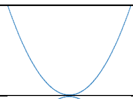
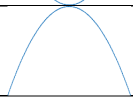
# Additional Mathematics Notes

## Functions and Graphs

| | | |
|---|---|---|
| $2x^2 + 2$<br>$a > 0$ |  | Turning point = Minimum point |
| $-2x^2 + 2$<br>$a < 0$ |  | Turning point = Maximum point |

Line of symmetry always passes through the turning point
($x$-coordinate of turning point/mid-value of $x$-intercepts)

| | | |
|---|---|---|
| y-intercept, x=0 | $y = ax^2 + bx + c$ | $y = a(0)^2 + b(0) + c$ |
| x-intercept, y=0 | | $0 = ax^2 - bx + c$ |

## Completing the Square

| | | | |
|---|---|---|---|
| $ax^2 + bx + c$ | $a(x^2 + \frac{b}{a}x + \frac{c}{a})$ | $a\left[(x + \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ | $a(x+h)^2 + k$<br>$a(x-h)^2 + k$ |
| Coefficient of $x^2$ must be +1 | | | |
| $ax^2 - bx + c$ | $a(x^2 - \frac{b}{a}x + \frac{c}{a})$ | $a\left[(x - \frac{b}{2a})^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ | |

## Conditions for Curve to Lie Completely Above or Below x-axis

| | | | |
|---|---|---|---|
| Above | Coefficient of $x^2 > 0$ | Minimum value is positive | $b^2 - 4ac < 0$ |
| ∴Since the coefficient of $x^2 = 1 > 0$ and the minimum value of $y = 4$, which is positive, the curve lies completely above the x-axis. | | | |
| Below | Coefficient of $x^2 < 0$ | Minimum value is negative | |
| ∴Since the coefficient of $x^2 = -1 < 0$ and the maximum value of $y = -4$, which is negative, the curve lies completely below the x-axis. | | | |

## Quadratic Formula (Given)

| | |
|---|---|
| $ax^2 + bx + c$ | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |

## Nature of Roots of Quadratic Equations

| | |
|---|---|
| $b^2 - 4ac \geq 0$ | Real roots |
| $b^2 - 4ac > 0$ | 2 real and distinct roots |
| $b^2 - 4ac < 0$ | Non-real roots |
| $b^2 - 4ac = 0$ | 2 real and equal roots |

## Intersection Between Line and Curve

| | |
|---|---|
| $b^2 - 4ac > 0$ | Line cuts curve at 2 distinct points |
| $b^2 - 4ac = 0$ | Line cuts curve at 1 point (Line is tangent to curve) |
| $b^2 - 4ac < 0$ | Line does not intersect curve |

## Finding Range of Values

| | |
|---|---|
| E.g. $(p-6)(p+2) > 0$ | Draw number line and curve |

## Laws of Surds

| | | | |
|---|---|---|---|
| $\sqrt{ab}$ | | $\sqrt{a} \times \sqrt{b}$ | |
| $\sqrt{\frac{a}{b}}$ | | $\frac{\sqrt{a}}{\sqrt{b}}$ | |
| $\sqrt{a} \times \sqrt{a},\ (\sqrt{a})^2$ | | $a$ | |
| $p\sqrt{a} + q\sqrt{a}$ | | $(p+q)\sqrt{a}$ | |
| $p\sqrt{a} - q\sqrt{a}$ | | $(p-q)\sqrt{a}$ | |
| $p^2 + (q\sqrt{a})^2$ | | $p^2 + q^2a$ | |
| $\frac{1}{\sqrt{2}+3}$ | $\frac{1}{\sqrt{2}+3} \times \frac{\sqrt{2}-3}{\sqrt{2}-3}$ | $\frac{\sqrt{2}-3}{2-9}$ | $-\frac{\sqrt{2}-3}{7}$ |

## Equality of Surds

| | | | |
|---|---|---|---|
| $a + b\sqrt{n}$ | $5 + 4\sqrt{n}$ | $a = 5$ | $b = 4$ |

**Polynomials**

| | |
|---|---|
| Degree of $P(x) \times Q(x)$ | Degree of $P(x)$ + Degree of $Q(x)$ |

