JC1 H2 Mathematics (9758)
Term 4 Revision Topical Quick Check
Chapter 3 Functions

Revision Guide Chapter 3 Page 3-4

6. **One-one functions**

(b) **Horizontal Line Test (MUST SKETCH GRAPH)**

The function f is one-one if **any** horizontal line $y = b$ ($b \in \mathbb{R}$) cuts the graph of f **at most once.**

- If not one-one: find a counter-example (sketch a horizontal line $y = k$ that cuts the graph twice (providing the value of k) or find $a, b \in \mathrm{D_f}$ such that $\mathrm{f}(a) = \mathrm{f}(b)$).

Inverse Functions

7. The inverse of a function f is denoted by f^{-1}.

(Do not confuse **inverse function** $\left(\mathrm{f}^{-1}\right)$ with **reciprocal graph** $\left(\dfrac{1}{\mathrm{f}(x)}\right)$)

8. f^{-1} exists $\Leftrightarrow$ f is one-one.
9. $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R_f}$ and $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D_f}$
10. The graph of f^{-1} can be obtained by reflecting the graph of f about the line $y = x$ (provided that f has an inverse).
11. $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x) = x$; but ff^{-1} and $\mathrm{f}^{-1}\mathrm{f}$ have different domains.
    (a) $\mathrm{D}_{\mathrm{ff}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R_f}$
    (b) $\mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}} = \mathrm{D_f}$
12. If $\pm\sqrt{\ }$ occurs in the process of finding inverse, use the domain of f to determine whether to choose $\sqrt{\ }$ (positive square root) or $-\sqrt{\ }$ (negative square root).

Composite Functions

13. If f and g are two functions such that $\mathrm{R_f} \subseteq \mathrm{D_g}$, then the composite function g of f (i.e. gf) exists and is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$ for all $x \in \mathrm{D_f}$.
14. For gf to exist, $\mathrm{R_f} \subseteq \mathrm{D_g}$.
15. Domain of composite function gf, $\mathrm{D_{gf}} = \mathrm{D_f}$.
16. Range of composite function gf, $\mathrm{R_{gf}} \subseteq \mathrm{R_g}$.
    (a) Method 1: Sketch both functions and use $\mathrm{D_f} \xrightarrow{\mathrm{f}} \mathrm{R_f} \xrightarrow{\mathrm{g}} \mathrm{R_{gf}}$
    [**Using $\mathrm{R_f}$ as restricted domain of g]
    (b) Method 2: Find the composite function and sketch the graph, keeping in mind of the domain of the first function, i.e., $\mathrm{D_f}$ since $\mathrm{D_{gf}} = \mathrm{D_f}$.

Let's Try Now

1 **YIJC Promo 9758/2022/Q9 (modified)**

The function f is defined by

$$\mathrm{f} : x \mapsto x^2 - 4x - 5 \text{, for } x \in \mathbb{R},\ x \leq 2.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(b) On the same diagram, sketch the graphs of f and f^{-1}. [3]

(c) Find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto 1 - x^2 \text{, for } x \in \mathbb{R},\ x < 1.$$

(d) Explain why the composite function fg exists and find the range of fg. [3]

1 YIJC Promo 9758/2022/Q9 (modified)

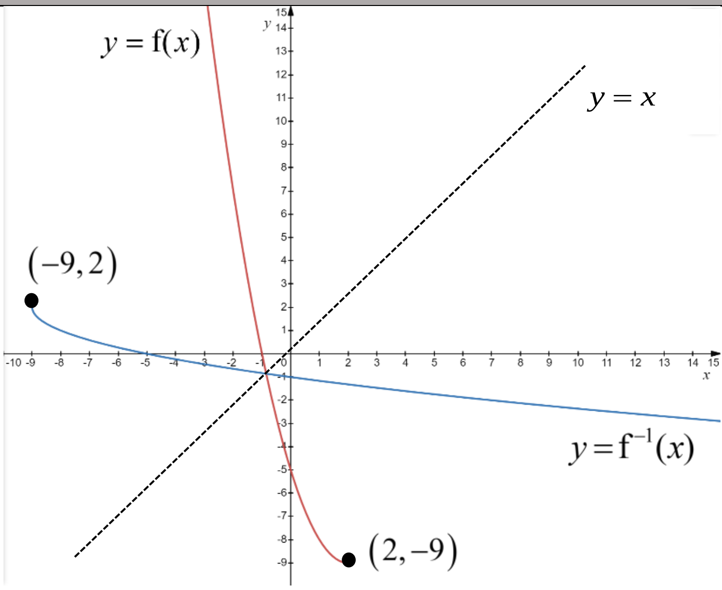
The function f is defined by

$$\mathrm{f} : x \mapsto x^2 - 4x - 5, \text{ for } x \in \mathbb{R},\ x \le 2.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

