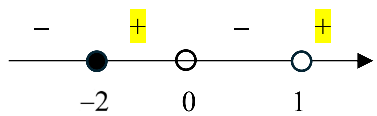
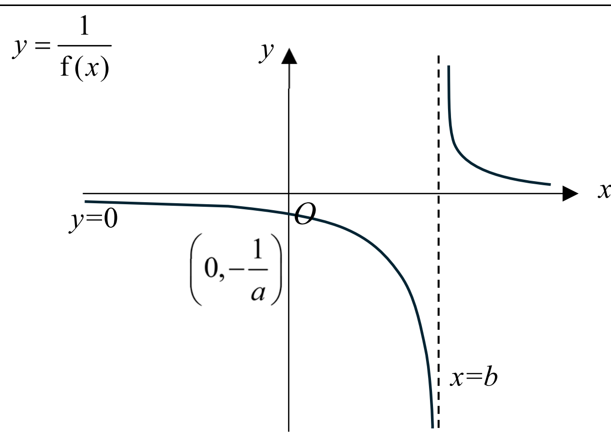
2024 H2 MATH PROMO SOLUTION

No.	Solution
1	Let x, y and z be the number of dresses, blouses and skirts produced per week respectively. $45x+50y+20z=3150$ $70x+60y+25z=4300$ $8x+6y+3z=480$ By GC, $x=30$, $y=20$, $z=40$. The shop produces 30 dresses, 20 blouses and 40 skirts each week.

No.	Solution
2(a)	$\frac{2x^2+3x+10}{x^2-x}\geq 2$ $\frac{2x^2+3x+10-2\left(x^2-x\right)}{x^2-x}\geq 0$ $\frac{5x+10}{x\left(x-1\right)}\geq 0$  $\therefore -2\leq x<0$ or $x>1$
(b)	Replace x in part **(a)** with $-e^x$ $-2\leq -e^x<0$ or $-e^x>1$ (Reject since $-e^x<0$ for all real x) $0<e^x\leq 2$ $x\leq \ln 2$ $\therefore \{x\in\mathbb{R}: x\leq \ln 2\}$

No.	Solution
3(a)(i)	$A'(-a,0)$ and $B'(0,b)$
3(a)(ii)	$B''(b+2,0)$
3(a)(iii)	$A'''(0,-a)$ and $B'''\left(\frac{b}{3},0\right)$
3(b)	

<table>
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<tr><td>4(a)(i)</td><td>

$S_n - S_{n-1} = 75 - \frac{3^{n+1}}{5^{n-2}} - \left[75 - \frac{3^n}{5^{n-3}}\right]$

$u_n = \frac{3^n}{5^{n-3}} - \frac{3^{n+1}}{5^{n-2}}$

$= \frac{3^{n-1}}{5^{n-1}}\left(\frac{3}{5^{-2}} - \frac{3^2}{5^{-1}}\right)$

$= \left(\frac{3}{5}\right)^{n-1}(75-45)$

$= 30\left(\frac{3}{5}\right)^{n-1}$

This is a GP with common ratio $r = \frac{3}{5}$ and first term $a = 30$

$S_\infty = \frac{30}{1-\frac{3}{5}} = 75$

OR

$S_n = 75 - \frac{3^{n+1}}{5^{n-2}} = 75 - 27\left(\frac{3}{5}\right)^{n-2}$

As $n \to \infty$, $\left(\frac{3}{5}\right)^{n-2} \to 0$. Hence, $S_\infty = 75$

</td></tr>
<tr><td>4(b)</td><td>

$\sum_{i=1}^{n} v_i = \frac{2n}{2n+1} = 1 - \frac{1}{2n+1}$

As $n \to \infty$, $\frac{1}{2n+1} \to 0$. Hence, $\sum_{i=1}^{\infty} v_i = 1$.

