RAFFLES INSTITUTION
2024 YEAR 5 PROMOTION EXAMINATION
Higher 2

MATHEMATICS 9758

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF26)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

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Mathematics Department

1 A curve D has equation

$$y = ax + b\ln x + \frac{c}{x}, \quad x > 0,$$

where a, b and c are constants. It is given that D has a stationary point at $x = \frac{3}{2}$ and the tangent to D at the point where $x = 1$ is $y = x - 5$. Find the values of a, b and c. [4]

2 With respect to the origin O, the fixed points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel.

(a) The variable point R has position vector $\mathbf{r} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a real parameter. Describe geometrically the set of all possible positions of R. [1]

It is given that the angle AOB is $90°$.

(b) Explain why $\mathbf{a}.\mathbf{b} = 0$. [1]

(c) Among the set of all possible points R, the point R^* is the closest to the origin O. Find the position vector of R^*. Hence state the ratio $AR^* : BR^*$ in terms of magnitudes of $\mathbf{a}$ and $\mathbf{b}$. [4]

3 **Do not use a calculator in answering this question.**

Solve the inequality

$$\frac{6x^2 + 2x - 3}{2x - 1} \geq 2(x+1).$$ [4]

Hence solve $\dfrac{6x^2 + 2|x| - 3}{2|x| - 1} \geq 2(|x|+1)$. [3]

4 A curve C has parametric equations

$$x = \frac{\lambda}{1+\lambda^3} \text{ and } y = \frac{\lambda^2}{1+\lambda^3}, \quad \text{for } \lambda > 0.$$

The point P is a variable point on C.

(a) With reference to the origin O, OP forms the diagonal of the rectangle $OQPR$, where vertices Q and R lie on the x- and y-axis respectively. Using differentiation, find the value of λ which maximises the area of rectangle $OQPR$. You need to show that your answer gives a maximum. [5]

When the area of rectangle $OQPR$ is at its maximum, the rate of change of the x-coordinate of the point P is 1 unit per second.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence determine the rate of change of the y-coordinate of the point P at this instant. [4]

5 **(a)** Find $\sum_{r=0}^{n}\left[(n+2)r+n\right]$, giving your answer in terms of n. [3]

[You may use the result $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for the rest of this question.]

(b) By writing down the first two and the last two terms in the series, find $\sum_{r=1}^{n}(r+2)^3$, giving your answer in terms of n. [3]

(c) Find $1^3 - 2^3 + 3^3 - 4^3 + 5^3 - 6^3 + \ldots + (2n-1)^3 - (2n)^3$, giving your answer in terms of n. [3]

6 It is given that $e^{y} = 1 + \sin 3x$.

(a) Show that

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 9 = 9e^{-y}.$$

Hence find the series expansion of y in ascending powers of x, up to and including the term in x^3, simplifying your answer. [5]

(b) Using standard series from the List of Formulae, verify that the series expansion obtained in **part (a)** is correct. [3]

7 **Do not use a calculator in answering this question.**

(a) Given that $x = 1 + 3i$ is a root of the equation

$$x^3 + ax^2 + 18x + b = 0,$$

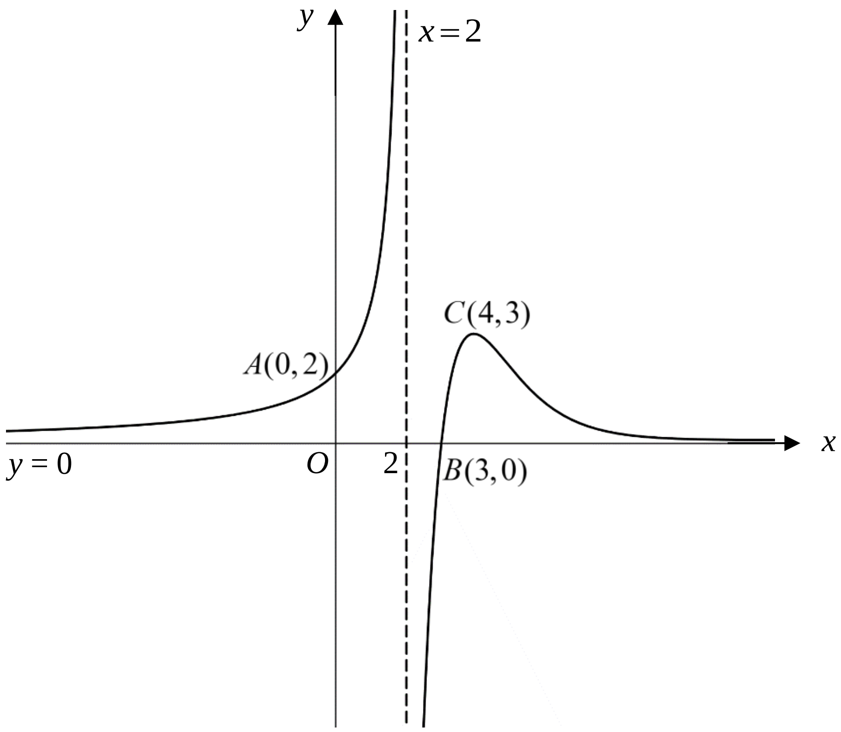
find the values of the real numbers a and b, and the other roots. [5]

