RAFFLES INSTITUTION
2023 YEAR 6 PRELIMINARY EXAMINATION

FURTHER MATHEMATICS
Paper 1

9649/01
September 2023
3 hours

Additional materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet and a graph paper booklet will be provided with this question paper. You should follow the instructions on the front cover of both booklets. If you need additional answer paper or graph paper ask the invigilator for a continuation booklet or graph paper booklet.

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

1 **(a)** Show, without sketching a graph, that there is exactly one root, α, of the equation $x^3 - e^{-2x} - 2 = 0$ in the interval [1, 2]. [2]

(b) Use linear interpolation once, on the interval [1, 2], to find an approximation, α_1, to α, giving one decimal place in your answer. [2]

(c) Use the Newton-Raphson method, with the starting value α_1, to find the value of α correct to 3 decimal places. [2]

2 A sequence of real numbers $u_1, u_2, u_3, \ldots$ satisfies the recurrence relation

$$u_1 = a \quad \text{and} \quad u_{n+1} = \frac{8u_n - 21}{u_n - 2}$$

for $a > 2$ and $n \geq 1$.

(a) Prove algebraically that, if the sequence converges, then it converges to either 3 or 7. [2]

(b) Use an algebraic method to prove that $u_n < u_{n+1} < 7$ if $3 < u_n < 7$. [4]

(c) Describe the behaviour of the sequence for $u_1 = 4$. [2]

3 The terms in the sequence $x_0, x_1, x_2, \ldots$ satisfy the recurrence relation

$$2ax_{n+2} + x_n = 2\sqrt{a}\, x_{n+1}$$

where a is a positive constant and $n \geq 0$.

(a) Find the general solution of this recurrence relation. [3]

(b) It is given that $a = 3$, $x_0 = 3$ and $x_1 = \frac{1}{\sqrt{3}}$. Show that

$$x_n = \mathrm{f}(n)\left[C\cos\left(\mathrm{g}(n)\right) + D\sin\left(\mathrm{g}(n)\right)\right]$$

where C and D are constants, and $\mathrm{f}(n)$ and $\mathrm{g}(n)$ are expressions of n, to be determined. [3]

(c) Determine, with reasons, the range of positive values of a such that x_n tends to zero as n tends to infinity. [2]

4 It is given that $\mathbf{A}$ is a 4×4 matrix with eigenvalues λ_i, $i=1, 2, 3, 4,$ such that $|\lambda_1|>|\lambda_2|>|\lambda_3|>|\lambda_4|>0$, and that $\mathbf{x}_i$ is an eigenvector of $\mathbf{A}$ associated with λ_i. It is known that if the eigenvalues are distinct, then the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is linearly independent.

(a) Show clearly why, for any $\mathbf{x}\in\mathbb{R}^4$ and positive integer k,

$$\mathbf{A}^k\mathbf{x}=\lambda_1^k\left(c_1\mathbf{x}_1+c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2+c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k\mathbf{x}_3+c_4\left(\frac{\lambda_4}{\lambda_1}\right)^k\mathbf{x}_4\right),$$

where c_1, c_2, c_3, c_4 are constants. [3]

(b) Explain why $\mathbf{A}^k\mathbf{x}\approx\lambda_1^k c_1\mathbf{x}_1$ for large values of k. [1]

The matrix $\mathbf{M}$ is defined to be

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and its eigenvalues are -1, and corrected to 5 decimal places, $-1.24698, 0.44504, 1.80194$.

(c) Find an eigenvector of $\mathbf{M}$ associated with the eigenvalue -1. [1]

(d) Let the eigenvalues of $\mathbf{M}$ be λ_i, $i=1, 2, 3, 4,$ such that $|\lambda_1|>|\lambda_2|>|\lambda_3|>|\lambda_4|>0$. Find the least integer value of k such that $\left|\left(\frac{\lambda_i}{\lambda_1}\right)^k\right|<0.0001$ for $i=2, 3, 4$. Use this value of k to find $\mathbf{M}^k\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, and deduce an approximate eigenvector of $\mathbf{M}$ associated with λ_1 in the form $\begin{pmatrix}u_1\\u_2\\1\\u_4\end{pmatrix}$, where u_1, u_2, u_4 are positive constants to be determined, correct to 2 decimal places. [3]

[Turn over

5 It is given that $\mathbf{A}=\begin{pmatrix} a & b+a \\ b-a & -a \end{pmatrix}$ where a and b are positive real constants.