$\sum_{i=11}^{\infty} v_i = \sum_{i=1}^{\infty} v_i - \sum_{i=1}^{10} v_i = 1 - \frac{20}{21} = \frac{1}{21}$

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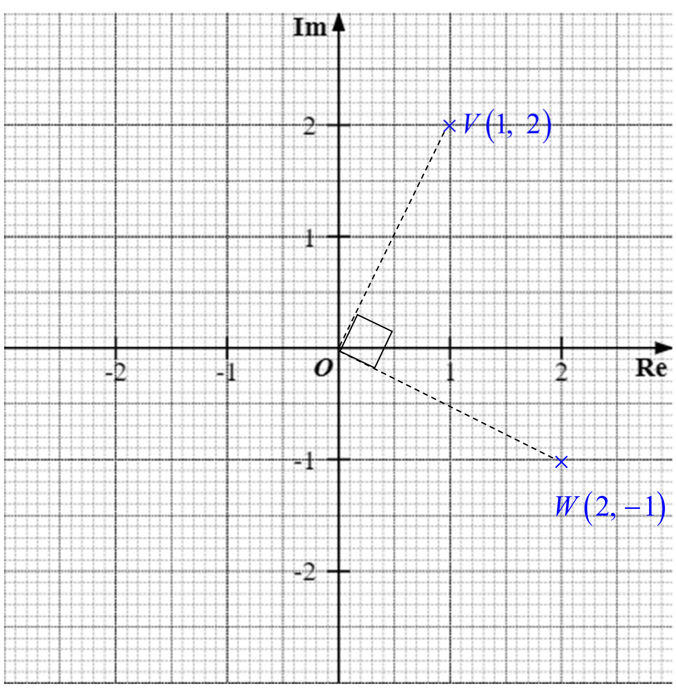
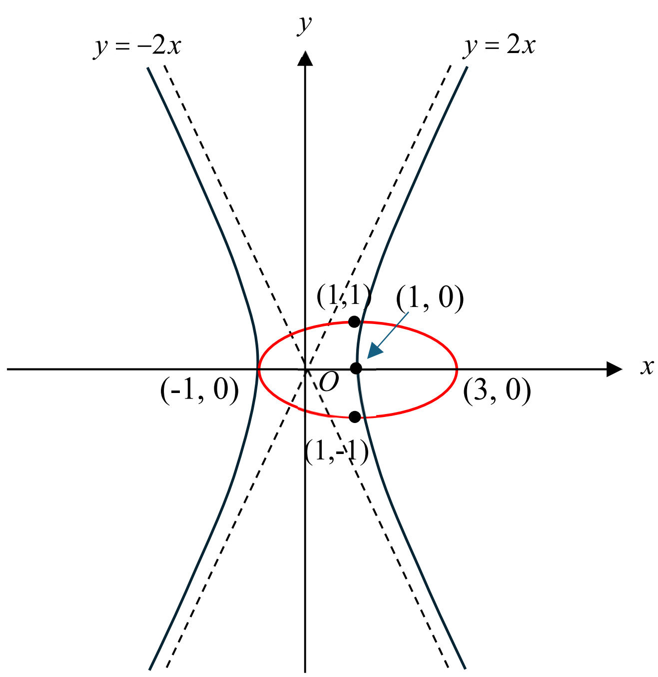
No.	Solution
5(a)	$3x^2+2xy-4y^2=1$ Differentiating wrt to x: $6x+\left(2x\frac{dy}{dx}+2y\right)-8y\frac{dy}{dx}=0$ $8y\frac{dy}{dx}-2x\frac{dy}{dx}=6x+2y$ $\frac{dy}{dx}=\frac{6x+2y}{8y-2x}$ $\frac{dy}{dx}=\frac{3x+y}{4y-x}$ (shown)
5(b)	Gradient of tangent at (1, 1) = $\frac{3+1}{4-1}=\frac{4}{3}$ Angle required = $\tan^{-1}\left(\frac{4}{3}\right)=53.1°$ (0.927 radian)
5(c)	For tangent parallel to x-axis, $\frac{dy}{dx}=0$ $3x+y=0 \Rightarrow y=-3x$ **<u>OR</u>** $x=-\frac{1}{3}y$ Sub into eqn of C: $3x^2+2x(-3x)-4(-3x)^2=1$ $-39x^2=1 \Rightarrow x^2=-\frac{1}{39}$ (no soln as $x^2 \geq 0$ for all $x \in \mathbb{R}$) **<u>OR</u>** $3\left(-\frac{1}{3}y\right)^2+2\left(-\frac{1}{3}y\right)(y)-4y^2=1$ $-\frac{13}{3}y^2=1 \Rightarrow y^2=-\frac{3}{13}$ (no soln as $y^2 \geq 0$ for all $y \in \mathbb{R}$) Since there is **<u>no solution for *x*</u>** (or y) such that $\frac{dy}{dx}=0$. There is no point on C at which the tangent to curve is parallel to the x-axis.

