RAFFLES INSTITUTION
2023 Year 5 H2 Mathematics Promotion Exam
Questions and Solutions with comments

1 The first 3 terms of a sequence are given by $u_1 = 1823$, $u_2 = 200$ and $u_3 = 2023$. Given that u_n is a quadratic polynomial in n, find u_n in terms of n. [4]

1	Let $u_n = an^2 + bn + c$. $u_1 = 1823:\ a + b + c = 1823$ $u_2 = 200:\ 4a + 2b + c = 200$ $u_3 = 2023:\ 9a + 3b + c = 2023$ Using GC, we have $a = 1723,\ b = -6792,\ c = 6892.$ Therefore, $u_n = 1723n^2 - 6792n + 6892.$	Mostly well done. Some students mistook "quadratic polynomial" as "quartic polynomial" and were hence unable to solve the question. Students should take note that a sequence is not restricted to only AP or GP. It can just be a generic sequence.

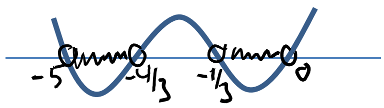
2 **Do not use a calculator in answering this question.**

Solve the inequality $x^2+6x+5 \le \dfrac{x+5}{3x+1}$. [4]

Hence solve

(a) $x^4+6x^2+5 \le \dfrac{x^2+5}{3x^2+1}$, [2]

(b) $(\ln x)^2+6\ln x+5 \le \dfrac{5+\ln x}{1+3\ln x}$. [2]

2	$x^2+6x+5-\dfrac{x+5}{3x+1}\le 0$ $(x+5)(x+1)-\dfrac{x+5}{3x+1}\le 0$ $(x+5)\dfrac{(x+1)(3x+1)-1}{3x+1}\le 0$ $\dfrac{(x+5)\left[3x^2+4x+1-1\right]}{3x+1}\le 0$ $\dfrac{(x+5)(x)(3x+4)}{3x+1}\le 0$  $\therefore -5\le x\le -\dfrac{4}{3}$ or $-\dfrac{1}{3}<x\le 0$	Most students were able to tackle this question successfully. Students are advised to check their work to avoid careless mistakes.
(a)	Replace x with x^2, $x^4+6x^2+5\le \dfrac{x^2+5}{3x^2+1}$ $\therefore -5\le x^2\le -\dfrac{4}{3}$ or $-\dfrac{1}{3}<x^2\le 0$ No solution, since $x^2\ge 0$ or $x=0$ $\Rightarrow x=0$	Students are reminded that complex numbers cannot be ordered. Hence they should not appear in inequalities.
(b)	Replace x with $\ln x$, $(\ln x)^2+6\ln x+5\le \dfrac{\ln x+5}{3\ln x+1}$ $\therefore -5\le \ln x\le -\dfrac{4}{3}$ or $-\dfrac{1}{3}<\ln x\le 0$ $\Rightarrow e^{-5}\le x\le e^{-4/3}$ or $e^{-1/3}<x\le 1$	This part was well done. Students could see the replacement and understand that the ln function is an increasing one, hence there will be no change to the inequality signs.

3 Vectors **a** and **b** are such that the magnitude of **a** is 2 and **b** is a unit vector perpendicular to **a**.

(a) Find the area of the parallelogram with adjacent sides formed by the vectors $\mathbf{a}+3\mathbf{b}$ and $5\mathbf{a}-4\mathbf{b}$. [3]

A vector **c** is such that $\mathbf{b}\times\mathbf{c}=21(\mathbf{a}\times\mathbf{b})$.

(b) Show that $\mathbf{c}=\lambda\mathbf{b}-21\mathbf{a}$, where λ is a constant. [2]

(c) Give the geometrical meaning of $|\mathbf{b}.\mathbf{c}|$ and find the possible values of λ if $|\mathbf{b}.\mathbf{c}|=5$. [3]