(a) Show that b is an eigenvalue of $\mathbf{A}$ and find an eigenvector of $\mathbf{A}$ associated with b. [3]

(b) Find $\mathbf{A}^2$ and $\mathbf{A}^{-1}$. [3]

(c) Let n be a positive integer. Conjecture an expression for $\mathbf{A}^n$ when

- n is odd,
- n is even.

Use mathematical induction to prove the correctness of your conjecture. [5]

6 Let α be a real constant such that $\sin\alpha \neq 0$.

(a) Show that the polar equation

$$\frac{1}{r}=A\cos\theta+B\cos\left(\theta-\alpha\right), \qquad (*)$$

represents a line with cartesian equation $mx+ny=1$, where m and n can be expressed in terms of A, B and α. [2]

(b) A conic E, with focus at the origin O, has polar equation

$$r=\frac{a}{1+e\cos\theta}$$

where e is the eccentricity and a is a nonzero constant. Points P and Q lie on E corresponding to $\theta=\alpha-\beta$ and $\theta=\alpha+\beta$ respectively, for $0<\beta<\frac{\pi}{2}$.

(i) Using (*) in part **(a)**, show that the polar equation of the chord PQ may be expressed as

$$\frac{1}{r}=\frac{e}{a}\cos\theta+\frac{\sec\beta}{a}\cos\left(\theta-\alpha\right). \qquad [4]$$

(ii) Find the polar equation of the tangent to E at the point where $\theta=\alpha$. [1]

7 **(a)** Given that $z = w + \frac{1}{w}$, where $w = k(\cos\theta + i\sin\theta)$ as θ varies, and k is a positive constant, express the real part and the imaginary part of z in terms of k and θ. [2]

Hence show that the point representing z in an Argand diagram lies on a conic section and state the eccentricity of this conic section in terms of k. [3]

(b) On a single Argand diagram, sketch the loci

(i) $z = w + \frac{1}{w}$, where $w = 2(\cos\theta + i\sin\theta)$ as θ varies, [2]

(ii) $|z-2| = \frac{5}{2}$. [2]

Given that the complex number v satisfies parts **(b)(i)** and **(b)(ii)**, deduce the possible exact value(s) of

(iii) $\arg\left(v - \frac{3}{2}\right)$, in the interval $(-\pi, \pi]$, [2]

(iv) $|v+2| + |v-2|$. [1]

8 The matrix $\mathbf{A}$ is defined by

$$\begin{pmatrix} a & -1 & -1 \\ 1 & b & 3 \\ 0 & 1 & c \end{pmatrix}$$

where a, b and c are constants, and $ab \neq -1$.

It is given that the determinant of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is 0.

(a) Show that the dimension of the null space of $\mathbf{A}$ is 1 and that $c = \dfrac{1+3a}{1+ab}$. [4]

In the rest of the question, it is given further that $a = 1$ and $b = 0$.

(b) Give a geometric interpretation of the column space of $\mathbf{A}$. [1]

In the rest of the question, let $\mathbf{x}$ be a 3×1 matrix and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

(c) It is given that $\mathbf{Ax} = \mathbf{b}$ has no solution. Without any computation, justify whether $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ belongs to the column space of $\mathbf{A}$. [1]

When the equation $\mathbf{Ax} = \mathbf{b}$ has no solution, we can use the *Least Squares Method* to find an approximate solution. This approximate solution is called a *least squares solution*.

A least squares solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{v}$ such that $\mathbf{x} = \mathbf{v}$ is a solution to $\mathbf{A}^{\mathrm{T}}\mathbf{Ax} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$.

(d) By considering a system of linear equations, find all the possible least squares solutions, $\mathbf{v}$, to $\mathbf{Ax} = \mathbf{b}$. [3]

(e) By considering the scalar product of $\mathbf{Av}$ and $\mathbf{Av} - \mathbf{b}$, draw a diagram to illustrate the relation between $\mathbf{b}$ and $\mathbf{Av}$. [3]

(f) With reference to the origin O, point P has coordinates $(1, 0, 1)$ and set $W = \{(x, y, z) \mid x + z = y, \text{ where } x, y, z \in \mathbb{R}\}$.

Without any further calculations, explain clearly how to find the point in W that is the closest to P, and hence find this point.