No.	***Solution***
6(a)(i)	$u_1 = p = 2.4$ $u_2 = 12 - \frac{24}{2.4} = 2$ $u_3 = 12 - \frac{24}{2} = 0$ u_4 will be undefined and we are not able to generate an infinite sequence. Hence, p cannot be 2.4.
6(a)(ii)	If the sequence is constant, then all terms will be p. $p = 12 - \frac{24}{p}$ $p^2 - 12p + 24 = 0$ $p = \frac{12 \pm \sqrt{144 - 4(24)}}{2} = 6 \pm 2\sqrt{3}$ Since $p > 5$, $p = 6 + 2\sqrt{3}$.
6(b)	$3v_{10} = 5v_{19}$ $3(v_1 + 9d) = 5(v_1 + 18d)$ $3v_1 + 27d = 5v_1 + 90d$ $2v_1 = -63d$ $S_n = \frac{n}{2}(2v_1 + (n-1)d)$ $= \frac{n}{2}(-64d + nd)$ $= \frac{nd}{2}(n - 64)$ Since $d < 0$ and S_n is a quadratic expression, S_n is largest when $n = \frac{64}{2} = 32$. $S_{32} = \frac{32d}{2}(32 - 64) = -512d$ <u>Alternative</u> $S_n = \frac{n}{2}(-64d + nd)$ $= \frac{d}{2}(n^2 - 64)$ $= \frac{d}{2}\left[(n-32)^2 - 1024\right]$ Since $d < 0$, S_n is largest when $n = 32$. $S_{32} = \frac{-1024d}{2} = -512d$

No.	Solution
7(a)(i)	$v+2w=5 \Rightarrow 3v+6w=15$ ---------- (1) $3v-w^*=1+5\text{i}$ ------ (2) (1) − (2): $6w+w^*=14-5\text{i}$ Let $w=x+\text{i}y$ $6(x+\text{i}y)+(x-\text{i}y)=14-5\text{i}$ $7x+5y\text{i}=14-5\text{i}$ Comparing real and imaginary parts: $7x=14 \Rightarrow x=2$ $5y=-5 \Rightarrow y=-1$ Hence, $w=2-\text{i}$ Sub $w=2-\text{i}$ into $v+2w=5$: $v=5-2(2-\text{i})=1+2\text{i}$ **<u>Alternatively</u>** $v+2w=5 \Rightarrow v=5-2w$ and sub into eqn (2): $3(5-2w)-w^*=1+5\text{i}$ $15-6w-w^*=1+5\text{i}$ Let $w=x+y\text{i}$: $15-6(x+y\text{i})-(x-y\text{i})=1+5\text{i}$ $7x+5y\text{i}=14-5\text{i}$ and compare real and imaginary parts
7(a)(ii)	

	Transformation: W is rotated **anti-clockwise** through $\frac{\pi}{2}$ about O to obtain V **OR** W is rotated **clockwise** through $\frac{3\pi}{2}$ about O to obtain V
7(b)	Since ${z_1}^2+{z_2}^2+{z_3}^2+{z_4}^2<0$, the equation has at least one non-real root. In addition, since the coefficients of the polynomial equation are all real, by conjugate root theorem, there is at least 1 pair of complex conjugate roots. Hence, at most two of z_1, z_2, z_3 and z_4 are real.

No.	Solution
8(a)	Since $\tan^2\theta+1=\sec^2\theta$, $\left(\frac{y}{2}\right)^2+1=x^2$ $x^2-\frac{y^2}{2^2}=1$ $\left[\text{i.e. } a=1 \text{ and } b=2\right]$
8(b)	$(x-1)^2+4y^2=4 \Rightarrow \frac{(x-1)^2}{2^2}+\frac{y^2}{1^2}=1$ 