(b) The complex numbers z and w satisfy the following equations.

$$w^* + z = 4 - 6i$$
$$w - 2z = 1 + 10i$$

Find z and w, giving your answers in the form $c + id$, where c and d are real numbers. [5]

8 **(a)** The diagram shows the curve $y = \mathrm{f}(x)$ with a maximum point at $C(4,3)$. The curve crosses the axes at the points $A(0,2)$ and $B(3,0)$. The lines $x = 2$ and $y = 0$ are the asymptotes of the curve.



Sketch the graph of $y = \mathrm{f}'(x)$, clearly stating the equations of the asymptotes and the coordinates of the points corresponding to A, B and C where appropriate. [3]

(b) The curve C_1 has equation $y = \dfrac{ax^2 + bx - 8}{x - 2}$, where a and b are constants. It is given that C_1 has an asymptote $y = 3 - 2x$.

(i) State the value of a and show that $b = 7$. [3]

(ii) Sketch C_1, clearly stating the equations of any asymptotes, the coordinates of any stationary points and of any points where C_1 crosses the axes. [3]

(iii) The curve C_1 is transformed by a translation of 2 units in the negative x-direction, followed by a stretch with scale factor $\dfrac{1}{2}$ parallel to the y-axis, to form the curve C_2. Find the equation of C_2. [2]

9 The planes p and q have equations

$$\mathbf{r} = (2+\lambda-2\mu)\mathbf{i} + (-3\lambda+\mu)\mathbf{j} + (3-\lambda+2\mu)\mathbf{k} \quad \text{and} \quad \mathbf{r}.\begin{pmatrix} a \\ -1 \\ b \end{pmatrix} = 1 \text{ respectively,}$$

where a and b are constants and λ and μ are parameters.

The line l passes through the point $(5,4,0)$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$. The planes p and q meet in the line l.

(a) Show that $a=1$ and $b=\frac{1}{2}$. [2]

(b) Find the exact acute angle between the planes p and q. [3]

(c) Find the distance from the point $A(2,0,3)$ to the plane q. Hence deduce the shortest distance from A to l. [4]

The plane q is reflected about the plane p to obtain the plane q'.

(d) Find a cartesian equation of the plane q'. [3]

10 It is given that

$$f(x) = \begin{cases} \frac{1}{2}x^2, & \text{for } 1 \le x < 2, \\ \frac{1}{4}(-3x+14), & \text{for } 2 \le x < 4, \end{cases}$$

and that $f(x) = f(x+3)$ for all real values of x.

(a) State the value of $f(0)$. [1]

(b) Sketch the graph of $y = f(x)$ for $0 \le x \le 5$. [3]

(c) The function g is given by $g(x) = \frac{4}{3}|x-2|$ for $x \in \mathbb{R}$. By sketching the graph of $y = g(x)$ on the same diagram as in **part (b)**, solve the inequality $f(x) > g(x)$. [4]

(d) The function h is given by $h(x) = \sqrt{3}\sin(\pi x) + \cos(\pi x) + 1$ for $0 \le x \le 2$. Explain why the composite function hf exists and find its range in exact form. [4]

11 A financial institution, Future Investments Inc., has introduced a new investment scheme. The scheme pays a compound interest of 5% per annum at the end of the year, based on the amount in the account at the beginning of each year.

John and Sarah are both interested in this scheme.

(a) John invests $\$x$ at the start of the first year and a further $\$x$ on the first day of each subsequent year. He chooses to leave the money in his account for the interest to accumulate.

(i) Write down the amount in John's account, including the interest, at the end of the first year. [1]

(ii) Show that John will have a total of $\$\,21x\left(1.05^n-1\right)$ in his account at the end of n years. [3]

(iii) If John invests $\$10\,000$ at the start of every year, find the number of years for the total in his account to first exceed $\$500\,000$. Determine if this happens at the start or at the end of that year. [4]

(b) Sarah invests $\$6\,000$ at the start of the first year. On the first day of each subsequent year, she invests $\$400$ more than the amount invested at the start of the previous year.

(i) Explain why the amount in Sarah's account at the end of n years can be given by $\sum_{r=1}^{n}\left[a+b(r-1)\right](1.05)^{n-(r-1)}$, and determine the values of the constants a and b. [2]

(ii) Hence determine the number of complete years for the total amount in her account to first exceed $\$500\,000$. [2]

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