[2]

9 A new flower-bed is being designed for a large garden. The flower-bed will occupy a shape defined by the curve C with parametric equations

$$x = 5\cos\theta + \cos 5\theta,$$
$$y = 5\sin\theta - \sin 5\theta.$$

Two of the lines of symmetry of C are $y = \frac{\sqrt{3}}{3}x$ and $y = \sqrt{3}x$.

(a) Sketch C and state the cartesian equations of the rest of the lines of symmetry. [4]

(b) **(i)** C can be represented by the polar equation $r = \mathrm{f}(\alpha)$. S and T are points on C, corresponding to $\alpha = 0$ and $\alpha = \tan^{-1}\frac{3}{2}$ respectively. Let O be the centre of the flower bed, corresponding to the origin in the graph of C. The gardener decides to plant geranium in Region A, the enclosed region bounded by ST on C, and the lines OS and OT.

Explain why the area of Region A cannot be determined by using the integral

$$\int_0^{\tan^{-1}\frac{3}{2}} \frac{1}{2}\left((5\cos\theta + \cos 5\theta)^2 + (5\sin\theta - \sin 5\theta)^2\right) \mathrm{d}\theta.$$ [1]

(ii) It is given that for any curve defined parametrically in parameter u,

$$\int_{u_1}^{u_2} \frac{1}{2}\left(x\frac{\mathrm{d}y}{\mathrm{d}u} - y\frac{\mathrm{d}x}{\mathrm{d}u}\right) \mathrm{d}u = \int_{\alpha_1}^{\alpha_2} \frac{1}{2}r^2 \ \mathrm{d}\alpha,$$

where

- r and α are the corresponding polar variables, and
- $\alpha = \alpha_1$ when $u = u_1$ and $\alpha = \alpha_2$ when $u = u_2$.

Use this result to find the area of Region A. [3]

(c) An artist decides to create a metal art piece in the middle of the flower bed using the curve C' defined by the parametric equations

$$x = k(5\cos\theta + \cos 5\theta),$$
$$y = k(5\sin\theta - \sin 5\theta),$$

for $0 < k < 1$ and $0 \le \theta \le 2\pi$.

This art piece is formed by rotating C' π radians about the y-axis and the art piece is then coated with a special paint to protect it from the weather.

Show that the exact surface area to be coated with the special protective paint is $\frac{d}{2}\pi k^2$, where d is an integer to be determined. [5]

[Turn over

10 **(a)** A satellite is moving in an elliptical orbit with the centre of the Earth at one of its foci. If r and R represent the least and greatest distance, respectively, between the satellite and the centre of the Earth, find the eccentricity of the orbit in terms of r and R. [3]

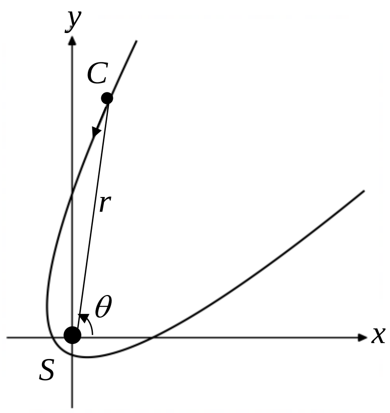
(b) A polar satellite has an elliptical orbit with the centre of the Earth at one of its foci. The *perigee*, the position of the polar satellite that is nearest to the Earth, is 4200 km above the South Pole and the *apogee*, the position of the polar satellite that is furthest away to the Earth, is 54000 km above the North Pole. Assume that the radius of the Earth is 6400 km.

(i) Find the height of the polar satellite above the surface of the Earth when the polar satellite crosses the equatorial plane. [3]

(ii) *Kepler's Second Law of Planetary Motion* states that the line segment joining the centre of the Earth to the satellite sweeps out equal areas in equal times.

The polar satellite takes 0.5 hours to travel from the perigee to the position when it first crosses the equatorial plane. Find the time taken for the satellite to travel from the position when it first crosses the equatorial plane to the apogee. [3]

(c) The diagram shows the parabolic orbit of a comet C with the Sun S at its focus. The direction of motion of the comet is indicated in the diagram.



Taking S as the pole, the distance SC is given by r, and the polar angle that the line SC makes with the initial line is given by θ. Two successive observations were made as this comet approached the Sun:

$$r = 50\,\text{AU when } \theta = \frac{\pi}{3} \quad \text{and} \quad r = 80\,\text{AU when } \theta = \frac{\pi}{4},$$

where 1 AU is the radius of the Earth's orbit (assumed circular) around the Sun. Find the shortest distance between the comet and the Sun. [4]

End of Paper