between the two T's

Number of ways = ${}^{7}C_{5} \times \frac{8!}{3!} \times 3! = 846720$

Total number of ways $= 120960 + 2(846720) = 1814400$

(iii) Method 1

Case 1: Both I together but both T separated



7 single letters (excluding Is and Ts) and 1 block of 2Is.

Number of ways = $8! \times {}^{9}C_{2} = 1451520$

Case 2: Both T together but I separated

Number of ways = $8! \times {}^{9}C_{2} = 1451520$ (same approach as case 1)

Case 3: Both I together and both T together

II TT D S R B U O N

7 single letters (excluding Is and Ts),1 block of Is and 1 block of Ts.

Number of ways = $9! = 362880$

Total number of ways in complement

$= (1451520 \times 2) + 362880 = 3265920$

Method 2

Number of ways in which both T are together = $\frac{10!}{2!}$

Number of ways in which both I are together = $\frac{10!}{2!}$

Number of ways in which both pairs of identical letters are together = 9!

Total number of ways in complement $= 2 \times \frac{10!}{2!} - 9! = 3265920$

Required probability = $1 - \frac{3265920}{\frac{11!}{2!2!}} = 0.673$

Source of Question: AJC JC2 Mid-Year CT 9758/2018/02/Q2

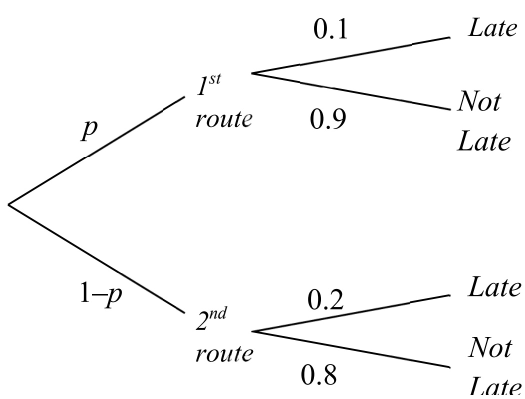
3 A student has two routes to get to school. The probability that he chooses the first route on any day is p where $0 < p < 1$, and the probability of him being late for school is 0.1 if he chooses the first route and 0.2 for the second route. There is a probability of 0.14 that a student is late for school on any given day.

(i) Find the value of p. [2]

(ii) Given that he is not late for school, find the probability that he chose the first route. [2]

(iii) Find the probability that the student is late for school more than once in three days. [2]

Solution:

(i)	 P(Late) $= 0.14$ $p\,(0.1) + (1-p)\,0.2 = 0.14$ $\Rightarrow 0.1\,p = 0.06 \;\Rightarrow\; p = 0.6$
(ii)	P (1st route | Not Late) $\dfrac{\mathrm{P}(\text{1st route} \cap \text{Not Late})}{\mathrm{P}(\text{Not Late})}$ $= \dfrac{0.6 \times 0.9}{1-0.14} = \dfrac{27}{43}$ (*or* 0.628)
(iii)	Let Y be the number of days a student is late for school in 3 days. $Y \sim \mathrm{B}(3, 0.14)$ $\mathrm{P}(Y > 1) = 1 - \mathrm{P}(Y \leq 1)$ $= 1 - 0.946688$ $= 0.0533$ (3 s.f.) <u>Alternative Method</u> P(student is late for school more than once in three days) =P(student late 2 times in 3 days) + P(student late all 3 days) $= {}^{3}C_{2}(0.14)^{2}(0.86) + (0.14)^{3}$ $= 0.0533$

Source of Question: HCI JC2 CT1 9758/2018/Q9

4 **(a)** In a particular college, students must choose 4 subjects to read. Subjects are classified as either 'Arts' or 'Science'. The available 'Arts' subjects are Economics, Geography, History and Literature, and the available 'Science' subjects are Biology, Chemistry, Mathematics and Physics. Each student must read at least one 'Arts' subject and at least one 'Science' subject.

(i) Find the number of ways that a student can choose his 4 subjects. [2]

The subjects are further classified into 2 levels, H1 and H2. Each student must read 3 subjects at H2 level and 1 subject at H1 level, and he cannot read the same subject at both H1 and H2 levels.

(ii) Find the number of ways that a student can choose his 3 H2 and 1 H1 subjects. [1]

In addition to the rules above, a student who reads H2 Biology, H2 Chemistry or H2 Physics must also read H2 Mathematics.

(iii) Find the number of ways that a student can choose his 3 H2 and 1 H1 subjects with this additional rule. [3]

(b) It is known that among the students in a college, 5% read Geography and 8% read History. There is a 20% chance that a student reads Geography given that he reads History.