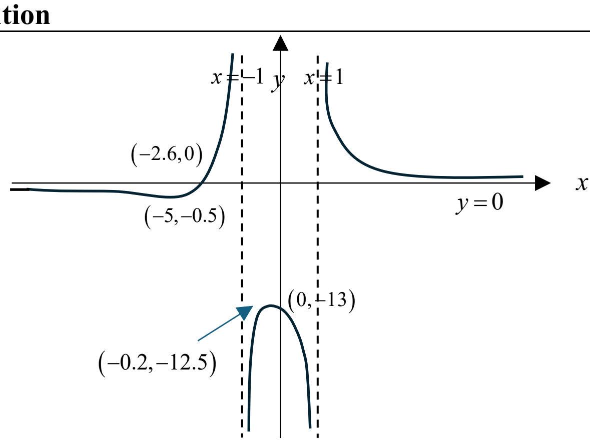
8(c)	$k(x-1)^2+4y^2=4 \quad \Rightarrow \quad \dfrac{(x-1)^2}{\left(\frac{2}{\sqrt{k}}\right)^2}+\dfrac{y^2}{1^2}=1$ To cut C_2 at most twice, $\dfrac{2}{\sqrt{k}}<2$. $k>1$
8(d)	Translate the graph $y = \mathrm{f}(x)$ by 1 unit in the negative x-direction. Stretch the resultant graph by factor 0.5 parallel to x-axis, with y-axis invariant. **OR** Stretch the graph $y = \mathrm{f}(x)$ by factor 0.5 parallel to x-axis, with y-axis invariant. Then translate the resultant graph by 0.5 unit in the negative x-direction.

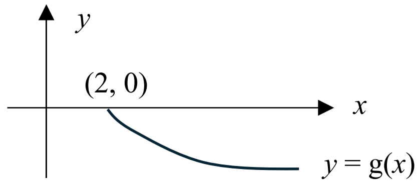
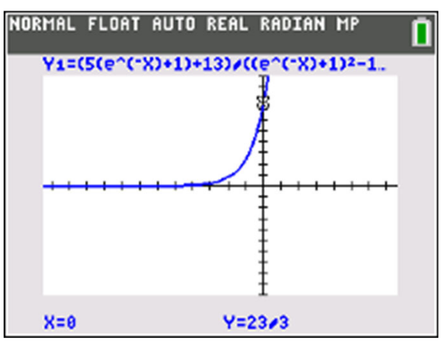
No.	Solution
9(a)	$\dfrac{1}{x^2+a^2}=\dfrac{1}{a^2}\left(1+\dfrac{x^2}{a^2}\right)^{-1}$ $=\dfrac{1}{a^2}\left[1+(-1)\dfrac{x^2}{a^2}+\left(\dfrac{-1\times-2}{2!}\right)\left(\dfrac{x^2}{a^2}\right)^2+\dots\right]$ $=\dfrac{1}{a^2}\left(1-\dfrac{x^2}{a^2}+\dfrac{x^4}{a^4}+\dots\right)$ OR $\dfrac{1}{a^2}-\dfrac{1}{a^4}x^2+\dfrac{1}{a^6}x^4+\dots$ $\dfrac{\cos ax}{x^2+a^2}=(\cos ax)\left(1+\dfrac{x^2}{a^2}\right)^{-1}$ $=\dfrac{1}{a^2}\left(1-\dfrac{a^2x^2}{2!}+\dfrac{a^4x^4}{4!}+\dots\right)\left(1-\dfrac{x^2}{a^2}+\dfrac{x^4}{a^4}+\dots\right)$ $=\dfrac{1}{a^2}\left(1-\dfrac{x^2}{a^2}+\dfrac{x^4}{a^4}-\dfrac{a^2x^2}{2!}+\dfrac{x^4}{2!}+\dfrac{a^4x^4}{4!}+\dots\right)$ $=\dfrac{1}{a^2}\left[1-\left(\dfrac{a^2}{2}+\dfrac{1}{a^2}\right)x^2+\left(\dfrac{1}{a^4}+\dfrac{1}{2}+\dfrac{a^4}{24}\right)x^4+\dots\right]$ $=\dfrac{1}{a^2}-\dfrac{1}{a^2}\left(\dfrac{a^2}{2}+\dfrac{1}{a^2}\right)x^2+\dfrac{1}{a^2}\left(\dfrac{1}{a^4}+\dfrac{1}{2}+\dfrac{a^4}{24}\right)x^4+\dots$ $=\dfrac{1}{a^2}-\left(\dfrac{a^4+2}{2a^4}\right)x^2+\left(\dfrac{24+12a^4+a^8}{24a^6}\right)x^4+\dots$ $c_1=\dfrac{1}{a^2}$, $c_2=-\left(\dfrac{a^4+2}{2a^4}\right)$ and $c_3=\dfrac{24+12a^4+a^8}{24a^6}$