Division Algorithm: $P(x) = D(x) \times Q(x) + R(x)$

| | | | |
|---|---|---|---|
| $P(x)$ = Dividend | $D(x)$ = Divisor | $Q(x)$ = Quotient | $R(x)$ = Remainder |

**Remainder and Factor Theorem**

| | |
|---|---|
| $f(x) \div (ax+b)$ | $Remainder = f(-\frac{b}{a})$ |

Factor Theorem, Remainder=0

**Cubic Expressions and Equations**

$$ax^3 + bx^2 + cx + d$$

| | |
|---|---|
| $(ax+b)(hx+k)(px+q)$ | $(ax+b)(px^2+qx+r)$ |

| | |
|---|---|
| Step 1 | Factorise $f(x)$ using Factor Theorem [Mode-3-4] |
| Step 2 | Use Synthetic Division/ Long Division/ Comparing Coefficients to factorise $f(x)$ completely |

**Synthetic Division**

| | | | | | |
|---|---|---|---|---|---|
| $f(x) = 2x^3 - 11x^2 - 7x + 6$ | -1 | 2 | -11 | -7 | 6 |
| $(x+1)$ | | | -2 | 13 | -6 |
| $f(x) = (x+1)(2x^2 - 13x + 6)$ | | 2 | -13 | 6 | 0 |

**Cubic Identities**

| | |
|---|---|
| Sum of cubes | $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ |
| Difference of cubes | $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ |

**Partial Fractions**

| | |
|---|---|
| Step 1: | Fraction must be a proper fraction (Degree of numerator < Degree of denominator) |
| Step 2: | Factorise denominator |
| Step 3: | Express proper algebraic fraction in partial fractions |

| Case | Denominator | Proper Fraction | Partial Fraction |
|---|---|---|---|
| 1 | Distinct linear factors | $\frac{px+q}{(ax+b)(cx+d)}$ | $\frac{A}{ax+b} + \frac{B}{cx+d}$ |
| 2 | Repeated linear factors | $\frac{px+q}{(ax+b)^2}$ | $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$ |
| 3 | Quadratic factor (cannot be factorised) | $\frac{px^2+qx+r}{(ax+b)(x^2+c^2)}$ | $\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$ |

| | |
|---|---|
| Step 4: | Solve for unknown constant |

**Binomial Theorem (Given)**

| | |
|---|---|
| Binomial expansion of $(a+b)^n$ | $\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$ |
| General Term | $T_{r+1} = \binom{n}{r}a^{n-r}b^r$ |

**Law of Indices**

| | |
|---|---|
| $a^0 = 1$ | $a^m \times a^n = a^{m+n}$ |
| $a^{-n} = \frac{1}{a^n}$ | $a^m \div a^n = a^{m-n}$ |
| $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ | $(a^m)^n = a^{mn}$ |
| $(\frac{a}{b})^n = \frac{a^n}{b^n}$ | $(ab)^n = a^n \times b^n$ |