3(a)	Area of the parallelogram $=\lvert(\mathbf{a}+3\mathbf{b})\times(5\mathbf{a}-4\mathbf{b})\rvert$ $=\lvert 5(\mathbf{a}\times\mathbf{a})+15(\mathbf{b}\times\mathbf{a})-4(\mathbf{a}\times\mathbf{b})-12(\mathbf{b}\times\mathbf{b})\rvert$ $=\lvert 19(\mathbf{b}\times\mathbf{a})\rvert$ $=19\lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\sin 90°$ $=38$	Common mistakes • $\lvert(\mathbf{a}+3\mathbf{b}).(5\mathbf{a}-4\mathbf{b})\rvert$ • $(\mathbf{a}+3\mathbf{b})\times(5\mathbf{a}-4\mathbf{b})$ (without the modulus) • $=\lvert 5(\mathbf{a}\times\mathbf{a})+15(\mathbf{a}\times\mathbf{b})-4(\mathbf{a}\times\mathbf{b})-12(\mathbf{b}\times\mathbf{b})\rvert$ (note that $\mathbf{b}\times\mathbf{a}\neq\mathbf{a}\times\mathbf{b}$)
(b)	$\mathbf{b}\times\mathbf{c}=21(\mathbf{a}\times\mathbf{b})$ $\Rightarrow(\mathbf{b}\times\mathbf{c})-21(\mathbf{a}\times\mathbf{b})=\mathbf{0}$ $\Rightarrow(\mathbf{b}\times\mathbf{c})+21(\mathbf{b}\times\mathbf{a})=\mathbf{0}$ $\Rightarrow\mathbf{b}\times(\mathbf{c}+21\mathbf{a})=\mathbf{0}$ <u>**Case 1: If** $\mathbf{c}=-21\mathbf{a}$</u> Note that $\mathbf{c}=-21\mathbf{a}$ satisfy the equation $\mathbf{b}\times(\mathbf{c}+21\mathbf{a})=\mathbf{0}$ <u>**Case 2: If** $\mathbf{c}\neq-21\mathbf{a}$</u> Since **b** and $\mathbf{c}+21\mathbf{a}$ are nonzero vectors, then $\mathbf{c}+21\mathbf{a}$ is parallel to **b** and so $\mathbf{c}+21\mathbf{a}=\lambda\mathbf{b}$, where λ is a nonzero constant. $\mathbf{c}=\lambda\mathbf{b}-21\mathbf{a}$ Note that $\lambda=0$ corresponds to Case 1. Thus we can say that $\mathbf{c}=\lambda\mathbf{b}-21\mathbf{a}$, where λ is a constant. (Shown) [Note that **a** and **b** are nonzero vectors and since $\mathbf{a}\perp\mathbf{b}$, $\mathbf{a}\times\mathbf{b}\neq\mathbf{0}$. And so, **c** is a nonzero vector.]	Common mistakes • $21(\mathbf{a}\times\mathbf{b})=21\mathbf{a}\times21\mathbf{b}$ (refer to Chap 4B notes page 15 property 4: $\lambda(\mathbf{a}\times\mathbf{b})=(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})$) • $(\mathbf{b}\times\mathbf{c})-21(\mathbf{a}\times\mathbf{b})=\mathbf{b}\times(\mathbf{c}-21\mathbf{a})$ (refer to Chap 4B notes page 15 property 3: $(\mathbf{b}\times\mathbf{a})\pm(\mathbf{b}\times\mathbf{c})=\mathbf{b}\times(\mathbf{a}\pm\mathbf{c})$) • $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ (should be $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$)
(c)	$\lvert\mathbf{b}\cdot\mathbf{c}\rvert=\lvert\mathbf{c}\cdot\mathbf{b}\rvert$ is the length of projection of **c** onto **b**. $\lvert\mathbf{b}\cdot\mathbf{c}\rvert=\lvert\mathbf{b}\cdot(\lambda\mathbf{b}-21\mathbf{a})\rvert$ $=\lvert\lambda(\mathbf{b}\cdot\mathbf{b})-21(\mathbf{b}\cdot\mathbf{a})\rvert$ Refer to Chap 4B notes page 9 property 3. $=\lvert\lambda\lvert\mathbf{b}\rvert^2-21(0)\rvert$ $=\lvert\lambda\rvert$ Since $\lvert\mathbf{b}\cdot\mathbf{c}\rvert=5$, $\lambda=\pm5$.	Refer to Chap 4B notes page 13: The length of projection of **a** onto **b** is given by $\lvert\mathbf{a}.\hat{\mathbf{b}}\rvert$, where $\hat{\mathbf{b}}$ is a unit vector in the direction of **b**. In this question, the unit vector is **b**. So, length projection $\lvert\mathbf{b}.\mathbf{c}\rvert$ is the length of projection of **c** onto **b**, and not **b** onto **c**.

4 **(a)** Write $\dfrac{1}{r(r+1)}$ in partial fractions. [1]

(b) Using your answer to part **(a)**, find $\displaystyle\sum_{r=1}^{n}\frac{1}{r^2+r}$. [2]

Hence find

(i) $\displaystyle\sum_{r=n+1}^{2n}\frac{1}{r^2+r}$, [2]

(ii) $\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+\dfrac{1}{4\times 5}+\cdots$ [2]

4(a)	$\dfrac{1}{r(r+1)}=\dfrac{1}{r}-\dfrac{1}{r+1}$	This part is well-done.
(b)	$\displaystyle\sum_{r=1}^{n}\frac{1}{r^2+r}=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$ $=\dfrac{1}{1}-\cancel{\dfrac{1}{2}}$ $+\cancel{\dfrac{1}{2}}-\cancel{\dfrac{1}{3}}$ $+\cancel{\dfrac{1}{3}}-\cancel{\dfrac{1}{4}}$ . . . $+\cancel{\dfrac{1}{n-1}}-\cancel{\dfrac{1}{n}}$ $+\cancel{\dfrac{1}{n}}-\dfrac{1}{n+1}$ $=1-\dfrac{1}{n+1}$	The part is well-done. In fact, this is Example 5 in Chap 6B notes.

(i)	$\sum_{r=n+1}^{2n}\frac{1}{r^2+r}$ $=\sum_{r=1}^{2n}\frac{1}{r^2+r}-\sum_{r=1}^{n}\frac{1}{r^2+r}$ $=\left(1-\frac{1}{2n+1}\right)-\left(1-\frac{1}{n+1}\right)$ $=\frac{1}{n+1}-\frac{1}{2n+1}$ $=\frac{2n+1-n-1}{(n+1)(2n+1)}$ $=\frac{n}{(n+1)(2n+1)}$	Quite a number of scripts did this part by using method of differences. However, the question states **“Hence”**, so should make use of the previous part answer to do part **(i)**. Common mistakes • $\sum_{r=n+1}^{2n}\frac{1}{r^2+r}=\sum_{r=1}^{2n}\frac{1}{r^2+r}-\sum_{r=1}^{n+1}\frac{1}{r^2+r}$ (refer to Chap 6B notes page 4 property 2: $\sum_{r=k}^{m}u_r=\left(\sum_{r=1}^{m}u_r\right)-\left(\sum_{r=1}^{k-1}u_r\right)$) • $\sum_{r=n+1}^{2n}\frac{1}{r^2+r}=\sum_{r=n+1-n}^{2n-n}\frac{1}{r^2+r}=\sum_{r=1}^{n}\frac{1}{r^2+r}$ (no such rule)
(ii)	$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots$ $=\sum_{r=2}^{\infty}\frac{1}{r(r+1)}$ $=\sum_{r=1}^{\infty}\frac{1}{r(r+1)}-\frac{1}{1\times 2}$ $=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)-\frac{1}{2}$ $=\frac{1}{2}$	The presentation for this part is not well-done. Refer to Chap 6B notes Example 8(iii) on page 12 for the correct presentation. Quite a number of scripts did not realized that this represents an infinite sum and wrote $\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots=\sum_{r=2}^{n}\frac{1}{r(r+1)}$