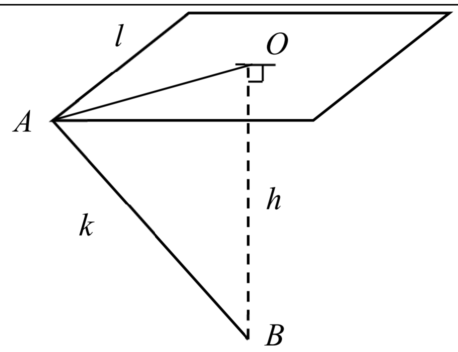
9(b)	Expansion is valid for $\left\lvert \frac{x^2}{a^2} \right\rvert < 1$ $x^2 < a^2$ $x^2 - a^2 < 0$ $(x-a)(x+a) < 0$ $-a < x < a$
9(c)	$\int_0^1 \frac{\cos 2x}{x^2+4}\ \mathrm{d}x \approx \int_0^1 \frac{1}{4} - \frac{1}{4}\left(\frac{4}{2}+\frac{1}{4}\right)x^2 + \frac{1}{4}\left(\frac{1}{16}+\frac{1}{2}+\frac{16}{24}\right)x^4\ \mathrm{d}x$ $= \int_0^1 \frac{1}{4} - \frac{9}{16}x^2 + \frac{59}{192}x^4\ \mathrm{d}x$ $= \left[\frac{1}{4}x - \frac{3}{16}x^3 + \frac{59}{960}x^5\right]_0^1$ $= \frac{119}{960}$

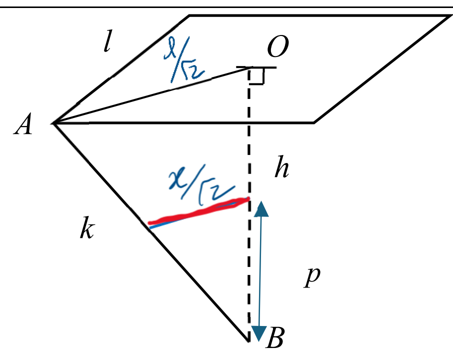
No.	**Solution**
10(a)(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\tan x}\right) = \mathrm{e}^{\tan x}\sec^2 x$
10(a)(ii)	$\int \mathrm{e}^{\tan x}\sec^3 x \sin x\ \mathrm{d}x = \int \left(\mathrm{e}^{\tan x}\sec^2 x\right)\tan x\ \mathrm{d}x$ $= \mathrm{e}^{\tan x}\tan x - \int \mathrm{e}^{\tan x}\sec^2 x\ \mathrm{d}x$ $= \mathrm{e}^{\tan x}\tan x - \mathrm{e}^{\tan x} + C$ Side working: $u = \tan x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$ $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{\tan x}\sec^2 x \quad \Rightarrow \quad v = \mathrm{e}^{\tan x}$
10(b)	$1 - 4x = A(3-8x) + B$ By observation, $-8A = -4 \Rightarrow A = \frac{1}{2}$ $3A + B = 1 \Rightarrow B = 1 - \frac{3}{2} = -\frac{1}{2}$ $\int \frac{1-4x}{\sqrt{1-4x^2+3x}}\ \mathrm{d}x$ $= \int \frac{\frac{1}{2}(3-8x) - \frac{1}{2}}{\sqrt{1-4x^2+3x}}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{3-8x}{\sqrt{1-4x^2+3x}}\ \mathrm{d}x - \frac{1}{2}\int \frac{1}{\sqrt{1-4x^2+3x}}\ \mathrm{d}x$