$$y = a^x, \quad y > 0, \quad a > 0$$

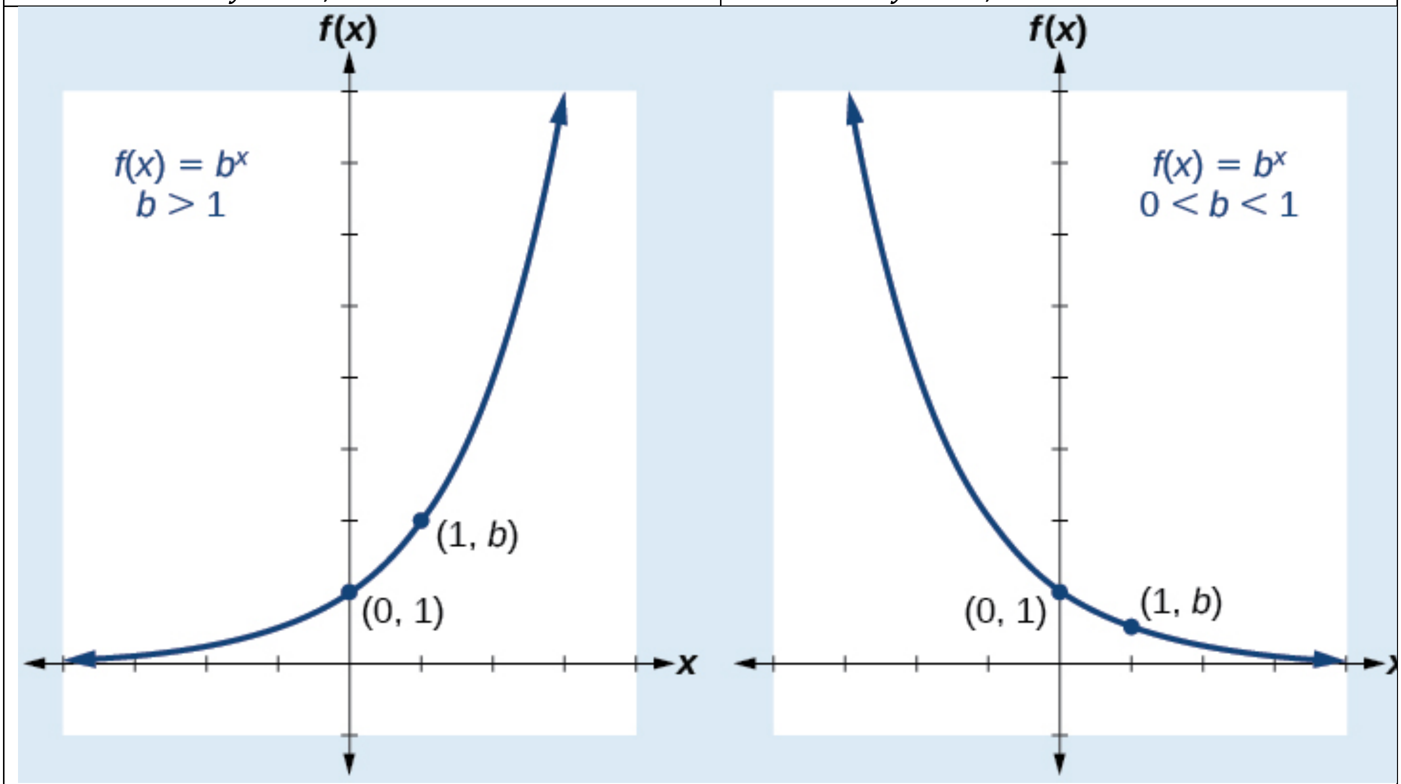
**Converting Exponential to Logarithmic**

| | |
|---|---|
| $y = a^x$ | $x = \log_a y$ |

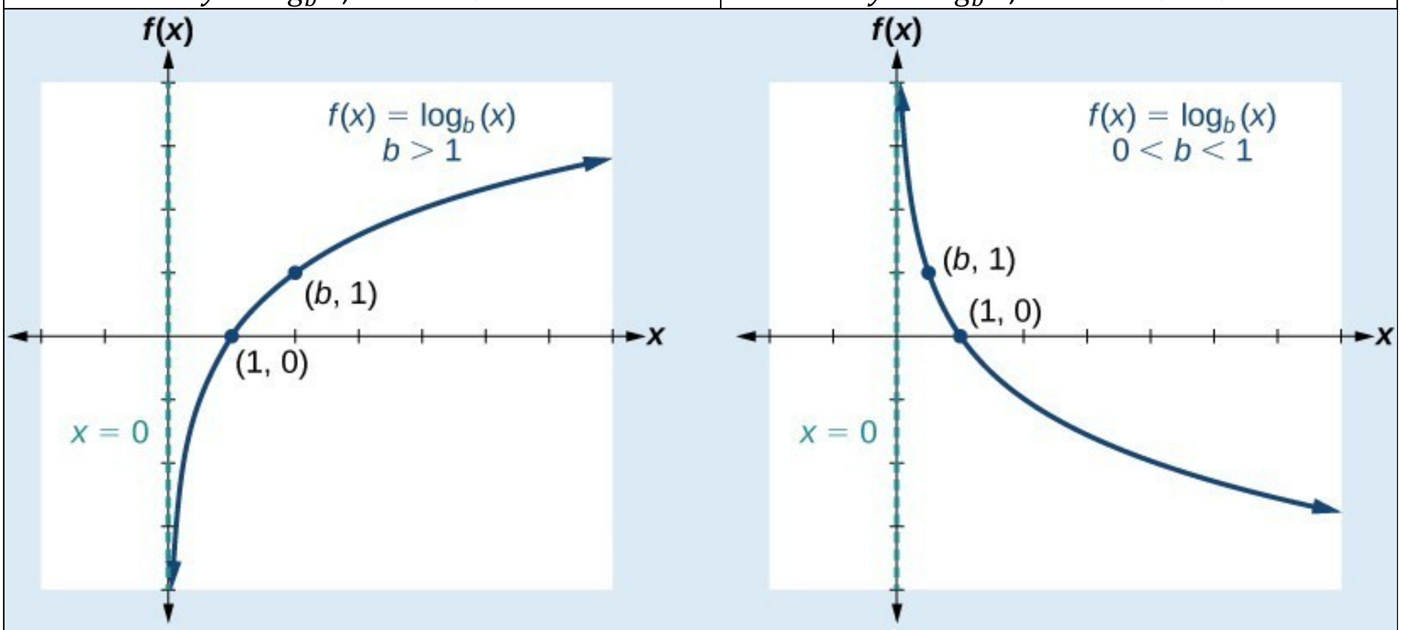
| **Law of Logarithms** | |
|---|---|
| $\log_a 1 = 0$ | $\log_a x + \log_a y = \log_a x\,y$ |
| $\log_a a = 1$ | $\log_a x - \log_a y = \log_a \frac{x}{y}$ |
| $a^{\log_a y} = y,$<br>$y > 0$ | $\log_a x^r = r \log_a x$ |