5 In this question you may use expansions from the List of Formulae (MF26).

(a) Find the first four non-zero terms of the Maclaurin series of $e^{-x}(1+\cos 3x)$, in ascending powers of x. [4]

(b) It is given that the first three terms of this series are equal to the first three terms in the series expansion, in ascending powers of x, of $\frac{1}{a+bx}+cx^2$. Find the values of a, b and c. [4]

<table>
<tr>
<td>5(a)</td>
<td>
$e^{-x}(1+\cos 3x)$
$=\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)\left(1+1-\frac{(3x)^2}{2!}+\cdots\right)$
$=\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots\right)\left(2-\frac{9x^2}{2}+\cdots\right)$
$=2-2x+\left(1-\frac{9}{2}\right)x^2+\left(-\frac{1}{3}+\frac{9}{2}\right)x^3+\cdots$
$=2-2x-\frac{7}{2}x^2+\frac{25}{6}x^3+\cdots$

Alternative method:
Let $y=e^{-x}(1+\cos 3x)$
$ye^x=1+\cos 3x$
$e^x\frac{dy}{dx}+e^x y=-3\sin 3x$
$e^x\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}+e^x\frac{dy}{dx}+e^x y=-9\cos 3x$
$e^x\frac{d^2y}{dx^2}+2e^x\frac{dy}{dx}+e^x y=-9\cos 3x$
$e^x\frac{d^3y}{dx^3}+3e^x\frac{d^2y}{dx^2}+3e^x\frac{dy}{dx}+e^x y=27\sin 3x$
When $x=0$, $y=2$, $\frac{dy}{dx}=-2$, $\frac{d^2y}{dx^2}=-7$, $\frac{d^3y}{dx^3}=25$,
$e^{-x}(1+\cos 3x)=2-2x-\frac{7}{2!}x^2+\frac{25}{3!}x^3+\cdots$
$=2-2x-\frac{7}{2}x^2+\frac{25}{6}x^3+\cdots$
</td>
<td>
Easier to use standard series expansions of e^{-x} and $\cos 3x$ from MF26.

There are various way to perform the repeated differentiation, e.g. use product rule on the RHS.
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</table>

(b)	$\frac{1}{a+bx}+cx^2$ $=a^{-1}\left(1+\frac{b}{a}x\right)^{-1}+cx^2$ $=\frac{1}{a}\left(1-\frac{b}{a}x+\left(\frac{b}{a}\right)^2x^2+\cdots\right)+cx^2$ $=\frac{1}{a}-\frac{b}{a^2}x+\left(\frac{b^2}{a^3}+c\right)x^2+\cdots$ Comparing with the series expansion in **(a)**, $\frac{1}{a}=2 \quad \Rightarrow \quad a=\frac{1}{2}$ $-\frac{b}{a^2}=-2 \quad \Rightarrow \quad b=2a^2=\frac{1}{2}$ $\frac{b^2}{a^3}+c=-\frac{7}{2} \quad \Rightarrow \quad c=-\frac{7}{2}-\frac{b^2}{a^3}=-\frac{7}{2}-2=-\frac{11}{2}$	Use the expansion of $(1+x)^n$. Factor out a and remember to apply power of –1 to it.

6 **(a)** Let $y=a^x$, where a is a positive constant. Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=a^x\ln a$. [2]

(b) A curve has equation

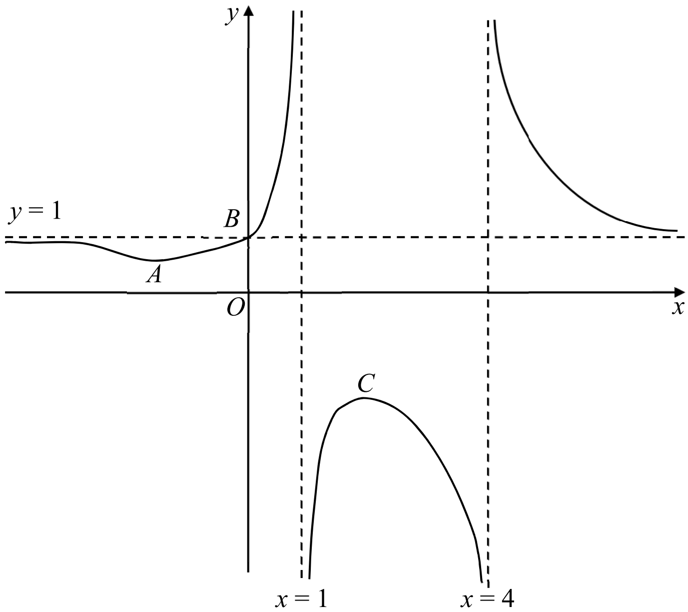
$$y\left(3^x\right)+2^{y-1}=2.$$

Find the equation of the normal to the curve at the point $(0,1)$. Give your answer in the form $y=Ax+B$, where A and B are constants in the exact form. [6]