Q1	Solution
(a)	Let $y = x^2 - 4x - 5$. Then $y = (x-2)^2 - 9$ $y + 9 = (x-2)^2$ $x - 2 = \pm\sqrt{y+9}$ $x = 2 \pm \sqrt{y+9}$ Since $\mathrm{D_f} = (-\infty, 2]$, $x = 2 - \sqrt{y+9}$. Hence, $\mathrm{f}^{-1}(x) = 2 - \sqrt{x+9}$ $\mathrm{D}_{\mathrm{f}^{-1}} = [-9, \infty)$

(b) On the same diagram, sketch the graphs of f and f^{-1}. [3]

Q1	Solution
(b)	

(c) Find the exact solution of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [3]

Q1	**Solution**
(c)	$\mathrm{f}(x)=x$ $x^2-4x-5=x$ $x^2-5x-5=0$ $x=\dfrac{-(-5)\pm\sqrt{(-5)^2-4(1)(-5)}}{2(1)}$ $x=\dfrac{5\pm3\sqrt{5}}{2}$ Since $\mathrm{D_f}=(-\infty,2]$, then we reject $x=\dfrac{5+3\sqrt{5}}{2}$. Therefore, $x=\dfrac{5-3\sqrt{5}}{2}$.

The function g is defined by

$$\mathrm{g}:x\mapsto 1-x^2,\text{ for } x\in\mathbb{R},\ x<1.$$

(d) Explain why the composite function fg exists and find the range of fg. [3]

Q1	**Solution**
(d)	$\mathrm{R_g}=(-\infty,1]$ $\mathrm{D_f}=(-\infty,2]$ Since $\mathrm{R_g}\subseteq\mathrm{D_f}$, then the composite function fg exists. $\mathrm{R_{fg}}=[-8,\infty)$

Let's Try Now

2 HCI Promo 9758/2022/Q6

The function h is defined as follows.

$$\mathrm{h} : x \mapsto \frac{1}{8}\left(x^3 - 6x^2 + 32\right), \qquad x \in \mathbb{R}.$$

(i) Explain why h does not have an inverse. [1]

(ii) If the domain of h is restricted to $0 \le x \le k$, state the largest value of k for which the function h^{-1} exists. [1]

Use the domain of h in part **(ii)** for the rest of this question.

(iii) Sketch the graphs of h and h^{-1} on the same diagram, showing clearly the relationship between the two graphs, and the coordinates of the end points for both graphs. [3]

(iv) Deduce the solution(s) of the equation $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. [2]

(v) The function g is defined as follows.

$$\mathrm{g} : x \mapsto \ln\left((x-3)^2 + 1\right), \qquad x \in \mathbb{R}.$$

Given that the composite function gh exists, find the exact range of gh. [2]

2 HCI Promo 9758/2022/Q6

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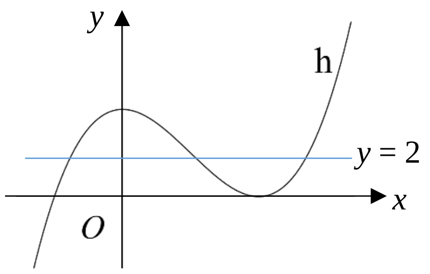
(i) Explain why h does not have an inverse. [1]

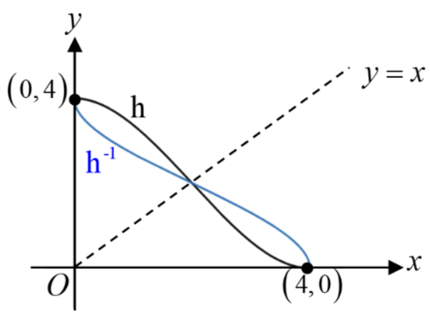
2	Solution
(i)	 Since $y=2$ cuts the graph of h more than once, h is not one to one function, hence h^{-1} doesn't exist.

(ii) If the domain of h is restricted to $0 \le x \le k$, state the largest value of k for which the function h^{-1} exists. [1]

2	Solution
(ii)	Largest value of k is 4.

Use the domain of h in part **(ii)** for the rest of this question.