| **Change of Base** |
|---|
| $\log_a b = \frac{\log_c b}{\log_c a}$ |

| **Exponential Functions and Graphs** | |
|---|---|
| $y = b^x$, where b$> 1$ | $y = b^x$, where $0 < b < 1$ |




| **Logarithmic Functions and Graphs** | |
|---|---|
| $y = \log_b x$, where $b > 0$ | $y = \log_b x$, where $0 < b < 1$ |



**Coordinate Geometry**

| | |
|---|---|
| Midpoint of Line | $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ |
| Length of Line | $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ |
| Gradient of line | $\frac{y_2-y_1}{x_2-x_1}$ |
| | $\tan\theta$ |
| Gradient of Perpendicular Lines | $m_1 \times m_2 = -1$ |
| Equation of Straight Line | $y = mx + c$ |
| | $y - y_1 = m(x - x_1)$ |
| Area of Polygon | $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$ (anti-clockwise) |
| Equation of Circle | $(x-a)^2+(y-b)^2=r^2$ |
| | $x^2+y^2+2gx+2fy+c=0$, where $g^2+f^2-c>0$, Centre $(-g,-f)$, Radius $r=\sqrt{g^2+f^2-c}$ |

**Linear Law**

| | |
|---|---|
| $Y = mX + c$ | $Y$ and $X$ only contains $x$ and $y$ |
| | $m$ and $c$ only contains constants |

**Sketching Pointers**

**Radian Measure**

$$\pi\, rad = 180°$$

**Special Angles**

| | $0°\ (0)$ | $30°\ \left(\frac{\pi}{6}\right)$ | $45°\ \left(\frac{\pi}{4}\right)$ | $60°\ \left(\frac{\pi}{3}\right)$ | $90°\ \left(\frac{\pi}{2}\right)$ |
|---|---|---|---|---|---|
| $\sin\theta$ | $\frac{\sqrt{0}}{2}=0$ | $\frac{\sqrt{1}}{2}=\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{4}}{2}=1$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ |
| $\tan\theta$ | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | $undefined$ |

$$\sin\theta \to \frac{\sqrt{\ }}{2}, 0\ to\ 4$$

$\cos\theta \to \sin\theta$ flipped opposite

$$\tan\theta \to \frac{\sin\theta}{\cos\theta}$$

**General Angles and Basic Angles**

| Basic angle, $\alpha \to$ General angle, $\theta$ | | |
|---|---|---|
| 1st Quadrant | $\theta = \alpha$ | $\theta = \alpha$ |
| 2nd Quadrant | $\theta = 180° - \alpha$ | $\theta = \pi - \alpha$ |
| 3rd Quadrant | $\theta = 180° + \alpha$ | $\theta = \pi + \alpha$ |
| 4th Quadrant | $\theta = 360° - \alpha$ | $\theta = 2\pi - \alpha$ |