6(a)	$y=a^x \Rightarrow \ln y = x\ln a$. Differentiate with respect to x: $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\ln a \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=y\ln a=a^x\ln a$ (shown). Alternatively, $y=a^x=\mathrm{e}^{\ln a^x}=\mathrm{e}^{x\ln a}$. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x\ln a}\left(\ln a\right)=a^x\ln a$ (shown).	Generally most of the students are able to show the result.
(b)	$y\left(3^x\right)+2^{y-1}=2$ Differentiate with respect to x: $\dfrac{\mathrm{d}y}{\mathrm{d}x}3^x+y3^x\ln 3+2^{y-1}\dfrac{\mathrm{d}y}{\mathrm{d}x}\ln 2=0$ $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-y3^x\ln 3}{3^x+2^{y-1}\ln 2}$. At $(0,1)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-(1)(1)\ln 3}{(1)+(1)\ln 2}=\dfrac{-\ln 3}{1+\ln 2}$. Equation of normal at $(0,1)$ is $y-1=\dfrac{1+\ln 2}{\ln 3}(x-0)$ $\Rightarrow y=\left(\dfrac{1+\ln 2}{\ln 3}\right)x+1$, where $A=\dfrac{1+\ln 2}{\ln 3}$, $B=1$ (shown).	Many students have stated that $\ln\left[y\left(3^x\right)+2^{y-1}\right]$ $=\ln\left[y\left(3^x\right)\right]+\ln\left[2^{y-1}\right]$ which is wrong. **Note that** $\ln(A+B)\neq \ln A+\ln B$. Also Chain rule was not applied correctly, resulting in $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2^{y-1}\right)=2^{y-1}\ln 2$ instead of $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2^{y-1}\right)=2^{y-1}\dfrac{\mathrm{d}y}{\mathrm{d}x}\ln 2$.

7 **(a)** A curve C has equation $y = \dfrac{ax-3}{(x-3)(x-1)}$ where a is a real constant such that $a \neq 1,\ 3$. Determine the range of values of a for which the curve C has no turning points. [4]

(b)



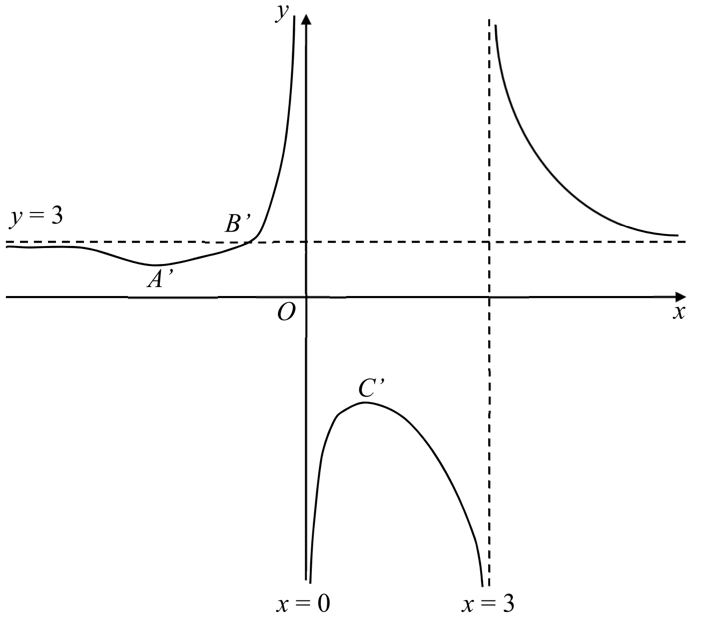
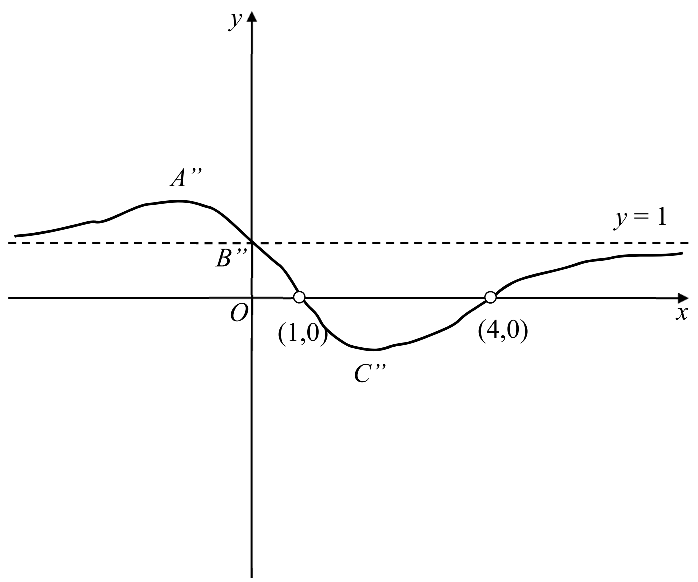
The diagram above shows the graph of $y = \mathrm{f}(x)$. It has asymptotes $y = 1$, $x = 1$ and $x = 4$. The curve cuts the y-axis at the point $B(0, 1)$, has a minimum at the point $A\left(-2, \dfrac{2}{3}\right)$ and a maximum at the point $C(2, -2)$.