(i) Show that the probability that a randomly chosen student reads both Geography and History is 0.016. [1]

(ii) Find the probability that a randomly chosen student reads either Geography or History but not both. [2]

(iii) State, with a reason, whether the events 'a student reads Geography' and 'a student reads History' are independent. [1]

Solutions:

(a) (i)	Method 1 (Complement) ${}^{8}C_{4}-2=68$
	Method 2 (Direct) ${}^{4}C_{1}\,{}^{4}C_{3}+{}^{4}C_{2}\,{}^{4}C_{2}+{}^{4}C_{3}\,{}^{4}C_{1}=68$
(ii)	$68\times 4=272$
(iii)	Method 1 (Direct) Case 1: H2 Math, anything except all science $({}^{7}C_{3}-1)\times 3=102$ Case 2: H1 Math, no other science ${}^{4}C_{3}=4$ Case 3: no Math, H1 science and 3 arts ${}^{3}C_{1}\times {}^{4}C_{3}=12$ Total = 4 + 102 + 12 = 118 ways
	Method 2 (Direct – finer subcases)

	Case 1: H2 Math, 2 other sciences, 1 arts ${}^3C_2 \times {}^4C_1 \times 3 = 36$ Case 2: H2 Math, 1 other science, 2 arts ${}^3C_1 \times {}^4C_2 \times 3 = 54$ Case 3: H2 Math, 3 arts ${}^4C_3 \times 3 = 12$ Case 4: H1 Math, 3 arts ${}^4C_3 = 4$ Case 5: H1 science (not Math), 3 arts ${}^3C_1 \times {}^4C_3 = 12$ Total = 36 + 54 + 12 + 4 + 12 = 118 ways
	<u>Method 3 (Complement)</u> Case 1: no Math, 1, 2 or 3 H2 science ${}^3C_1 \times {}^4C_3 \times 3 + {}^3C_2 \times {}^4C_2 \times 4 + {}^3C_3 \times {}^4C_1 \times 4$ $= 36 + 72 + 16 = 124$ Case 2: H1 Math, 1 or 2 H2 science ${}^3C_1 \times {}^4C_2 + {}^3C_2 \times {}^4C_1$ $= 18 + 12 = 30$ Total = 272 – 124 – 30 = 118 ways
(b) **(i)**	Let *G* and *H* represent the events that a student reads Geography and History respectively. P(*G*) = 0.05, P(*H*) = 0.08, $\mathrm{P}(G \mid H) = 0.2$ $\mathrm{P}(G \cap H) = \mathrm{P}(G \mid H) \times \mathrm{P}(H)$ $= 0.2 \times 0.08 = 0.016$
(ii)	<u>Method 1</u> $\mathrm{P}(G \cup H) - \mathrm{P}(G \cap H)$ $= \left[\mathrm{P}(G) + \mathrm{P}(H) - \mathrm{P}(G \cap H)\right] - \mathrm{P}(G \cap H)$ $= \mathrm{P}(G) + \mathrm{P}(H) - 2\mathrm{P}(G \cap H)$ $= 0.05 + 0.08 - 2(0.016)$ $= 0.098$
	<u>Method 2</u>

	$\mathrm{P}(G \cap H') + \mathrm{P}(G' \cap H)$ $= [\mathrm{P}(G) - \mathrm{P}(G \cap H)] + [\mathrm{P}(H) - \mathrm{P}(G \cap H)]$ $= \mathrm{P}(G) + \mathrm{P}(H) - 2\mathrm{P}(G \cap H)$ $= 0.05 + 0.08 - 2(0.016)$ $= 0.098$
(iii)	Since $\mathrm{P}(G) = 0.05 \neq \mathrm{P}(G \mid H) = 0.2$, OR $\mathrm{P}(G) \times \mathrm{P}(H) = 0.05 \times 0.08 = 0.004$ $\mathrm{P}(G \cap H) = 0.016 \neq \mathrm{P}(G) \times \mathrm{P}(H)$, a student reading Geography and reading History are not independent events.