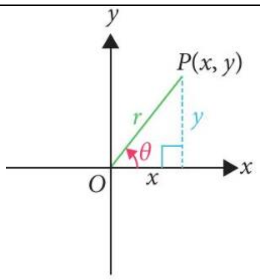
**Cosecant, Secant and Cotangent**

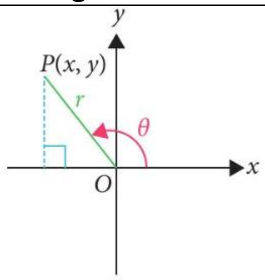
$$\operatorname{cosec}\theta \to \frac{1}{\sin\theta}$$

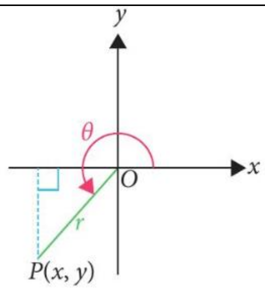
$$\sec\theta \to \frac{1}{\cos\theta}$$

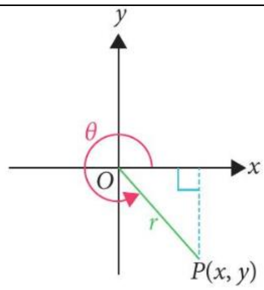
$$\cot\theta \to \frac{1}{\tan\theta} \to \frac{\cos\theta}{\sin\theta}$$

## Trigonometric Ratios of General Angles




| | | |
|---|---|---|
| $\sin\theta = \frac{y}{r}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{r}{y}$ | $\cos\theta = \frac{x}{r}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$ | $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$ |

$x = x$-coordinate of point $P$

$y = y$-coordinate of point $P$

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \text{ where } \cos\theta \neq 0$$

## Signs of Trigonometric Ratios & Trigonometric Ratios of Related Angles (ASTC)

| 2nd Quadrant (S) | | | 1st Quadrant (A) | | |
|---|---|---|---|---|---|
| $x < 0$<br>$y > 0$<br>$r > 0$ | **$\sin\theta \to$ positive**<br>$\cos\theta \to$ negative<br>$\tan\theta \to$ negative | $\sin\theta = \sin\alpha$<br>$\cos\theta = -\cos\alpha$<br>$\tan\theta = -\tan\alpha$ | $x > 0$<br>$y > 0$<br>$r > 0$ | $\sin\theta \to$ positive<br>$\cos\theta \to$ positive<br>$\tan\theta \to$ positive | $\sin\theta = \sin\alpha$<br>$\cos\theta = \cos\alpha$<br>$\tan\theta = \tan\alpha$ |
| **3rd Quadrant (T)** | | | **4th Quadrant (C)** | | |
| $x < 0$<br>$y < 0$<br>$r > 0$ | $\sin\theta \to$ negative<br>$\cos\theta \to$ negative<br>**$\tan\theta \to$ positive** | $\sin\theta = -\sin\alpha$<br>$\cos\theta = -\cos\alpha$<br>$\tan\theta = \tan\alpha$ | $x > 0$<br>$y < 0$<br>$r > 0$ | $\sin\theta \to$ negative<br>**$\cos\theta \to$ positive**<br>$\tan\theta \to$ negative | $\sin\theta = -\sin\alpha$<br>$\cos\theta = \cos\alpha$<br>$\tan\theta = -\tan\alpha$ |

## Relationship between Trigonometric Ratio of General Angle $\theta$ and its Negative Angle $-\theta$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