By showing clearly the equations of asymptotes and the coordinates of the points corresponding to A, B and C where possible, sketch, on **separate diagrams**, the graphs of

(i) $y = 3\mathrm{f}(x+1)$, [4]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$. [4]

7(a)	$y = \dfrac{ax-3}{(x-1)(x-3)}$ $\dfrac{dy}{dx} = \dfrac{a(x^2-4x+3)-(2x-4)(ax-3)}{\left[(x-1)(x-3)\right]^2}$ $= \dfrac{-ax^2+6x+3a-12}{\left[(x-1)(x-3)\right]^2}$ To find turning points, $\dfrac{dy}{dx} = 0$ So, $-ax^2+6x+3a-12=0$ For the curve to have no turning points, discriminant of $-ax^2+6x+3a-12=0$ is <0	Some students think of $\dfrac{dy}{dx} > 0$ or $\dfrac{dy}{dx} < 0$ but not $\dfrac{dy}{dx} = 0$. Factorisation of the quadratic expression can definitely be improved.

	$36-4(-a)(3a-12)<0$ $12a^2-48a+36<0$ $12(a-1)(a-3)<0$ $\Rightarrow 1<a<3$	
(b) **(i)**	 $A'(-3,2)$, $B'(-1,3)$ and $C'(1,-6)$.	This question involves simple transformations about x and y axis. Majority of students were able to perform the transformations. However, some are still confused over the directions.
(ii)	 $A''(-2,\frac{3}{2})$, $B''(0,1)$ and $C''(2,-\frac{1}{2})$.	The shape of graph drawn by some students was not smooth. Some forgot that x intercepts are not included. The concept behind drawing of reciprocal graphs definitely can be improved upon by the cohort. Not that well done in general.

8 **Do not use a calculator in answering this question.**

(a) One root of the equation $2z^3 - 5z^2 + \alpha z - 5 = 0$, where $\alpha \in \mathbb{R}$, is $z = 1 - 2\text{i}$. Find the value of α and the other roots. [5]

(b) The complex number z is given by

$$z = \frac{\left(-2 + 2\sqrt{3}\text{i}\right)^2}{\cos\frac{1}{12}\pi + \text{i}\sin\frac{1}{12}\pi}.$$

Find $|z|$ and $\arg(z)$. [4]

8(a)	Since coefficients of equation are real, $z = 1 + 2\text{i}$ is also a root. So, the quadratic factor is $\left(z - (1 - 2\text{i})\right)\left(z - (1 + 2\text{i})\right) = z^2 - 2z + 5$. Thus, by observation, $2z^3 - 5z^2 + \alpha z - 5 = \left(z^2 - 2z + 5\right)(2z - 1)$ Comparing coefficients of z, $\alpha = 10 + 2 = 12$. The other roots of the equation are $1 + 2\text{i}$ and $\frac{1}{2}$.	This question was generally well done. Almost all recognizing that $z = 1 + 2\text{i}$ was a root. However algebraic slips by some in finding the quadratic factor corresponding to the pair of complex conjugate roots. Most common error was not observing that the coefficient of z^3 was "2" so a linear factor is of the form $(2z - k)$. Some students not aware of the difference between "root" which is $\frac{1}{2}$ and "linear factor" $(2z - 1)$
	Alternatively, $(1 - 2\text{i})^2 = 1 - 4\text{i} - 4 = -3 - 4\text{i}$ $(1 - 2\text{i})^3 = (-3 - 4\text{i})(1 - 2\text{i}) = -3 + 2\text{i} - 8 = -11 + 2\text{i}$ Substitute $z = 1 - 2\text{i}$ into $2z^3 - 5z^2 + \alpha z - 5 = 0$, $2(-11 + 2\text{i}) - 5(-3 - 4\text{i}) + \alpha(1 - 2\text{i}) - 5 = 0$ $\alpha(1 - 2\text{i}) = 12 - 24\text{i} = 12(1 - 2\text{i})$ $\alpha = 12$ Since coefficients of equation are real, $z = 1 + 2\text{i}$ is also a root. So, the quadratic factor is $\left(z - (1 - 2\text{i})\right)\left(z - (1 + 2\text{i})\right) = z^2 - 2z + 5$. Thus, $2z^3 - 5z^2 + 12z - 5 = \left(z^2 - 2z + 5\right)(2z - 1)$ $z = \frac{1}{2}$ is the third root of the equation.	Question states **"do not use a calculator"** so working to find $(1 - 2\text{i})^2$ and $(1 - 2\text{i})^3$ is needed.

(b)	$\left\|-2+2\sqrt{3}\mathrm{i}\right\| = \sqrt{2^2+\left(2\sqrt{3}\right)^2} = \sqrt{4+12} = 4$ $\arg\left(-2+2\sqrt{3}\mathrm{i}\right) = \frac{2\pi}{3}$ So, $\|z\| = \frac{4^2}{1} = 16$ $\arg(z) = 2\left(\frac{2\pi}{3}\right) - \frac{\pi}{12} - 2\pi = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$ Alternatively, $z = \frac{\left(-2+2\sqrt{3}\mathrm{i}\right)^2}{\cos\frac{1}{12}\pi + \mathrm{i}\sin\frac{1}{12}\pi} = \frac{\left(4\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}\right)^2}{\mathrm{e}^{\frac{\pi\mathrm{i}}{12}}} = 16\mathrm{e}^{\left(\frac{4\pi}{3}-\frac{\pi}{12}\right)\mathrm{i}} = 16\mathrm{e}^{\left(\frac{5\pi}{4}\right)\mathrm{i}}$ $z = 16\mathrm{e}^{\left(-\frac{3\pi}{4}\right)\mathrm{i}}$ $\|z\| = 16$ and $\arg(z) = -\frac{3\pi}{4}$	Many students appeared to struggle with this question failing to represent the "numerator" in modulus - argument form. Several also forgot the restriction that $-\pi < \arg(z) \le \pi$. Students should not omit "-2π": $\arg(z) = \frac{5\pi}{4} = -\frac{3\pi}{4}$ Solutions using exponential form appeared shorter/neater.