(iii) Sketch the graphs of h and h^{-1} on the same diagram, showing clearly the relationship between the two graphs, and the coordinates of the end points for both graphs. [3]

2	Solution
(iii)	

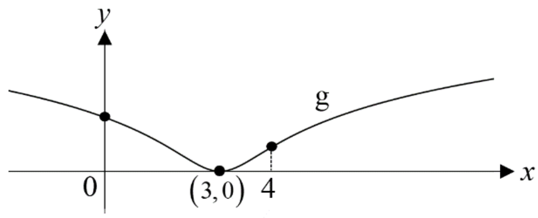
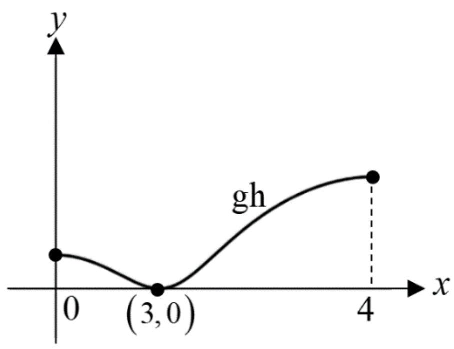
(iv) Deduce the solution(s) of the equation $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. [2]

2	Solution
(iv)	From GC, $\mathrm{h}(x) = x \quad \Rightarrow x = 2$ $\therefore x = 2$ or $x = 0$ or $x = 4$.

(v) The function g is defined as follows.

$$\mathrm{g}: x \mapsto \ln\left((x-3)^2+1\right), \qquad x \in \mathbb{R}.$$

Given that the composite function gh exists, find the exact range of gh. [2]

2	**Solution**
(v)	 $$\mathrm{R_h} = [0,4] \xrightarrow{\mathrm{g}} [0, \ln 10]$$ $\mathrm{R_{gh}} = [0, \ln 10]$ **OR** $\mathrm{gh}(x) = \ln\left(\left(\dfrac{x^3-6x^2+32}{8}-3\right)^2+1\right)$  $\mathrm{R_{gh}} = [0, \mathrm{gh}(4)]$ $\mathrm{R_{gh}} = [0, \ln 10]$

Let's Try Now

3 CJC Promo 9758/2022/Q10(b)

Functions g and h are defined by

$$\mathrm{g}: x \mapsto \frac{1}{1-x^2}, \qquad x \in \mathbb{R}, x > 1,$$
$$\mathrm{h}: x \mapsto 1-2x, \qquad x \in \mathbb{R}.$$

(i) Explain why the composite function gh does not exist. [2]

(ii) Find $\mathrm{hg}(x)$. [1]

(iii) Find the range of $\mathrm{hg}(x)$. [2]

(iv) By using the result in part **(ii)**, or otherwise find $(\mathrm{hg})^{-1}(4)$. [3]

3 CJC Promo 9758/2022/Q10(b)

Functions g and h are defined by

$$\mathrm{g}: x \mapsto \frac{1}{1-x^2}, \qquad x \in \mathbb{R}, x > 1,$$
$$\mathrm{h}: x \mapsto 1-2x, \qquad x \in \mathbb{R}.$$

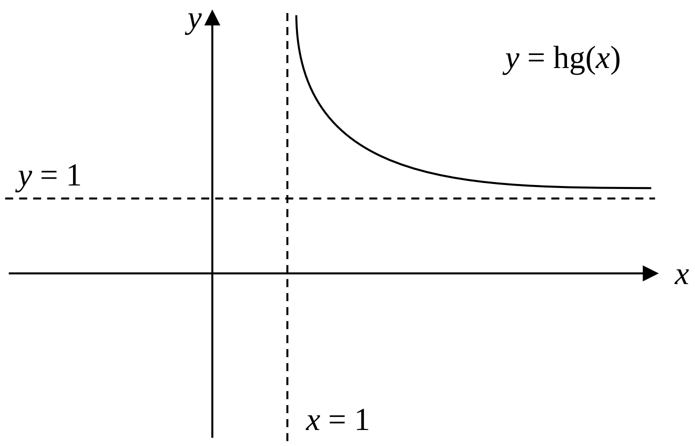
(i) Explain why the composite function gh does not exist. [2]