## Trigonometric Ratios of Complementary Angles

$$\sin(90° - \theta) = \cos\theta$$

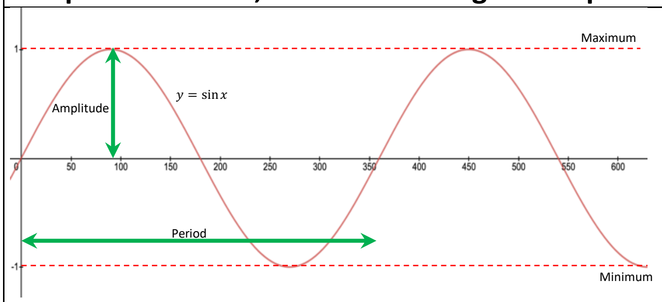
$$\cos(90° - \theta) = \sin\theta$$

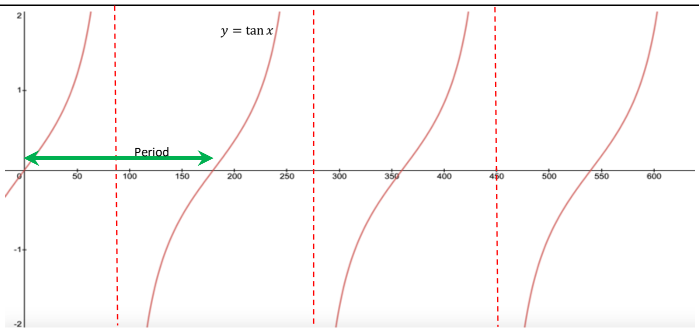
$$\tan(90° - \theta) = \frac{1}{\tan\theta}$$

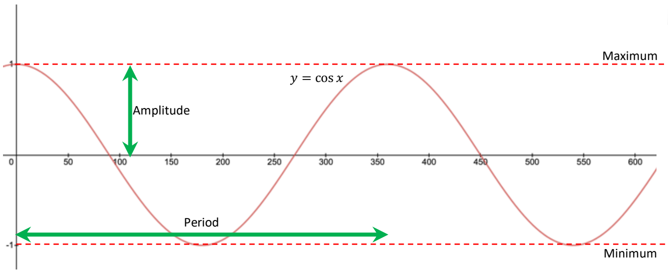
$$\downarrow$$

$$\cot\theta \to \frac{1}{\tan\theta} \to \frac{\cos\theta}{\sin\theta}$$

## Properties of Sine, Cosine and Tangent Graphs




| | $y = \tan x$ |
|---|---|
| Special Values of 1 | $\tan x = 1$<br>When $x = 45°, 225°$<br>$x = \frac{\pi}{4}, \frac{5\pi}{4}$ |
| Special Values of -1 | $\tan x = -1$<br>When $x = 135°, 315°$<br>$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ |

**Properties of Sine, Cosine and Tangent Graphs (Basic Graph)**

| | $y=\sin x$ | $y=\cos x$ | $y=\tan x$ |
|---|---|---|---|
| Zero Value | $\sin x=0$,<br>When $x=0°, 180°, 360°$<br>$x=0, \pi, 2\pi$ | $\cos x=0$<br>When $x=$<br>$90°, 270°, \frac{\pi}{2}, \frac{3\pi}{2}$ | $\tan x=0$<br>When $x=0°, 180°, 360°$<br>$x=0, \pi, 2\pi$ |
| Maximum Value | $\sin x=1$,<br>When $x=90°, \frac{\pi}{2}$ | $\cos x=1$<br>When $x=0°, 360°, 0, 2\pi$ | Can be any real number,<br>No range or max/min<br>value |
| Minimum Value | $\sin x=-1$<br>When $x=270°, \frac{3\pi}{2}$ | $\cos x=-1$<br>When $x=180°, 2\pi$ | |
| Range | $-1 \leq \sin x \leq 1$ | $-1 \leq \cos x \leq 1$ | |
| Axis of Curve | $y=0$ | $y=0$ | |
| Period | $360°,\ 2\pi$ | $360°,\ 2\pi$ | $180°,\ \pi$ |
| Amplitude | $1$ | $1$ | |
| Symmetry | Rotational symmetry of order 2 about the origin<br>$\sin(-x)=-\sin x$ | Symmetrical about $y$-axis<br>$\cos(-x)=\cos x$ | Rotational symmetry of order 2 about the origin<br>$\tan(-x)=-\tan x$ |