9 It is given that

$$f : x \mapsto \left| \frac{1}{x-4} \right| \text{, where } x \in \mathbb{R}, x \neq 4,$$

$$g : x \mapsto \ln(x+2) \text{ , where } x \in \mathbb{R}, x > -2.$$

(a) Explain why the composite function gf exists and find gf in a similar form. [3]

(b) Find the range of gf. [1]

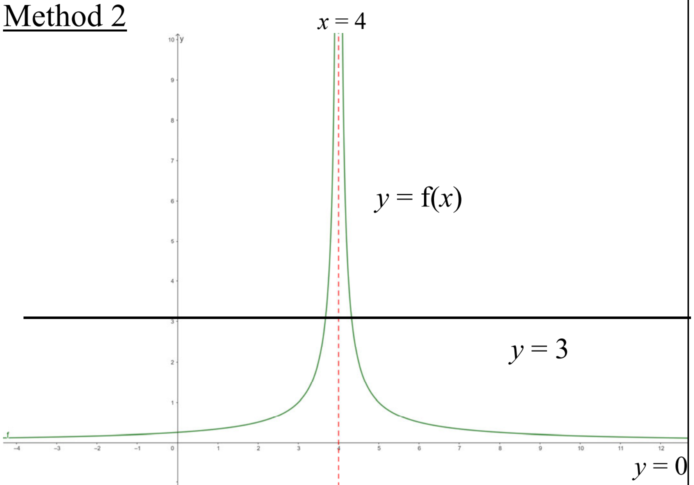
(c) Explain why f does not have an inverse. [1]

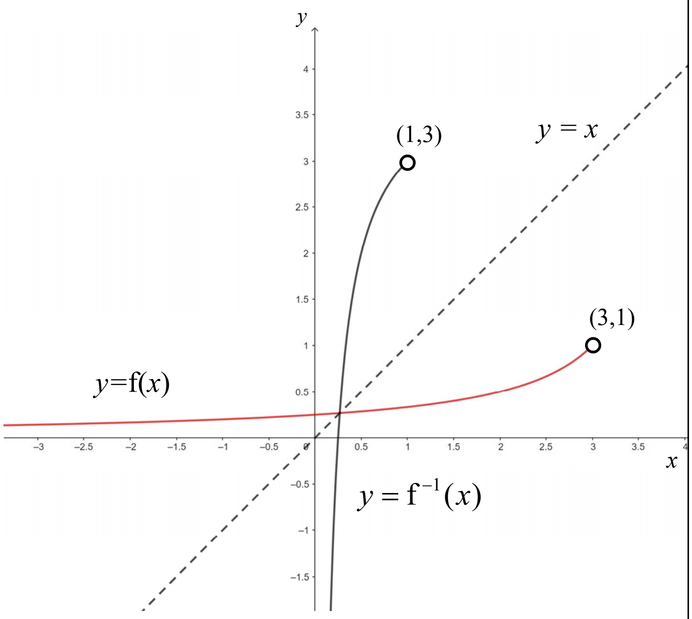
(d) If the domain of f is further restricted to $x < k$, state the maximum value of k such that the function f^{-1} exist. [1]

In the rest of this question, the domain of f is $x \in \mathbb{R},\ x < 3$.

(e) Find $f^{-1}(x)$ and state the domain of f^{-1}. [3]

(f) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram and state the geometrical relation between the two graphs. [3]

9(a)	Since $R_f = (0,\infty) \subseteq D_g = (-2,\infty)$, gf exists. $gf : x \mapsto \ln\left(\left\|\frac{1}{x-4}\right\|+2\right)$, where $x \in \mathbb{R}, x \neq 4$	To find gf in a <u>similar form</u>, many students forgot about specifying the domain of gf. Some **<u>wrong</u>** set notations observed : • gf = … ✗ ; gf(x) = … ✓ • $R_f \in D_g$ ✗ ; $R_f \subseteq D_g$ ✓ • $R_f \in (0,\infty)$ ✗ ; $R_f = (0,\infty)$ ✓ • $R_f > 0$ ✗ ; $R_f = (0,\infty)$ ✓ • $r_f = (0,\infty)$ ✗ ; $R_f = (0,\infty)$ ✓ • $x\ \varepsilon\ \mathbb{R}$ ✗ ; $x \in \mathbb{R}$ ✓
(b)	$R_{gf} = (\ln 2, \infty)$	Generally, well-done.
(c)	<u>Method 1</u> Since $3, 5 \in D_f$ and $3 \neq 5$ such that $f(3) = 1 = f(5)$, f is not one-one. Hence f^{-1} does not exist.	<u>Mtd 1</u> is analytical in nature. Many students attempted this approach missed 1 or 2 conditions.
	<u>Method 2</u>  Since the horizontal line $y = 3$ cuts the graph of f twice, f is not one-one. Hence f^{-1} does not exist.	<u>Mtd 2</u> is graphical in nature. Many students correctly mentioned "… *cuts the graph of* …", but the graph is nowhere to be seen. Yes, you ought to provide a sketch of y = f(x) to support your answer, using this approach. Some **<u>wrong</u>** expressions observed : • graph of f(x) ✗ graph of f ✓ graph of y = f(x) ✓ • cuts f ✗ cuts graph of f ✓ • f(x) is not 1-1 ✗ f is not 1-1 ✓