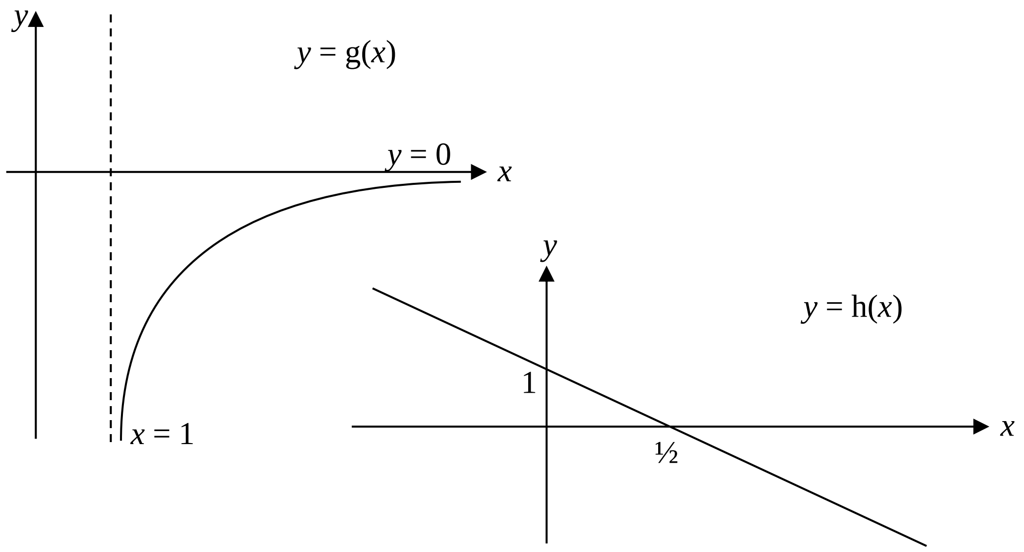
Q3	Solution
(i)	For gh to exist, $\mathrm{R_h} \subseteq \mathrm{D_g}$. $\mathrm{R_h} = (-\infty, \infty)$ $\mathrm{D_g} = (1, \infty)$ Since $\mathrm{R_h} \not\subseteq \mathrm{D_g}$, gh does not exist.

(ii) Find $\mathrm{hg}(x)$. [1]

Q3	Solution
(ii)	$\mathrm{hg}(x) = \mathrm{h}\left(\frac{1}{1-x^2}\right) = 1-2\left(\frac{1}{1-x^2}\right) = 1-\frac{2}{1-x^2}$

(iii) Find the range of $\mathrm{hg}(x)$. [2]

Q3	Solution
(iii)	**<u>Method ①:</u> Composite Function** $\mathrm{D_{hg}} = \mathrm{D_g} = (1, \infty)$  $\mathrm{R_{hg}} = (1, \infty)$ **<u>Method ②:</u> Mapping Method**



$$\mathrm{D_g} = (1, \infty) \xrightarrow{\mathrm{g}} \mathrm{R_g} = (-\infty, 0) \xrightarrow{\mathrm{h}} \mathrm{R_{hg}} = (1, \infty)$$

(iv) By using the result in part **(ii)**, or otherwise find $(\mathrm{hg})^{-1}(4)$. [3]

Q3	Solution
(iv)	<u>**Hence**</u> Let $(\mathrm{hg})^{-1}(4) = a$ $4 = \mathrm{hg}(a)$ $4 = 1 - \frac{2}{1-a^2}$ $\frac{2}{1-a^2} = -3$ $2 = -3 + 3a^2$ $3a^2 = 5$ $a^2 = \frac{5}{3}$ $a = \sqrt{\frac{5}{3}} \text{ or } -\sqrt{\frac{5}{3}}$ Since $\mathrm{D_{hg}} = \mathrm{D_g} = (1, \infty)$, $a = \sqrt{\frac{5}{3}}$

	<u>Otherwise</u> Let $y=1-\frac{2}{1-x^2}$ $\frac{2}{1-x^2}=1-y$ $1-x^2=\frac{2}{1-y}$ $x^2=1-\frac{2}{1-y}$ $x=\sqrt{1-\frac{2}{1-y}}$ or $-\sqrt{1-\frac{2}{1-y}}$ Since $\mathrm{D}_{(\mathrm{hg})^{-1}}=\mathrm{R}_{\mathrm{hg}}=(1,\infty)$, $x=\sqrt{1-\frac{2}{1-y}}$ $\therefore (\mathrm{hg})^{-1}(x)=\sqrt{1-\frac{2}{1-x}}$ $(\mathrm{hg})^{-1}(4)=\sqrt{1-\frac{2}{1-4}}=\sqrt{\frac{5}{3}}$