**$y=a\sin bx+c$, $y=a\cos bx+c$ and $y=a\tan bx+c$**

| | $y=a\sin bx+c$, $y=a\cos bx+c$ | $y=a\tan bx+c$ |
|---|---|---|
| Maximum Value | $a+c$ | |
| Minimum Value | $-a+c$ | |
| Amplitude | $a=\frac{max-min}{2}$ | |
| Period | $\frac{360°}{b}$ or $\frac{2\pi}{b}$ | $\frac{180°}{b}$ or $\frac{\pi}{b}$ |
| Interval | $\frac{Period}{4}$ | |
| Vertical Shift | $c=\frac{max+min}{2}$ | $c$ |

**Sketching of Graphs**

| | |
|---|---|
| Step 1: | Find the period |
| Step 2: | Find the interval |
| Step 3: | Recall 5 critical points for the correct trigonometric function |

| Graph | **5 Critical Points** | | | | |
|---|---|---|---|---|---|
| Sine | 0 | 1 | 0 | -1 | 0 |
| Cosine | 1 | 0 | -1 | 0 | 1 |
| Tangent | 0 | 1 | Undefined | -1 | 0 |

| | |
|---|---|
| Step 4: | Multiply 5 critical points by $a$ |
| Step 5: | Add $c$ to the values in Step 4 |
| Step 6: | Use values in Step 5 to mark and sketch out the graph |

**Principle Values**

| | Degree | Radian | Value of $x$ |
|---|---|---|---|
| $\sin^{-1} x$ | $-90° \leq \sin^{-1} x \leq 90°$ | $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$ | Where $x=-1 \leq x \leq 1$ |
| $\cos^{-1} x$ | $0° \leq \cos^{-1} x \leq 180°$ | $0 \leq \cos^{-1} x \leq \pi$ | |
| $\tan^{-1} x$ | $-90° \leq \tan^{-1} x \leq 90°$ | $-\frac{\pi}{2} \leq \tan^{-1} x \leq \frac{\pi}{2}$ | Where $x$ is any real value |

**Basic Trigonometric Identities (Given)**

$$\sin^2\theta+\cos^2\theta=1$$

$$\sec^2\theta=1+\tan^2\theta$$

$$\mathrm{cosec}^2\theta=1+\cot^2\theta$$

| For $0° \leq x \leq 360°$, | $0$ | $1$ | $-1$ |
|---|---|---|---|
| $\sin x$ | $\sin x = 0$,<br>$x = 0°, 180°, 360°$ | $\sin x = 1$,<br>$x = 90°$ | $\sin x = -1$<br>$x = 270°$ |
| $\cos x$ | $\cos x = 0$,<br>$x = 90°, 270°$ | $\cos x = 1$<br>$x = 0°, 360°$ | $\cos x = -1$,<br>$x = 180°$ |
| $\tan x$ | $\tan x = 0$,<br>$x = 0°, 180°$ | $\tan x = 1$,<br>$x = 45°, 225°$ | $\tan x = -1$,<br>$x = 135°, 315°$ |
| For $0 \leq x \leq 2\pi$ | $0$ | $1$ | $-1$ |
| $\sin x$ | $\sin x = 0$,<br>$x = 0, \pi, 2\pi$ | $\sin x = 1$,<br>$x = \frac{\pi}{2}$ | $\sin x = -1$<br>$x = \frac{3\pi}{2}$ |
| $\cos x$ | $\cos x = 0$,<br>$x = \frac{\pi}{2}, \frac{3\pi}{2}$ | $\cos x = 1$<br>$x = 0, 2\pi$ | $\cos x = -1$,<br>$x = \pi$ |
| $\tan x$ | $\tan x = 0$,<br>$x = 0, \pi$ | $\tan x = 1$,<br>$x = \frac{\pi}{4}, \frac{5\pi}{4}$ | $\tan x = -1$,<br>$x = \frac{3\pi}{4}, \frac{7\pi}{4}$ |

**Solving Trigonometry Equations Questions Example**

Solve the equation $\sin x = 0.5$, where $0° \leq x \leq 360°$.

$\sin x = 0.5$
$0° \leq x \leq 360°$
$x$ lies in 1st or 2nd quadrant
Basic angle, $\sin \alpha = 0.5$

$$\alpha = \sin^{-1} 0.5$$
$$\alpha = 30°$$

$x = 30°, \ 180° - 30°$
$\quad = 30°, \ 150°$