(d)	Maximum $k = 4$	Generally, well-done. It is also okay to write just "4".
(e)	Let $y = \left\lvert \frac{1}{x-4} \right\rvert = -\frac{1}{x-4}$, since $x < 3$ $x - 4 = -\frac{1}{y}$ $x = 4 - \frac{1}{y}$ $\therefore f^{-1}(x) = 4 - \frac{1}{x}$ $D_{f^{-1}} = R_f = (0,1)$	We ought to provide the reason to simplify to $y = -\frac{1}{x-4}$. Some students forgot to find the domain of f^{-1}.
(f)	 The graph of $y = f(x)$ is a reflection of the graph of $y = f^{-1}(x)$ in the line $y = x$.	• You must label – the 2 graphs. – the axes – the end-points (empty-circled) • Use the <u>same-scale</u> for both axes, and sketch $y = x$ (dotted) so that the symmetrical relationship can be clearly depicted. The line $y = x$ should pass though the origin and the point of intersection between the graphs of $y = f(x)$ and $y = f^{-1}(x)$. • Clearly, the values "1" and "3" are *highlights*. It is not right to sketch with "1" being so close to "3", but far away from "0".

10 In long-track speed skating, races are run counter-clockwise on a 400-metre two-lane oval rink. A full 400-metre round the rink is known as a lap. The current world records for the 10000 metres men's race and 500 metres men's race are 12 minutes 30.74 seconds (11 Feb 2022) and 33.61 seconds (9 Mar 2019) respectively.

Abel and Caine are 2 skaters training for the 10000 metres men's race. During a particular training session for Abel, he completes the first lap in b seconds and he takes 10% longer to complete each succeeding lap than he does in the previous lap.

(a) Write down an expression for the time taken by Abel to complete n laps, in terms of b and n.
Hence find the value of b that will enable Abel to complete 10000 metres in 13 minutes. [3]

(b) Comment on the feasibility of this value of b in the context of the question. [1]

After training for 6 months, Abel and Caine both entered a 10000 metres men's race. During the race, Abel completes the first lap in k seconds and then he takes 1% longer to complete each succeeding lap than he does in the previous lap. Caine also completes the first lap in k seconds and on each subsequent lap he spends d seconds more than he spent on the previous lap. They arrive at the finish line at the same time.

(c) Find the value of $\frac{d}{k}$, correct to 5 decimal places. [4]

(d) Given that Abel completes his 25th lap in p seconds and Caine completes his 25th lap in q seconds, evaluate $\frac{q}{p}$, giving your answer correct to 3 decimal places.
Hence determine the skater who has the faster time for the first 24 laps. [4]

10 (a)	The lap times for Abel forms a GP with first term b and common ratio 1.1. Time taken to complete n laps, $S_{\text{n}} = \dfrac{b(1.1^n - 1)}{1.1 - 1}$ $= 10b(1.1^n - 1)$ 10000 metres equals 25 laps, so we let $S_{25} = 13 \times 60$ $10b(1.1^{25} - 1) = 780$ $b = \dfrac{78}{(1.1^{25} - 1)}$ $= 7.9311 = 7.93 \quad (3 \text{ s.f.})$	Simplify the expression for S_{n} Answer the question explicitly. Some students wrote both u_n and S_{n} and so it was not clear which was the answer to the question. Quite a number also missed out this part and only did the numerical workings.
(b)	The current world record for 500 metres is 33.61 seconds and this means that the average time per lap is around 26.89 seconds. Hence, Abel's training strategy is not feasible as he is unlikely to complete the first 400 metres in 7.93 seconds.	Make reference to the information given in the question.
(c)	Abel's times form a GP with first term k and common ratio 1.01. Caines's times form an AP with first term k and common difference d.	

	We have $$S_{25}(\text{Abel}) = S_{25}(\text{Caine})$$ $$\frac{k(1.01^{25}-1)}{1.01-1} = \frac{25}{2}\left(2k+24d\right)$$ $$100k(1.01^{25}-1) = 25k+300d$$ $$k\left(100(1.01^{25})-125\right) = 300d$$ $$\frac{d}{k} = \frac{100(1.01^{25})-125}{300}$$ $$\approx 0.01081 \quad (5\text{ dp})$$	Give your answer correct to the accuracy as specified by the question.
(d)	We have $$\frac{q}{p} = \frac{k+24d}{k(1.01^{24})}$$ $$= \frac{1}{1.01^{24}} + \frac{24}{1.01^{24}}\left(\frac{d}{k}\right)$$ $$\approx \frac{1}{1.01^{24}} + \frac{24}{1.01^{24}}\left(0.01081\right)$$ $$\approx 0.99190 = 0.992 \ \left(3\text{ dp}\right).$$ Hence, $q \approx 0.992p < p$. This means Abel spent more time on the last lap than Caine. Since both of them arrived at the finish line at the same time, Abel is the skater who has the faster time for the first 24 laps.	Give your answer correct to the accuracy as specified by the question. Answer the question explicitly and with some reasoning. As the question states 'hence', the conclusion must be inferred from the fact that $q < p$.

11 The lines l_1 and l_2 have equations

$$\mathbf{r}=\begin{pmatrix}10\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\1\end{pmatrix} \quad \text{and} \quad \mathbf{r}=\begin{pmatrix}4\\4\\5\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

respectively, where λ and μ are parameters.