	$=\frac{1}{2}\int(3-8x)\left(1-4x^2+3x\right)^{-\frac{1}{2}}\ \mathrm{d}x-\frac{1}{2}\int\frac{1}{2\sqrt{\left(\frac{5}{8}\right)^2-\left(x-\frac{3}{8}\right)^2}}\ \mathrm{d}x$ $=\frac{1}{2}(2)\sqrt{1-4x^2+3x}-\frac{1}{4}\sin^{-1}\left(\frac{x-\frac{3}{8}}{\frac{5}{8}}\right)+C$ $=\sqrt{1-4x^2+3x}-\frac{1}{4}\sin^{-1}\left(\frac{8x-3}{5}\right)+C$
10(c)	$u=\sqrt{1+x}\Rightarrow x=u^2-1$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{1+x}}=\frac{1}{2u}$ When $x=3,\ u=\sqrt{1+3}=\sqrt{4}=2$. When $x=1,\ u=\sqrt{1+1}=\sqrt{2}$ $\int_1^3\frac{x^2}{\left(\sqrt{1+x}\right)^3}\ \mathrm{d}x=\int_{\sqrt{2}}^2\frac{\left(u^2-1\right)^2}{u^3}\ 2u\ \mathrm{d}u$ $=2\int_{\sqrt{2}}^2\frac{u^4-2u^2+1}{u^2}\ \mathrm{d}u$ $=2\int_{\sqrt{2}}^2 u^2-2+\frac{1}{u^2}\ \mathrm{d}u$ $=2\left[\frac{u^3}{3}-2u-\frac{1}{u}\right]_{\sqrt{2}}^2$ $=2\left[\left(\frac{8}{3}-4-\frac{1}{2}\right)-\left(\frac{2\sqrt{2}}{3}-2\sqrt{2}-\frac{\sqrt{2}}{2}\right)\right]$ $=2\left(-\frac{11}{6}+\frac{11\sqrt{2}}{6}\right)$ $=\frac{11}{3}\left(\sqrt{2}-1\right)$

No.	**Solution**
11(a)	

11(b)	$0 \in R_f$ but $0 \notin D_g$ Since $R_f \not\subset D_g$, gf does not exist.
11(c)	 Every horizontal line $y = k$ cuts the graph of $y = g(x)$ at most once. Hence, g is one-to-one. Inverse of g exists. $y = -\ln(x-1)$ $e^{-y} = x - 1$ $x = e^{-y} + 1$ $g^{-1}(x) = e^{-x} + 1$
11(c)	$R_{g^{-1}} = D_g = [2, \infty)$ Let $R_{g^{-1}}$ be the new domain of f Using the graph in part (a), $R_{fg^{-1}} = \left(0, \frac{23}{3}\right]$ **Alternatively** Sketch the graph $y = fg^{-1}(x) = \frac{5(e^{-x}+1)+13}{(e^{-x}+1)^2 - 1}$ for $D_{g^{-1}} = R_g = (-\infty, 0]$ and $R_{fg^{-1}} = \left(0, \frac{23}{3}\right]$ 

No.	Solution
12(a)	 By considering one corner of the square as shown above, $OA=\frac{\sqrt{l^2+l^2}}{2}=\frac{\sqrt{2l^2}}{2}=\frac{1}{\sqrt{2}}l$ Using ΔOAB: $k^2=\left(\frac{l}{\sqrt{2}}\right)^2+h^2 \;\Rightarrow\; h=\sqrt{k^2-\frac{l^2}{2}}$ $(\because h>0)$ $V=\frac{1}{3}l^2\sqrt{k^2-\frac{l^2}{2}}$ $V^2=\frac{1}{9}l^4\left(k^2-\frac{l^2}{2}\right)=\frac{k^2l^4}{9}-\frac{l^6}{18}$ (shown)
12(b)	$V^2=\frac{k^2l^4}{9}-\frac{l^6}{18}$ Differentiating with respect to *l*, $2V\frac{dV}{dl}=\frac{4}{9}k^2l^3-\frac{1}{3}l^5$ -------- Eq(1) When $\frac{dV}{dl}=0$, $\frac{4}{9}k^2l^3-\frac{1}{3}l^5=0$ $\frac{1}{9}l^3\left(4k^2-3l^2\right)=0$ $l=0$ (NA) or $l^2=\frac{4}{3}k^2$ $l=\frac{2}{\sqrt{3}}k$ (since $l>0$) $V^2=\frac{k^2\left(\frac{4}{3}k^2\right)^2}{9}-\frac{\left(\frac{4}{3}k^2\right)^3}{18}=\frac{16}{243}k^6$ $V=\frac{4}{9\sqrt{3}}k^3$