Source of Question: ACJC JC2 CT1 9758/2018/Q3 (modified)

5 Given that $\mathrm{P}(B \cup C) = \dfrac{3}{4}$ and $\mathrm{P}(B' \cap C) = \dfrac{1}{5}$,

(i) find $\mathrm{P}(C)$ if B and C are two mutually exclusive events, [1]

(ii) find $\mathrm{P}(C)$ if B and C are two independent events. [4]

Solution:

(i)	Since B and C are two mutually exclusive events, $\mathrm{P}(B \cap C) = 0$. $\mathrm{P}(C) = \mathrm{P}(B' \cap C) = \dfrac{1}{5}$.
(ii)	$\mathrm{P}(B) = \mathrm{P}(B \cup C) - \mathrm{P}(B' \cap C)$ $\mathrm{P}(B) = \dfrac{3}{4} - \dfrac{1}{5} = \dfrac{11}{20}$. Since $\mathrm{P}(B \cup C) = \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B \cap C)$ $\dfrac{3}{4} = \mathrm{P}(B) + \mathrm{P}(C) - [\mathrm{P}(B).\mathrm{P}(C)]$ (Note: Event B and C are independent events.) $\dfrac{3}{4} = \dfrac{11}{20} + \mathrm{P}(C) - \left[\dfrac{11}{20}.\mathrm{P}(C)\right]$ $\dfrac{3}{4} - \dfrac{11}{20} = \dfrac{9}{20}\mathrm{P}(C)$ $\mathrm{P}(C) = \dfrac{4}{9}$.

Source of Question: IJC Prelim 9758/2017/02/Q6

6 Seven red counters and two blue counters are placed in a bag. All the counters are indistinguishable except for their colours. Clark and Kara take turns to draw a counter from the bag at random with replacement. The first player to draw a blue counter wins the game and the game ends immediately.

If Clark draws first, find the probability that

(i) Clark wins the game at his third draw, [2]

(ii) Kara wins the game. [3]

Solution:

(i)	P(Clark wins in 3rd draw) $=\frac{7}{9}\times\frac{7}{9}\times\frac{7}{9}\times\frac{7}{9}\times\frac{2}{9}$ $=0.081322$ $=0.0813$
(ii)	P(Kara wins) $=\frac{7}{9}\times\frac{2}{9}+\left(\frac{7}{9}\right)^3\times\frac{2}{9}+\left(\frac{7}{9}\right)^5\times\frac{2}{9}+....$ $=\frac{2}{9}\left[\frac{7}{9}+\left(\frac{7}{9}\right)^3+\left(\frac{7}{9}\right)^5+...\right]$ $=\frac{2}{9}\left(\frac{\frac{7}{9}}{1-\left(\frac{7}{9}\right)^2}\right)$ $=0.4375\ \ or\ \ \frac{7}{16}$

Source of Question: RVHS JC2 Mid-Year CT 9758/2018/Q7

7 The numbers on the faces of a fair tetrahedron blue die are 1, 2, 3 and 4; while the numbers on the faces of another fair tetrahedron red die are –1, 0, 1 and 2. Both dice are rolled once and the numbers on the faces in contact with the table are noted.

The random variable X denotes the sum of the numbers obtained from the 2 dice.

Blue Die

	+	1	2	3	4
	–1				
Red Die	0				
	1				
	2				

(i) By completing the above table which shows the possible values of X, construct the probability distribution table for X. [2]

(ii) Find $\text{Var}(X)$. [2]

The following events A and B are defined as follows:

A : X is an odd number

B : X is more than 2

(iii) Find $\text{P}\left(A \middle| B\right)$ and determine if A and B are independent. [3]

Solution:

(i)

Blue Die

	+	1	2	3	4
	–1	0	1	2	3
Red	0	1	2	3	4
die	1	2	3	4	5
	2	3	4	5	6

From the above possibility diagram, we construct the probability distribution table for random variable X:

x	0	1	2	3	4	5	6
$\text{P}(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

(ii)

$\text{Var}(X) = \text{E}(X^2) - \left(\text{E}(X)\right)^2$

	$=\left(\sum_{x=0}^{6} x^2 \mathrm{P}\left(X=x\right)\right)-\left(\sum_{x=0}^{6} x\mathrm{P}\left(X=x\right)\right)^2$ $=\frac{184}{16}-\left(\frac{48}{16}\right)^2=2\frac{1}{2}$
(iii)	$\mathrm{P}\left(A\middle|B\right)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ $=\frac{6}{16}\div\frac{10}{16}$ $=\frac{3}{5}$ We note that $\mathrm{P}(A)=\frac{2+4+2}{16}=\frac{1}{2}\neq \mathrm{P}(A\vert B)$, hence, the events A and B are not independent. OR: $\mathrm{P}(A\cap B)=\frac{6}{16}$ $\mathrm{P}(A)\times \mathrm{P}(B)=\frac{8}{16}\times\frac{10}{16}=\frac{5}{16}$ Since $\mathrm{P}(A\cap B)\neq \mathrm{P}(A)\times \mathrm{P}(B)$, the events A and B are not independent.