(a) Without the use of a calculator, show that l_1 and l_2 are skew lines. [3]

(b) Find a vector, $\mathbf{n}$, that is perpendicular to both l_1 and l_2. [1]

Referred to the origin O, points P and Q have position vectors $\begin{pmatrix}4\\4\\5\end{pmatrix}$ and $\begin{pmatrix}10\\-1\\2\end{pmatrix}$ respectively.

(c) Find the exact length of projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$. [2]

A plane π has equation $3x+z=11$.

(d) Find, in degrees, the acute angles between l_1 and π, and between l_2 and π. Hence, or otherwise, determine which one of l_1 or l_2 is not intersecting π. [4]

(e) Find the exact distance between the line determined in part **(d)** and π. [2]

(a)	Since $\begin{pmatrix}3\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}-1\\2\\3\end{pmatrix}$ for all $k\in\mathbb{R}$, so l_1 and l_2 are not parallel. Consider $\begin{pmatrix}10\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\1\end{pmatrix}=\begin{pmatrix}4\\4\\5\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}$ $\Rightarrow\begin{cases}10+3\lambda=4-\mu\\-1-\lambda=4+2\mu\\2+\lambda=5+3\mu\end{cases}\Rightarrow\begin{cases}3\lambda+\mu=-6 & ----(1)\\\lambda+2\mu=-5 & ----(2)\\\lambda-3\mu=3 & ----(3)\end{cases}$ Using (1) and (2) to solve, we get $\lambda=-\frac{7}{5}$ and $\mu=-\frac{9}{5}$. LHS of (3) $=\lambda-3\mu=-\frac{7}{5}-3\left(-\frac{9}{5}\right)=4\neq$ RHS of (3) So l_1 and l_2 do not intersect. Hence, l_1 and l_2 are skew lines (shown).	Generally, showing parallel lines is well done. For finding inconsistent solutions (no solution) for simultaneous equations, the common mistakes are • using GC (question states that "without the use of a calculator"), • did not show inconsistency or contradiction clearly.
(b)	$\begin{pmatrix}3\\-1\\1\end{pmatrix}\times\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}-5\\-10\\5\end{pmatrix}=-5\begin{pmatrix}1\\2\\-1\end{pmatrix}$. $\therefore\mathbf{n}=\begin{pmatrix}1\\2\\-1\end{pmatrix}$.	Generally, well done, note that $5\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ are parallel but they are not equal vectors.

(c)	Length of projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$ $=\dfrac{\left|\left[\begin{pmatrix}10\\-1\\2\end{pmatrix}-\begin{pmatrix}4\\4\\5\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}\right|}{\sqrt{1+4+1}}$ $=\dfrac{\left|\begin{pmatrix}6\\-5\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\end{pmatrix}\right|}{\sqrt{6}}$ $=\dfrac{\lvert 6-10+3\rvert}{\sqrt{6}}=\dfrac{1}{\sqrt{6}}.$	Common mistakes: • careless in writing vectors such as missing negative sign or dot product error. It is important to check. • Missing modulus sign in the numerator. As the dot product can give negative value, and we are interested in the absolute value to find the length. So, the modulus sign is necessary.
(d)	$\pi:\ \mathbf{r}\cdot\begin{pmatrix}3\\0\\1\end{pmatrix}=11$ Acute angle between l_1 and π $=\sin^{-1}\dfrac{\left|\begin{pmatrix}3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\1\end{pmatrix}\right|}{\sqrt{10}\sqrt{11}}=\sin^{-1}\dfrac{10}{\sqrt{10\times 11}}=72.5^\circ$ (nearest 0.1°). Acute angle between l_2 and π $=\sin^{-1}\dfrac{\left|\begin{pmatrix}3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}\right|}{\sqrt{10}\sqrt{14}}=\sin^{-1}\dfrac{0}{\sqrt{10\times 14}}=0^\circ.$ Hence, either l_2 is on π or l_2 is parallel to π, but not on π. Since $\begin{pmatrix}4\\4\\5\end{pmatrix}\cdot\begin{pmatrix}3\\0\\1\end{pmatrix}=12+0+5=17\neq 11$, so l_2 is not on π but is parallel to π. Thus, l_2 is not intersecting π.	Common mistakes: • Use inverse cosine, but did not find its complement angle to get the correct answer • Did not give the required angular unit as stated in the question (in degrees) • Did not give the required accuracy (inexact answers in degree should be correct to 0.1°) Note that angle between a line and plane is 0° $\Rightarrow$ the line and the plane are parallel $\Rightarrow$ the line could lie on the plane OR the line and the plane does not intersect. Many students did not check that line l_2 does not lie on plane π which is a key step.
(e)	$(0,\ 0,\ 11)$ is on π since $\begin{pmatrix}0\\0\\11\end{pmatrix}\cdot\begin{pmatrix}3\\0\\1\end{pmatrix}=11.$ Distance between l_2 and π	Common mistakes are • Some chose a point on the line l_1 instead of line l_2 • Some did not find the difference between a point

	$= \dfrac{\left\lvert \left[\begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix} \right] \bullet \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{9+0+1}}$ $= \dfrac{\left\lvert \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\rvert}{\sqrt{10}}$ $= \dfrac{\lvert 12+0-6 \rvert}{\sqrt{10}}$ $= \dfrac{6}{\sqrt{10}}$ or $\dfrac{3\sqrt{10}}{5}$.	on the line l_2 and a point on the plane π