	<u>Method 1: 2nd derivative test</u> To show that V is maximum, differentiate Eq (1) with respect to l: $2\left(V\frac{\mathrm{d}^2V}{\mathrm{d}l^2}+\left(\frac{\mathrm{d}V}{\mathrm{d}l}\right)^2\right)=\frac{4}{3}k^2l^2-\frac{5}{3}l^4$ Substitute $l^2=\frac{4}{3}k^2$, $\frac{\mathrm{d}V}{\mathrm{d}l}=0$: $2V\frac{\mathrm{d}^2V}{\mathrm{d}l^2}=\frac{4}{3}k^2\left(\frac{4}{3}k^2\right)-\frac{5}{3}\left(\frac{4}{3}k^2\right)^2$ $=-\frac{32}{27}k^4$ $\frac{\mathrm{d}^2V}{\mathrm{d}l^2}=-\frac{16k^4}{27V}<0$ (since $V>0$ and $k^4>0$) Hence, the maximum value of V is $\frac{4}{9\sqrt{3}}k^3$. <u>Method 2: 1st deriviative test</u> $2V\frac{\mathrm{d}V}{\mathrm{d}l}=\frac{4}{9}k^2l^3-\frac{1}{3}l^5$ $\frac{\mathrm{d}V}{\mathrm{d}l}=\frac{4}{18V}l^3\left(k^2-\frac{3}{4}l^2\right)$ $=\frac{2}{9V}l^3\left(k-\frac{\sqrt{3}}{2}l\right)\left(k+\frac{\sqrt{3}}{2}l\right)$ <table><tr><td>l</td><td>$\left(\frac{2}{\sqrt{3}}k\right)^-$</td><td>$\frac{2}{\sqrt{3}}k$</td><td>$\left(\frac{2}{\sqrt{3}}k\right)^+$</td></tr><tr><td>$\frac{\sqrt{3}}{2}l$</td><td>$<k$</td><td>$k$</td><td>$>k$</td></tr><tr><td>$\frac{\mathrm{d}V}{\mathrm{d}l}$</td><td>+ve</td><td>0</td><td>-ve</td></tr></table> Hence, the maximum value of V is $\frac{4}{9\sqrt{3}}k^3$.
12(c)	 Let the sides of the surface area of the water be x at time t.

	By similar triangles, $\dfrac{x/\sqrt{2}}{l/\sqrt{2}} = \dfrac{p}{h} \Rightarrow x = \dfrac{p}{h}l$ Hence, volume of water $W = \dfrac{1}{3}x^2 p = \dfrac{p^3 l^2}{3h^2} = \dfrac{1}{3}p^3\left(\dfrac{\frac{4}{3}k^2}{\frac{1}{3}k^2}\right) = \dfrac{4}{3}p^3$ <u>Alternatively,</u> Dimensions of container: $l = \dfrac{2}{\sqrt{3}}k$ and $h = \sqrt{k^2 - \dfrac{l^2}{2}} = \dfrac{1}{\sqrt{3}}k$ Hence, $l = 2h$. This implies $x = 2p$ Hence, volume of water $W = \dfrac{1}{3}x^2 p = \dfrac{1}{3}\left(4p^2\right)p = \dfrac{4}{3}p^3$ Differentiate $W = \dfrac{4}{3}p^3$ w.r.t p, $\dfrac{\mathrm{d}W}{\mathrm{d}p} = 4p^2$ When $\dfrac{\mathrm{d}W}{\mathrm{d}t} = \dfrac{1}{3}$ and $p = \dfrac{k}{4}$, $\dfrac{\mathrm{d}p}{\mathrm{d}t} = \dfrac{\mathrm{d}p}{\mathrm{d}W} \times \dfrac{\mathrm{d}W}{\mathrm{d}t} = \dfrac{1}{4\left(\frac{k}{4}\right)^2} \times \dfrac{1}{3} = \dfrac{4}{3k^2}$ Rate of increase of p when $p = \dfrac{k}{4}$ is $\dfrac{4}{3k^2}$ cms^{-1} <u>Alternatively</u> Differentiate $W = \dfrac{4}{3}p^3$ w.r.t t, $\dfrac{\mathrm{d}W}{\mathrm{d}t} = 4p^2\left(\dfrac{\mathrm{d}p}{\mathrm{d}t}\right)$ When $\dfrac{\mathrm{d}W}{\mathrm{d}t} = \dfrac{1}{3}$ and $p = \dfrac{k}{4}$, $\dfrac{1}{3} = 4\left(\dfrac{k}{4}\right)^2\left(\dfrac{\mathrm{d}p}{\mathrm{d}t}\right)$ $\dfrac{\mathrm{d}p}{\mathrm{d}t} = \dfrac{4}{3k^2}$ Rate of increase of p when $p = \dfrac{k}{4}$ is $\dfrac{4}{3k^2}$ cms^{-1}